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We study limits on a primordial magnetic field arising from cosmological data, including that from big bang nucleosynthesis, cosmic microwave background polarization plane Faraday rotation limits, and large-scale structure formation. We show that the physically relevant quantity is the value of the effective magnetic field, and limits on it are independent of how the magnetic field was generated.

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I. INTRODUCTION

There is much interest in the origin of the coherent part of the large-scale μG magnetic fields in galaxies [1].¹ A leading possible explanation is that these large-scale magnetic fields are the amplified remnants of a primordial seed magnetic field generated in the early Universe [5–7]. Such early magnetogenesis models include those in which seed magnetic field generation occurs during inflation or shortly thereafter, or during a cosmological phase transition (such as the electroweak or QCD transition). Clearly the strength of the seed magnetic field should be small enough so as to not generate a larger than observed cosmological anisotropy. Magnetic field energy density contributes to the relativistic (radiation) energy density and thus another requirement is that it not exceed the big bang nucleosynthesis (BBN) bound on the radiation energy density.

There are two main questions that need to be answered: (1) Are the amplitude and statistical properties of any of these seed magnetic fields such that, after amplification by a realistic model, they can explain the strengths and correlation lengths of the observed magnetic fields in large-scale structures (LSSs) such as galaxies? and, (2) Are any of the seed magnetic fields detectable through cosmological observations, such as cosmic microwave background (CMB) measurements² or LSS observations? And, if yes, what are

the observational constraints on such primordial magnetic fields?

In this paper we focus on the second question and consider two cosmological consequences of a primordial magnetic field: Faraday rotation of the CMB polarization plane and the effect on LSS formation. As two of the effects of a primordial magnetic field, these have been widely discussed in the literature. See Refs. [8–13] for discussions of magnetic-field-induced CMB anisotropies,³ Refs. [15–17] for the Faraday rotation effect, and Refs. [18–22] for effects of a primordial magnetic field on LSS formation.

It has become conventional to derive the cosmological effects of a seed magnetic field by using a magnetic field spectral shape (parametrized by the spectral index n_B) and the smoothed value of the magnetic field (B_λ) at a given scale λ (which is usually taken to be 1 Mpc). We develop here a different and more correct formalism based on the effective magnetic field value that is determined by the total energy density of the magnetic field. As a striking consequence, we show that even an extremely small smoothed magnetic field of 10^{-29} G at 1 Mpc, with the Batchelor spectral shape ($n_B = 2$) at large scales, can leave detectable signatures in CMB or LSS statistics.⁴

We also show that the conventional approach based on the smoothed magnetic field results in some confusion when considering phase-transition generated magnetic fields [26] with spectral shape sharper (on large scales) than the white noise spectrum (i.e. with $n_B > 0$). In this

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¹On larger scales there have been two recent claims of an observed lower limit of order 10^{-15} G on the intergalactic magnetic field [2,3], as well as one claimed detection of a field of this strength [4]. Prior to these observations, the intergalactic magnetic field was observationally only limited to be smaller than a few nG.

²A cosmological magnetic field induces all three kinds of gravitational perturbations, scalar, vector, and tensor; all three of which contribute to CMB temperature and polarization anisotropies.

³The effects of a homogeneous magnetic field on the scalar mode of CMB fluctuations, including the resulting non-Gaussianity of the CMB temperature map, are discussed in Refs. [14].

⁴This strong limit on the primordial magnetic field is the consequence of the BBN bound and the sharp shape of the magnetic field at large scales [23,24]. The low efficiency of gravitational wave production by a cosmological magnetic field [24] results in a weaker limit on the seed field from the direct detection of the induced gravitational wave signal [25].

case the total energy density of the magnetic field, ρ_B , is mainly concentrated on large wave numbers (small length scales). In what follows we show that primordial magnetic field effects on cosmological scales are determined not by the amplitude of the magnetic field on these scales, but rather by the total energy density of the magnetic field.

Here we consider limits on a seed magnetic field from the observational constraint on the CMB polarization plane rotation angle and observational constraints on the formation of the first bound structures. Both tests give comparable limits on the effective magnetic field, ranging from 10^{-9} to 10^{-7} G, depending on the spectral shape of the magnetic field. Note that these limits are of order of the BBN bound. The best limit on the seed magnetic field is for the scale-invariant case that can be generated during inflation [27,28].

In a Universe with only scalar mode perturbations, the CMB B -polarization signal vanishes. The CMB B polarization signal can arise from vector or tensor perturbations, and thus B -polarization detection-based tests are powerful tools for probing nonstandard cosmological models and the relic gravitational wave background. Since a cosmological magnetic field can source a CMB B -polarization signal, a crucial test to limit the magnetic field is based on CMB B -polarization measurements. In a separate paper we will address the cosmological magnetic field energy density limits that can result from this test.

The structure of our paper is as follows. In Sec. II we review magnetic field statistical properties. In Sec. III we examine the CMB polarization plane Faraday rotation effect and the resulting magnetic field limits. In Sec. IV we consider the influence of a magnetic field on LSS statistics and determine the resulting limits on the magnetic field. We discuss our results and conclude in Sec. V.

II. MODELING THE PRIMORDIAL MAGNETIC FIELD

A stochastic Gaussian magnetic field is fully described by its two-point correlation function. For simplicity we consider here the case of the nonhelical magnetic field for which the two-point correlation function in wave number space is [11]

$$\langle B_i^*(\mathbf{k}) B_j(\mathbf{k}') \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}') P_{ij}(\hat{\mathbf{k}}) P_B(k). \quad (1)$$

Here i and j are spatial indices, $i, j \in (1, 2, 3)$, $\hat{k}_i = k_i/k$ a unit wave vector, $P_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - \hat{k}_i \hat{k}_j$ the transverse plane projector, $\delta^{(3)}(\mathbf{k} - \mathbf{k}')$ the Dirac delta function, and $P_B(k)$ is the power spectrum of the magnetic field.⁵

⁵We use

$$B_j(\mathbf{k}) = \int d^3x e^{i\mathbf{k} \cdot \mathbf{x}} B_j(\mathbf{x}), \quad B_j(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k} \cdot \mathbf{x}} B_j(\mathbf{k}), \quad (2)$$

when Fourier transforming between position and wave number spaces. We assume flat spatial hypersurfaces (consistent with current observational indications, [29]).

We define the smoothed magnetic field B_λ through the mean-square magnetic field [11]

$$B_\lambda^2 = \langle \mathbf{B}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x}) \rangle_\lambda, \quad (3)$$

where the smoothing is done on a comoving length λ with a Gaussian smoothing kernel function $\propto \exp[-x^2/\lambda^2]$. Corresponding to the smoothing length λ is the smoothing wave number $k_\lambda = 2\pi/\lambda$. The power spectrum $P_B(k)$ is assumed to be a simple power law on large scales, $k < k_D$ (where k_D is the cutoff wave number),

$$P_B(k) = P_{B0} k^{n_B} = \frac{2\pi^2 \lambda^3 B_\lambda^2}{\Gamma(n_B/2 + 3/2)} (\lambda k)^{n_B}, \quad (4)$$

and assumed to vanish on small scales where $k > k_D$.

The energy density of the magnetic field is [13]

$$\rho_B(\eta_0) = \frac{B_\lambda^2(k_D \lambda)^{n_B+3}}{8\pi \Gamma(n_B/2 + 5/2)}. \quad (5)$$

We define the effective magnetic field B_{eff} through $\rho_B = B_{\text{eff}}^2/(8\pi)$ and thus we get for the scale-invariant spectrum ($n_B = -3$ [27]) $B_{\text{eff}} = B_\lambda$ for all values of λ . The scale-invariant case is the only case where the values of the effective and smoothed fields coincide.

We need to define the magnetic field cutoff wave number k_D . We assume that the cutoff scale is determined by the Alfvén wave damping scale $k_D \sim v_A L_S$ where v_A is the Alfvén velocity and L_S the Silk damping scale [10]. Such a description is more appropriate when we are dealing with a homogeneous magnetic field and the Alfvén waves are the fluctuations $\mathbf{B}_1(\mathbf{x})$ with respect to a background homogeneous magnetic field \mathbf{B}_0 ($|\mathbf{B}_1| \ll |\mathbf{B}_0|$). In the case of the stochastic magnetic field we generalize the Alfvén velocity definition, see Ref. [11], by referring to the analogy between the effective magnetic field and the homogeneous magnetic field. Assuming that the Alfvén velocity is determined by B_{eff} , a simple computation gives the expression of k_D in terms of B_{eff} [24]:

$$\frac{k_D}{1 \text{ Mpc}^{-1}} = 1.4 \sqrt{\frac{(2\pi)^{n_B+3} h}{\Gamma(n_B/2 + 5/2)}} \left(\frac{10^{-7} \text{ G}}{B_{\text{eff}}} \right). \quad (6)$$

Here h is the Hubble constant in units of $100 \text{ km s}^{-1} \text{ Mpc}^{-1}$. The BBN limit on the effective magnetic field strength, $B_{\text{eff}} \leq 8.4 \times 10^{-7} \text{ G}$ [24], gives an upper limit on the cutoff wave number k_D ,

$$k_D^{\text{BBN}} \geq 0.17 h^{1/2} \frac{(2\pi)^{(n_B+3)/2}}{\Gamma^{1/2}(n_B/2 + 5/2)} \text{ Mpc}^{-1}. \quad (7)$$

In the case of an extremely large magnetic field it is possible to have $\lambda_D > 1 \text{ Mpc}$. At this point it would seem unreasonable (unjustified) to consider a smoothing scale $\lambda = 1 \text{ Mpc}$ as is conventionally done.

III. CMB POLARIZATION PLANE ROTATION

The presence of a primordial magnetic field during recombination causes a rotation of the CMB polarization plane through the Faraday effect [15]. The rms rotation angle $\alpha_{\text{rms}} = (\langle \alpha^2 \rangle)^{1/2}$ induced by a stochastic magnetic field with smoothed amplitude B_λ and spectral index n_B is given by

$$\langle \alpha^2 \rangle = \sum_l \frac{2l+1}{4\pi} C_l^\alpha, \quad (8)$$

where the rotation multipole power spectrum C_l^α is [17]

$$C_l^\alpha \simeq \frac{9l(l+1)}{(4\pi)^3 q^2 \nu_0^4} \frac{B_\lambda^2}{\Gamma(n_B/2 + 3/2)} \left(\frac{\lambda}{\eta_0}\right)^{n_B+3} \int_0^{x_S} dx x^{n_B} j_l^2(x). \quad (9)$$

Here η_0 is the conformal time today, ν_0 is the CMB photon frequency, $q^2 = 1/137$ is the squared elementary charge in cgs units, $j_l(x)$ is a Bessel function with argument $x = k\eta_0$, and $x_S = k_S\eta_0$ where $k_S = 2 \text{ Mpc}^{-1}$ is the Silk damping scale. In the case of an extreme magnetic field which just satisfies the BBN bound, k_D might become less than the Silk damping scale. In this case the upper limit in the integral above must be replaced by $x_D = k_D\eta_0$.

In terms of B_{eff} , Eq. (9) can be rewritten in the following form:

$$C_l^\alpha \simeq 1.6 \times 10^{-4} \frac{l(l+1)}{(k_D\eta_0)^{n_B+3}} \left(\frac{B_{\text{eff}}}{1 \text{ nG}}\right)^2 \left(\frac{100 \text{ GHz}}{\nu_0}\right)^4 \frac{n_B+3}{2} \times \int_0^{x_S} dx x^{n_B} j_l^2(x), \quad (10)$$

and, as a result,

$$\alpha_{\text{rms}} \simeq 0.14^\circ \left(\frac{B_{\text{eff}}}{1 \text{ nG}}\right) \left(\frac{100 \text{ GHz}}{\nu_0}\right)^2 \frac{\sqrt{n_B+3}}{(k_D\eta_0)^{(n_B+3)/2}} \times \left[\sum_{l=0}^{\infty} (2l+1)l(l+1) \int_0^{x_S} dx x^{n_B} j_l^2(x) \right]^{1/2}. \quad (11)$$

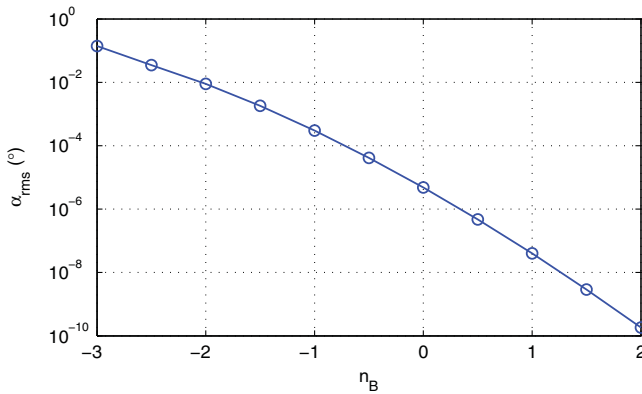


FIG. 1 (color online). rms rotation angle α_{rms} as a function of spectral index n_B for the case when $B_{\text{eff}} = 1 \text{ nG}$ and $\nu_0 = 100 \text{ GHz}$. Circles correspond to the computed values.

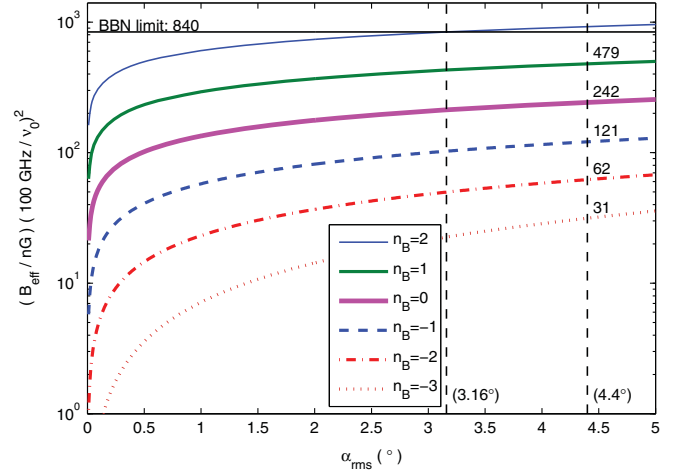


FIG. 2 (color online). Effective magnetic field limits set by the measurement of the rotation angle α_{rms} for different spectral indices ($n_B = -3, -2, -1, 0, 1, 2$, from bottom to top). The horizontal solid line shows the upper limit set by BBN. Vertical dashed lines correspond to the angles $\alpha_{\text{rms}} = 3.16^\circ$ that are set by the BBN limit on the effective magnetic field with spectral index $n_B = 2$ and $\alpha_{\text{rms}} = 4.4^\circ$ set by the WMAP 7-year data. The numerical values of the effective magnetic field constraints (in nG at 100 GHz) from the $\alpha_{\text{rms}} = 4.4^\circ$ limit are shown on the graph for each spectral index value.

It is of interest to compare Eq. (11) with the corresponding result, Eq. (2) of Ref. [15], derived for a homogeneous magnetic field and at frequency $\nu_0 = 30 \text{ GHz}$,

$$\alpha_{\text{rms}} \simeq 1.6^\circ \left(\frac{B_0}{1 \text{ nG}}\right) \left(\frac{30 \text{ GHz}}{\nu_0}\right)^2. \quad (12)$$

Both expressions agree for $n_B \rightarrow -3$ after accounting for $\sum_l (2l+1)j_l^2(x) = 1$ and the fact that Bessel functions peak at $x \sim l$ for given l (see the Appendix).

Figure 1 shows the rms rotation angle α_{rms} , Eq. (11), as a function of the spectral index n_B when the effective magnetic field is normalized to be 10^{-9} G . The WMAP 7-year data limits the rms rotation angle to be less than 4.4° at 95% C. L. [30]. This allows us to limit the effective magnetic field as shown in Fig. 2.

IV. LARGE-SCALE STRUCTURE

A primordial tangled magnetic field can also induce the formation of structures in the Universe. In particular, these fields can play an important role in the formation of first structures (see, e.g., Refs. [9,18–22,31,32]).

The magnetic-field-induced matter power spectrum $P(k)$ is $\propto k^4$ for $n_B > -1.5$ and $\propto k^{2n_B+7}$ for $n_B \leq -1.5$ [19,32]. The cutoff scale of the power spectrum is determined by the larger of the magnetic Jeans' wave number k_J and the thermal Jeans' wave number k_{therm} (for a detailed discussion, see, e.g., Ref. [22]). Here the magnetic Jeans' wave number is (see, e.g., Ref. [32])

$$k_J \simeq (230^{(n_B+3)/2} \times 13.8)^{2/(n_B+5)} \left(\frac{1 \text{ nG}}{B_{\text{eff}}} \right) \text{Mpc}^{-1}. \quad (13)$$

Unlike the Λ CDM matter power spectrum, the magnetic-field-induced matter power spectrum increases at small scales and can exceed the Λ CDM matter one at small scales (for a comparison of these two spectra, see, e.g. Fig. 3 of Ref. [19]). And, therefore, one of the more important contributions of the additional power induced by magnetic fields is to the formation of the first structures in the Universe (e.g. Refs. [18,20,21] and references therein).

In Fig. 3 we show the linear mass dispersion $\sigma(M)$ for matter power spectra induced by a primordial magnetic field with $B_{\text{eff}} = 6 \text{ nG}$ at $z = 10$ for different values of n_B . Notable features of Fig. 3 are (a) the mass dispersion on small scales is larger for a larger value of n_B ; and, (b) for $n_B \geq -1.5$, the mass dispersion drops more sharply at larger scales than for $n_B \leq -1.5$. We focus here on the mass dispersion on the smallest scales, as these scales are more relevant for the formation of the first structures in the Universe. These first structures were responsible for the reionization of the Universe at $z \simeq 10$. To obtain meaningful constraints on B_{eff} from the formation of first structures, we need to know how the curves shown in Fig. 3 vary as B_{eff} is changed and as the Universe evolves.

The mass dispersion $\sigma(M, z)$ evolves with the time dependence of the growing mode of the linear density perturbations sourced by the primordial magnetic field [19,32]. The growing mode is $\propto a(t)$, the scale factor, at high redshifts, the same as in the “standard” Λ CDM case without a magnetic field. To account for this evolution the curves corresponding to σ in Fig. 3 must be scaled by roughly a factor of $\simeq 11/(1+z)$ for redshifts $z \gg 1$.

It can be shown that the value of σ at the smallest scales ($M \simeq 10^6 \text{ M}_\odot$) is invariant under a change in B_{eff} if the cutoff scale is determined by k_J : an increase/decrease in

the value of B_{eff} is compensated by a decrease/increase in the value of k_J . However, if B_{eff} is decreased to a value at which $k_{\text{therm}} \leq k_J$, then the value of σ decreases with a decrease in B_{eff} , as the cutoff scale becomes independent of the value of B_{eff} .

It has been shown that the dissipation of magnetic fields in the post-recombination era can substantially alter the thermal and ionization history of the Universe [18,20,22]. In particular, this dissipation raises the matter temperature and therefore the Jeans’ scale in the IGM (intergalactic medium). For $B_{\text{eff}} \geq 1 \text{ nG}$ the matter temperature rises to $\simeq 10^4 \text{ K}$ as early as $z \geq 100$, [20], resulting in a steep rise in the Jeans’ scale as compared to the usual case. The Jeans’ wave number corresponding to this temperature is $k_{\text{therm}} \simeq 10 \text{ Mpc}^{-1}$ (see, e.g., Fig. 4 of Ref. [22]).

WMAP results show that the Universe reionized at $z \simeq 10$. This reionization was caused by the nonlinear collapse of the first structures, followed by star formation and the emission of UV photons from the collapsed halos. For a virialized structure in the spherical collapse model, the linear mass dispersion $\sigma \simeq 1.7$. This implies that the value of σ at the scales of interest at $z \simeq 10$ is not expected to be much higher than 1.7. Consider the $n_B = 2$ model in Fig. 3; the value of mass dispersion at the smallest scales is $\simeq 100$, which means that the first structures formed at $z \simeq 650$ in this case (the redshift of the collapse of first structures is $\simeq 6.5\sigma_{\text{max}}$, where σ_{max} is the maximum value of σ at $z \simeq 10$), which can certainly be ruled out by the WMAP data on CMB anisotropies. A similar argument can be used to rule out almost all the models shown in Fig. 3. Only the nearly scale-invariant models with $n_B \simeq -3$ do not put strong constraints on the strength of the magnetic field. As argued above, the value of mass dispersion at the smallest scales to collapse is nearly independent of the magnetic field strength unless B_{eff} decreases to a value such that $k_J = k_{\text{therm}}$. In this case, the value of σ decreases below those shown in Fig. 3. We have explored a wide range of B_{eff} for the range of spectral indices shown in Fig. 3. We find that the range of acceptable values is 1–3 nG. In Fig. 4 we show the B_{eff} corresponding to $k_J = k_{\text{therm}}$. Notwithstanding various complications discussed above, this figure gives a rough sense of the acceptable range of B_{eff} over the entire range of n_B .

In the foregoing, we neglect the impact of the Λ CDM model on the process of reionization. As the density fields induced by the Λ CDM model and the magnetic field are uncorrelated, the matter power spectra owing to these two physical phenomena would add in quadrature. The smallest structures to collapse at $z \simeq 10$ in the WMAP-normalized Λ CDM model are 2.5σ fluctuations of the density field as opposed to the magnetic field case where 1σ collapse is possible (Fig. 3). This means the number of collapsed halos is more abundant in the latter case. Therefore, depending on the star-formation history, if the magnetic-field-induced halo collapse made an important contribution to the reionization process, the far rarer halos from Λ CDM would

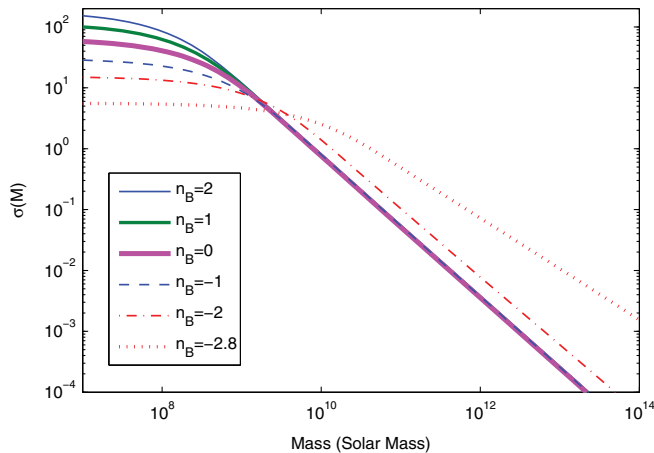


FIG. 3 (color online). The mass dispersion at $z = 10$ for $B_{\text{eff}} = 6 \text{ nG}$ as a function of magnetic field power spectral index n_B . From top to bottom (at the left-hand side of the plot), the curves correspond to $n_B = 2, 1, 0, -1, -2, -2.8$.

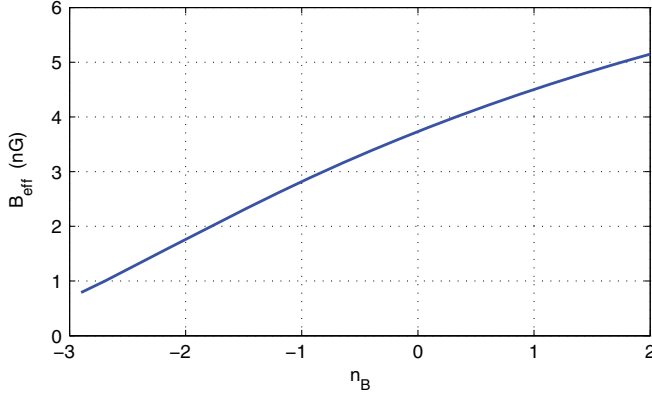


FIG. 4 (color online). Constraint on the magnetic field strength B_{eff} as a function of the power spectral index n_B .

have made a negligible impact (for further details and references see Ref. [21]).

V. CONCLUSIONS

In this paper we study the large-scale imprints of a cosmological magnetic field, such as the rotation of the CMB polarization plane and formation of the first bound structures. We derive the corresponding limits on a primordial magnetic field energy density, expressed as limits on the effective value of the magnetic field, B_{eff} . These limits are identical to limits on the smoothed magnetic field B_λ (smoothed over a length scale λ that is conventionally taken to be 1 Mpc) only in the case of the scale-invariant magnetic field (when $n_B = -3$). For a steep magnetic field with spectral index $n_B = 2$ the difference between $B_{\lambda=1 \text{ Mpc}}$ and B_{eff} is enormous (greater than 10^{15}). We show that using the smoothed magnetic field can result in some confusion; e.g. an extremely small smoothed magnetic field on large scales does not mean that this field cannot leave observable traces on cosmological scales.

An intergalactic magnetic field of effective value larger than 1–10 nG (with, depending on magnetic spectral index, corresponding values of $B_{\lambda=1 \text{ Mpc}}$ in the range 10^{-8} – 10^{-26} G) is ruled out by cosmological data. These limits of 1–10 nG are consistent with recent observational bounds on the intergalactic magnetic field [2–4] if the field was generated in the early Universe with spectral shape $n_B \leq 1$. This favors the inflationary magnetogenesis scenario.

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APPENDIX: EVALUATING THE RIGHT-HAND SIDE OF EQ. (11) WHEN $n_B \rightarrow -3$

The $\sqrt{n_B + 3}$ factor in the numerator of the right-hand side of Eq. (11) is compensated by a corresponding $1/\sqrt{n_B + 3}$ from the Bessel function integral when the spectral index $n_B \rightarrow -3$ and so the expression for α_{rms} remains finite in this limit. To establish this we use properties of the Bessel function. Recall that $j_l^2(x)$ peaks at $x \sim l$, as shown in Fig. 5. This allows us to replace the factor $l(l+1)j_l^2(x)$ by $x^2 j_l^2(x)$ (the accuracy of this approximation is of order 15%–20%). The next step is to perform the sum over l . It is obvious that there is cutoff multipole number l_C that corresponds to the cutoff wave number, $l_C \sim \min(x_D, x_S)$. Now $j_l^2(x)$ satisfies

$$\sum_{l=0}^{\infty} (2l+1)j_l^2(x) = 1, \quad (\text{A1})$$

while we are interested in computing $\sum_{l=0}^{l_C} (2l+1)j_l^2(x)$. The Silk damping scale cutoff multipole number is $l_S \simeq 16000$, [17]. Figure 6 shows that the sum to l_S converges to 1.

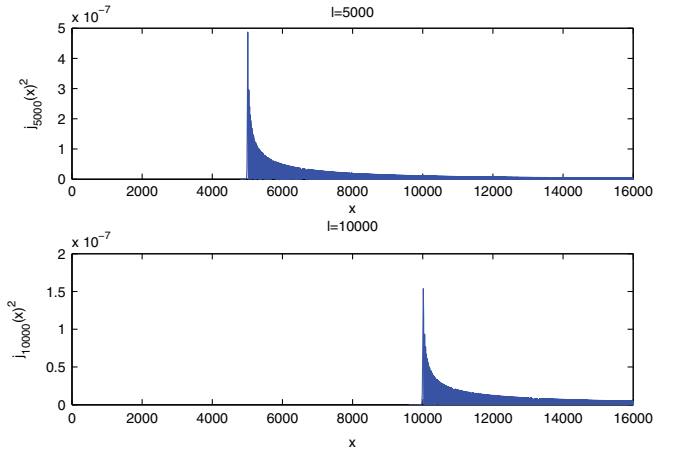


FIG. 5 (color online). The squared spherical Bessel functions $j_l^2(x)$ for $l = 5000$ (top) and $l = 10000$ (bottom). Clearly $j_l^2(x)$ peaks at $x \approx l$.

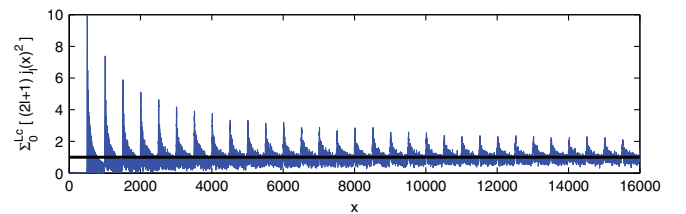


FIG. 6 (color online). The sum of the squared spherical Bessel function $\sum_{l=0}^{l_C} (2l+1)j_l^2(x)$ for $l_C = x_S \simeq 16000$. The sum converges to 1 (horizontal solid line).

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