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POTENTIAL METHOD IN THEORY OF THERMOELASTICITY OF BINARY MIXTURES

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In this paper, the boundary value problems of the linear theory of thermoelasticity of binary mixtures are investigated by means of the potential method (boundary integral method). The basic properties of thermoelastopotentials are treaded. The uniqueness and existence theorems of solutions of the interior and exterior boundary value problems by means potential method and multidimensional singular integral equations are proved. The Sommerfeld-Kupradze type radiation conditions are established. The existence of eigenfrequencies of the interior homogeneous boundary value problems of steady oscillations is studied.

Keywords: *binary mixtures, thermoelasticity, potential method, uniqueness and existence theorems.*

1 Introduction

A general thermodynamic theory of two interacting continua was constructed by Green and Naghdi [1], while the nonlinear theory of mixtures of two- or many-component elastic solids was developed by Green and Steel [2]. A linear variant of the latter theory was proposed by Steel [3]. In [2,3], the interaction force of two components depends on a difference in partial displacement velocities (the diffusion model). The theory of binary mixtures of thermoelastic solids, in which the component interaction force depends on a difference of partial displacements (the shift model), was constructed by Iesan [4]. The same author proved uniqueness theorems of the solutions for initial boundary value problems of the linear version of semigroup theory in [5]. In [2-5], the mixture components are assumed to have the same temperature value. In [6], fundamental solutions of steady oscillation equations of the diffusion and shift models of the linear theory of two thermoelastic solids are constructed in terms of elementary functions, and some basis properties of these solutions are established. The linear and nonlinear theory of two thermoelastic solids with components having different temperature values were respectively constructed by Khoroshun and Soltanov [7] and Iesan [8]. An extensive review of the results of the mixture theory can be found in books [9,10] and papers [11-13].

The investigation of boundary value problems of mathematical physics by the classical potential method has a hundred year history. The application of this method to the basic spatial boundary value problems of the elasticity theory reduces these problems to multidimensional singular integral equations [14]. In [15, 16], Muskhelishvili developed the theory of one-dimensional singular integral equations and, using this theory, studied plane boundary-value problems of the elasticity theory. Owing to the works of Mikhlin [17], Kupradze and his pupils [14, 18, 19], the theory of multidimensional singular integral equations has presently been worked out with sufficient completeness. This theory makes it possible to investigate three-dimensional problems not only of classical elasticity theory, but also problems of elasticity theory with conjugated fields. Two- and three-dimensional problems of the linear theory of binary mixtures are studied by means of the potential method in [20-23].

In this paper, the boundary value problems of the linear theory of thermoelasticity of binary mixtures are investigated by means of the potential method. The uniqueness and existence theorems of solutions of the interior and exterior boundary value problems by means potential method and multidimensional singular integral equations are proved. The Sommerfeld-Kupradze type radiation conditions are established. The existence of eigenfrequencies of the interior homogeneous boundary value problems of steady oscillations is studied.

2 Basic Equations

Let $x=(x_1,x_2,x_3)$ be the point of the Euclidean three-dimensional space E^3 , The system of equations of steady oscillations in the shift model of the linear theory of thermoelasticity of binary mixture is written as [8]

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$$a_1\Delta u + b_1 \operatorname{grad}\operatorname{div} u + c\Delta w + d \operatorname{grad}\operatorname{div} w + \omega^2\rho_1 u - \alpha(u-w) - \alpha_1 \operatorname{grad}\theta = 0,$$
$$c\Delta u + d \operatorname{grad}\operatorname{div} u + a_2\Delta w + b_2 \operatorname{grad}\operatorname{div} w + \omega^2\rho_2 w + \alpha(u-w) - \alpha_2 \operatorname{grad}\theta = 0, \qquad (1)$$
$$(\alpha_0\Delta + i\omega\alpha_3)\theta + i\omega\alpha_1 T_0 \operatorname{div} u + i\omega\alpha_2 T_0 \operatorname{div} w = 0,$$
where $u=(u_1,u_2,u_3)$ and $w=(w_1,w_2,w_3)$ are the partial displacements, θ is the temperature measured from the constant absolute temperature T_0; $a_1,a_2,b_1,b_2,c,d,\alpha_0,\alpha_1,\alpha_2,\alpha_3$ are constitutive coefficients; ρ_1 and ρ_2 are partial density constants; ω is the oscillation frequency, $\omega>0$, Δ is the Laplacian, α is the shift coefficient. The system (1) may be written as
$$A(D_x)U(x) = 0, \qquad (2)$$
where $A(D_x)$ is the matrix differential operator corresponding left-hand side of (1), $U=(u,w,\theta)$ $=(U_1,U_2,\ldots,U_7)$.

3 Basic Boundary Value Problems

Let S be the closed surface surrounding the finite domain Ω^+ in E^3. $S\in C^{2,\nu}$, $0<\nu\le 1$, $\bar{\Omega}^+=\Omega^+\cup S$, $\Omega^-=E^3\setminus\bar{\Omega}^+$.

A vector function U is called regular in Ω^- (or Ω^+) if

1. $U_l\in C^2(\Omega^-)\cap C^1(\bar{\Omega}^-)$ (or $U_l\in C^2(\Omega^+)\cap C^1(\bar{\Omega}^+)$);
2. $U_l(x)=\sum_{j=1}^{5}U_{lj}(x)$, $U_{lj}\in C^2(\Omega^-)\cap C^1(\bar{\Omega}^-)$;
3. $(\Delta+k_j^2)U_{lj}(x)=0$,

$$\left(\frac{\partial}{\partial|x|}-ik_j\right)U_{lj}(x)=e^{ik_j|x|}o(|x|^{-1}), \qquad (3)$$

for $|x|>>1$, where k_j is the wave number, $l=1,2,\ldots,7$, $j=1,2,\ldots,5$, $|x|=(x_1^2+x_2^2+x_3^2)^{1/2}$.

Equalities in (3) is a Sommerfeld-Kupradze type *radiation conditions* in the linear theory of binary mixtures of thermoelastic solids. The basic boundary value problems of steady oscillations of linear theory of thermoelasticity of binary mixture are formulated as follows.

Problem $(I)_f^+$: Find a regular solution to system (2) for $x\in\Omega^+$ that satisfies the boundary condition
$$\lim_{\Omega^+\ni x\to z\in S} U(x)\equiv\{U(z)\}^+ = f(z).$$

Problem $(I)_f^-$: Find a regular solution to system (2) for $x\in\Omega^-$ that satisfies the boundary condition
$$\lim_{\Omega^-\ni x\to z\in S} U(x)\equiv\{U(z)\}^- = f(z), \qquad (4)$$
where f is the known vector function on S.

4 Uniqueness Theorems

We have the following results.

Theorem 1. Exterior boundary value problem $(I)_f^-$ admits at most one regular solution.

Theorem 2. Interior homogeneous boundary value problem $(I)_0^+$ has a non-trivial solution $U=(u,w,0)$ in the class of regular vectors, where the vector $V=(u,w)$ is a solution to the system
$$a_1\Delta u + b_1 \operatorname{grad}\operatorname{div} u + c\Delta w + d \operatorname{grad}\operatorname{div} w + \omega^2\rho_1 u - \alpha(u-w) = 0,$$
$$c\Delta u + d \operatorname{grad}\operatorname{div} u + a_2\Delta w + b_2 \operatorname{grad}\operatorname{div} w + \omega^2\rho_2 w + \alpha(u-w) = 0, \qquad (5)$$
$$\alpha_1 \operatorname{div} u + \alpha_2 \operatorname{div} w = 0, \quad \text{for} \quad x\in\Omega^+$$
satisfying the boundary condition
$$\{V(z)\}^+ = 0; \qquad (6)$$
the problems $(I)_0^+$ and (5), (6) have the same eigenfrequencies.

5 Existence Theorem

In this section, the existence theorem of regular solution of the problem $(I)_f^-$ is proved by means of the potential method and theory of multidimensional singular integral equations. For the definitions a normal-type singular integral operator and index of operators see Ref. [17]. The basic theory of one and multidimensional singular integral equations is given in Ref. [14, 15].

In the sequel we use the following notation:

1. $Z^{(1)}(x,g)=\int_S \Gamma(x-y)g(y)d_yS$ is a single-layer potential;
2. $Z^{(2)}(x,g)=\int_S [P(D_y,n(y))\Gamma^*(x-y)]^* g(y)d_yS$ is a double-layer potential, where Γ is the fundamental matrix of the operator $A(D_x)$; g is seven-component vectors, P is a generalized stress operator, $n(y)$ is the external unit vector to S at y, and the superscript $*$ denotes transposition.

Remark. The matrix Γ is constructed in Ref. [6].

We seek a regular solution to problem $(I)_f^-$ in the form
$$U(x)=Z^{(2)}(x,g)+a'Z^{(1)}(x,g), \quad x\in\Omega^-, \qquad (7)$$
where $a'=a_1'+ia_2'$; a_1' and a_2' are the real numbers, $a_1'>0$, $a_2'<0$, and g is the required seven-component vector.

Obviously, the vector U is solution of equation
$$A(D_x)U(x)=0 \quad \text{for } x\in\Omega^-.$$

Keeping in mind the boundary condition (4), from (7) we obtain singular integral equation

$$Kg(z) \equiv -\frac{1}{2} g(z) + Z^{(2)}(z,g) + a' Z^{(1)}(z,g)$$

$$= f(z) \qquad \text{for} \quad z \in S\,, \tag{8}$$

where singular integral operator K is of the normal type and $\operatorname{ind} K = 0$.

Now we prove that the equation

$$Kg(z) = 0 \tag{9}$$

has only a trivial solution.

Indeed, let g a solution of Eq. (9) and $g \in C^{1,\nu}(S)$. The vector U defined by Eq. (7) is a regular solution of problem $(I)_0^-$. Using Theorem 1, we have

$$U(x) = 0 \quad \text{for} \quad x \in \Omega^-. \tag{10}$$

On the other hand from Eq. (7) we get

$$\{U(z)\}^- + \{U(z)\}^+ = -g(z), \tag{11}$$

$$\{P(D_z,n)U(z)\}^- - \{P(D_z,n)U(z)\}^+ = a' g(z), \tag{12}$$

where $z \in S$. Therefore, by (10), from Eqs. (11), (12) we obtain

$$\{P(D_z,n)U(z) + a'U(z)\}^+ = 0 \ \text{ for } \ z \in S\,. \tag{13}$$

Hence, the vector U is a solution of equation

$$A(D_x)U(x) = 0 \qquad \text{for} \ x \in \Omega^+, \tag{14}$$

satisfying the boundary condition (13). From Eqs. (13), (14) we obtain (for details see Ref. [23])

$$\{U(z)\}^+ = 0\,. \tag{15}$$

Finally, using Eqs. (10) and (15), from (11) we have $g(z) = 0$ for $z \in S$.

Thus, the homogeneous Eq. (9) has only a trivial solution, and therefore Eq. (8) is always solvable for an arbitrary vector f.

We have thereby proved the following theorem.

Theorem 3. If $S \in C^{2,\nu}$, $f \in C^{1,\nu'}(S)$, $0 < \nu' \le \nu \le 1$, then a regular solution of the problem $(I)_f^-$ exists, is unique, and is represented by sum (7), where g is a solution of the singular integral equation (8), which is always solvable for an arbitrary vector f.

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