Faraday rotation of the cosmic microwave background polarization by a stochastic magnetic field

Arthur Kosowsky,[1,*] Tina Kahniashvili,[2,†] George Lavrelashvili,[3,‡] and Bharat Ratra[4,§]

[1]*Department of Physics and Astronomy, Rutgers University, 136 Frelinghuysen Road, Piscataway, New Jersey 08854, USA*
[2]*Department of Physics, Kansas State University, 116 Cardwell Hall, Manhattan, Kansas 66506, USA and Center for Plasma Astrophysics, Abastumani Astrophysical Observatory, 2A Kazbegi Avenue, GE-0160 Tbilisi, Georgia*
[3]*Department of Theoretical Physics, A. Razmadze Mathematical Institute, GE-0193 Tbilisi, Georgia*
[4]*Department of Physics, Kansas State University, 116 Cardwell Hall, Manhattan, Kansas 66506, USA*

(Received 30 September 2004; published 28 February 2005)

A primordial cosmological magnetic field induces Faraday rotation of the cosmic microwave background polarization. This rotation produces a curl-type polarization component even when the unrotated polarization possesses only gradient-type polarization, as expected from scalar density perturbations. We compute the angular power spectrum of curl-type polarization arising from small Faraday rotation due to a weak stochastic primordial magnetic field with a power-law power spectrum. The induced polarization power spectrum peaks at arc minute angular scales. Faraday rotation is one of the few cosmological sources of curl-type polarization, along with primordial tensor perturbations, gravitational lensing, and the vector and tensor perturbations induced by magnetic fields; the Faraday rotation signal peaks on significantly smaller angular scales than any of these, with a power spectrum amplitude which can be comparable to that from gravitational lensing. Prospects for detection are briefly discussed.

DOI: 10.1103/PhysRevD.71.043006 PACS numbers: 98.70.Vc, 42.25.Ja, 98.80.–k

I. INTRODUCTION

A primordial cosmological seed magnetic field has been proposed to explain the existence of observed large-scale (~ 10 kpc), μG strength, ordered magnetic fields observed in galaxies and galaxy clusters [1,2]. Cosmological perturbations induced by such a magnetic field and the corresponding cosmic microwave background (CMB) temperature and polarization anisotropies have been the subject of a number of recent studies [3–17]. The strength of such a hypothetical cosmological magnetic field can be significantly constrained by comparing model predictions and microwave background observational data [18].

Several studies of the evolution of cosmological magnetic fields with power-law power spectra have been done [19,20]. The strongest constraints result from amplification of primordial magnetic fields via an inverse cascade mechanism, which takes power on small scales and transfers it to larger scales; this process is effective for fields with significantly increasing power on small scales [21,22]. Such "blue" power spectra result from magnetic fields generated in small scales at late time, for example, during cosmological phase transitions. In this work we consider general power-law magnetic fields, including fields generated during an early inflation epoch, which are not strongly constrained by their subsequent evolution. Observational and theoretical constraints on primordial fields are discussed in more detail in the concluding section of this paper.

A distinctive method of constraining a cosmological magnetic field is to study the rotation of the microwave background polarization orientation due to the Faraday effect, as the radiation propagates to us in the cosmological magnetic field. A homogeneous magnetic field of strength 10^{-9} G induces a measurable rotation of order 1° at a frequency of 30 GHz [23]. Faraday rotation is the classic method of probing large-scale magnetic fields in the universe, using quasars as sources of polarized radiation, e.g. [24]. Faraday rotation of radiation from discrete point sources can constrain the primordial magnetic field power spectrum [25], and can also probe the evolution of the magnetic field with redshift [26]. The microwave background in principle offers a great advantage because it provides a source of polarized light from a fixed redshift which covers the entire sky; with sufficiently precise measurements the projected magnetic field can be mapped in detail. The drawback is that the polarization of the microwave background is very small, only a part in a million, and the Faraday rotation expected from a cosmological magnetic field is also small and so challenging to detect.

In general, a polarization field has two independent components, known in the microwave background context as G and C [27] (for "gradient" and "curl"), or equivalently as E and B modes [28]. It is well known that primordial scalar density perturbations induce only G polarization, and that C polarization must arise from other production processes, notably primordial tensor perturbations [29,30], gravitational lensing of the G-polarization component [31], and vector and tensor perturbations from, e.g., magnetic fields [13]. Faraday rotation, which rotates polarization orientations, can also be a source of C polarization given an initial G-polarization field.

*Electronic address: kosowsky@physics.rutgers.edu
†Electronic address: tinatin@phys.ksu.edu
‡Electronic address: lavrela@itp.unibe.ch
§Electronic address: ratra@phys.ksu.edu

In this paper we compute the power spectrum of microwave background C polarization resulting from Faraday rotation by a stochastic primordial magnetic field with a given power spectrum and helicity spectrum. We find that the power spectrum distinctively peaks on arc minute angular scales, significantly smaller than other commonly considered sources of C polarization. Previous work on the polarization power spectrum has considered the simpler but less realistic special case of a constant magnetic field [23,32,33], or considered a stochastic magnetic field but only calculated the power spectrum of the rotation measure rather than of the microwave background polarization [34]. Sophisticated analysis techniques for extracting microwave background power spectra from maps of the sky have been developed, so measuring the power spectrum of Faraday rotation and then confirming its scaling with frequency will be much more efficient than requiring frequency information from the outset to measure the rotation. Measuring the power spectrum of Faraday rotation also sidesteps any systematic errors associated with comparing observations at different frequencies; this is a significant advantage since these measurements will all be dominated by systematic errors.

The rest of the paper is organized as follows. The next section defines the primordial magnetic field power spectrum and helicity spectrum and explains the approximation we employ to separate polarization generation from polarization orientation rotation at the surface of last scattering. Section III derives the power spectrum of the Faraday rotation measure, which distills the Faraday effect of the tangled primordial magnetic field into a simple scalar function on the sky. We give an explicit demonstration that any helical component of the magnetic field does not contribute to Faraday rotation. Then given this rotation field and a primordial G-polarization field, Sec. IV derives an analytic expression for resulting power spectrum of C polarization. Section V numerically evaluates these expressions and presents the power spectra for both the rotation field and the C polarization for a range of magnetic field power spectra, and the final section discusses these results in the context of future experiments to measure small-scale microwave background polarization. Two short appendixes contain some useful but technical mathematical results.

II. MAGNETIC FIELD MODEL

A. Magnetic field power spectrum

We will assume the existence of a cosmological magnetic field generated during or prior to the early radiation-dominated epoch, with the energy density of the magnetic field a first-order perturbation to the standard Friedmann-Lemaître-Robertson-Walker homogeneous cosmological spacetime model. The conductivity of the primordial plasma is high so we can work in the infinite-conductivity limit for all scales larger than the damping scale set by photon and neutrino diffusion. This results in a "frozen-in" magnetic field and a corresponding electric field $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ where $\mathbf{v}$ is the plasma fluid velocity. Since the fluid velocity is always small in the early universe on the scales considered, the electric field is always negligible compared to the magnetic field and will not be considered further. Neglecting fluid backreaction onto the magnetic field, the spatial and temporal dependence of the magnetic field separates, with magnetic flux conservation fixing the temporal dependence to be the simple scaling $\mathbf{B}(\mathbf{x}, \eta) = \mathbf{B}(\mathbf{x})/a^2$ where a is the scale factor and η conformal time.

We also assume the magnetic field is a Gaussian random field. Taking into account the possible helicity of the field [35], the magnetic field spectrum in wave number space is [14,16]

$$\langle B_i^*(\mathbf{k}) B_j(\mathbf{k}')\rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}')[P_{ij} P_B(k) + i\epsilon_{ijl}\hat{k}_l P_H(k)], \tag{1}$$

where $P_B(k)$ and $P_H(k)$ are the symmetric and helical parts of the magnetic field power spectrum, ϵ_{ijl} is the antisymmetric tensor, and the plane projector $P_{ij} \equiv \delta_{ij} - \hat{k}_i\hat{k}_j$ satisfies $P_{ij}P_{jk} = P_{ik}$ and $P_{ij}\hat{k}_j = 0$ with unit wave number components $\hat{k}_i = k_i/k$. We use the convention

$$B_j(\mathbf{k}) = \int d^3x e^{i\mathbf{k}\cdot\mathbf{x}} B_j(\mathbf{x}), \qquad B_j(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} B_j(\mathbf{k}) \tag{2}$$

when Fourier transforming between real and wave number spaces.

The power spectrum $P_B(k)$ is related to the energy density of the magnetic field, while the helicity part $P_H(k)$ is related to the spatial average $\langle \mathbf{B} \cdot (\nabla \times \mathbf{B})\rangle$ [14]. Transforming from an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 = \hat{\mathbf{k}}\}$ to the helicity basis [36]

$$\mathbf{e}^{\pm}(\mathbf{k}) = -\frac{i}{\sqrt{2}}(\mathbf{e}_1 \pm i\mathbf{e}_2), \qquad \mathbf{e}_3 = \hat{\mathbf{k}}, \tag{3}$$

the power spectra $P_B(k)$ and $P_H(k)$ can be expressed in terms of magnetic field components in the helicity basis as [14,16]

$$\langle B^+(\mathbf{k})B^+(-\mathbf{k}') + B^-(\mathbf{k})B^-(-\mathbf{k}')\rangle = -(2\pi)^3 P_B(k)\delta^{(3)}(\mathbf{k} - \mathbf{k}'), \tag{4}$$

$$\langle B^+(\mathbf{k})B^+(-\mathbf{k}') - B^-(\mathbf{k})B^-(-\mathbf{k}')\rangle = (2\pi)^3 P_H(k)\delta^{(3)}(\mathbf{k} - \mathbf{k}'). \tag{5}$$

We describe both the symmetric and helical parts by simple power laws

$$P_B(k) = A_B k^{n_B}, \qquad P_H(k) = A_H k^{n_H}. \tag{6}$$

If the magnetic field is generated in small scales, after

inflation, the power spectrum index is constrained to be $n_B \geq 2$ [37]. Also the amplitudes are generically constrained by $P_B(k) \geq |P_H(k)|$ [19,22,37], which implies $n_H \geq n_B$ [16,37]: a field cannot support a fixed helicity in the limit of zero field strength. Note that some authors define spectral indices which correspond to our $n_B + 3$ and $n_H + 3$ [19,38].

As a phenomenological normalization of the magnetic field, we smooth the field on a comoving scale λ by convolving with a Gaussian smoothing kernel $f_\lambda(\mathbf{x}) = N\exp(-x^2/2\lambda^2)$ to obtain the smoothed field $\mathbf{B}_\lambda(\mathbf{x})$, and introduce average values of energy density [13]

$$B_\lambda{}^2 \equiv \langle \mathbf{B}_\lambda(\mathbf{x}) \cdot \mathbf{B}_\lambda(\mathbf{x})\rangle = \frac{1}{\pi^2}\int_0^\infty dk k^2 e^{-k^2\lambda^2} P_B(k), \quad (7)$$

and helicity [16]

$$H_\lambda{}^2 \equiv \lambda|\langle \mathbf{B}_\lambda(\mathbf{x}) \cdot \{\nabla \times \mathbf{B}_\lambda(\mathbf{x})\}\rangle| = \frac{\lambda}{\pi^2}\int_0^\infty dk k^3 e^{-k^2\lambda^2}|P_H(k)|. \quad (8)$$

Equations (7) and (8) may be used to reexpress the power-law power spectra of Eqs. (6) in closed form as [13,14,16]

$$P_B(k) = \frac{(2\pi)^{n_B+5}}{2}\frac{B_\lambda^2}{\Gamma(n_B/2+3/2)}\frac{k^{n_B}}{k_\lambda^{n_B+3}}, \qquad k < k_D, \quad (9)$$

$$P_H(k) = \frac{(2\pi)^{n_H+5}}{2}\frac{H_\lambda^2}{\Gamma(n_H/2+2)}\frac{k^{n_H}}{k_\lambda^{n_H+3}}, \qquad k < k_D, \quad (10)$$

where the smoothing wave number $k_\lambda = 2\pi/\lambda$. We make the approximation that both spectra vanish for all wave numbers k larger than a damping wave number k_D. We assume that the magnetic field damping is due to Alfvén wave damping from photon viscosity, and the cutoff wave number to be [6,7,13]

$$\left(\frac{k_D}{\mathrm{Mpc}^{-1}}\right)^{n_B+5} \approx 2.9\times 10^4\left(\frac{B_\lambda}{10^{-9}\ \mathrm{G}}\right)^{-2}\left(\frac{k_\lambda}{\mathrm{Mpc}^{-1}}\right)^{n_B+3} h, \quad (11)$$

which will always be a much smaller scale than the Silk damping scale (thickness of the last scattering surface) for standard cosmological models (h is the Hubble constant in units of 100 km s^{-1} Mpc^{-1}).

B. Faraday rotation

A cosmological magnetic field at the epoch of last scattering will rotate the CMB polarization orientation in a given sky direction due to Faraday rotation (see, e.g., [39]). For a review of CMB polarization theory see [40]; for computational methods and statistics, see [27,28,41,42]. The time derivative of the orientation angle α of linearly polarized monochromatic radiation passing through a plasma in the presence of a magnetic field $\mathbf{B}(\mathbf{x}, \eta)$ is [23]

$$\omega_{\mathbf{B}}(\mathbf{x}, \mathbf{n}, \eta) = \dot{\alpha} \equiv \frac{d\alpha}{d\eta} = \frac{q^3 x_e n_e a}{2\pi m_e^2 \nu^2}\mathbf{B}(\mathbf{x}, \eta)\cdot\mathbf{n}. \quad (12)$$

Here $\mathbf{n}$ is the propagation direction of the radiation and ν the radiation frequency. The electron charge and mass are q and m_e, and n_e and x_e are the total electron number density and ionization fraction. We normalize the scale factor a by setting $a_0 = 1$ today. We employ cosmological units with $\hbar = 1 = c$.

In terms of the comoving magnetic field $\mathbf{B}(\mathbf{x}) = \mathbf{B}(\mathbf{x}, \eta)a^2$, Eq. (12) can be rewritten as

$$\omega_B(\mathbf{x}, \mathbf{n}, \eta) = \frac{3}{(4\pi)^2\nu_0^2 q}\dot{\tau}(\mathbf{x})\mathbf{B}(\mathbf{x})\cdot\mathbf{n} \quad (13)$$

with comoving frequency of the observed radiation ν_0 and the differential optical depth $\dot{\tau} = x_e n_e \sigma_T a$. Here $\sigma_T = 8\pi\alpha_{EM}^2/(3m_e^2)$ is the Thompson scattering cross section with the fine-structure constant $\alpha_{EM} = q^2 \approx 1/137$. We neglect inhomogeneities in the free electron density, assuming that $\dot{\tau}(\eta)$ is independent of $\mathbf{x}$; then the rotation angle observed today is

$$\alpha(\mathbf{n}, \eta_0) = \frac{3}{(4\pi)^2\nu_0^2 q}\int_{\eta_{\mathrm{dec}}}^{\eta_0} d\eta\dot{\tau}(\eta)\mathbf{B}(\mathbf{x})\cdot\mathbf{n}. \quad (14)$$

Throughout the rest of this paper, $\mathbf{B}$ represents the comoving value of the magnetic field.

Faraday rotation of the microwave background is a somewhat subtle problem, because the polarization is generated and rotated simultaneously in the region of the last scattering surface. In a rigorous treatment, these two effects must be computed together via the complete radiative transfer equations describing the evolution of polarization fluctuations. However, as long as the total rotation is small compared to $\pi/2$, the total rotated polarization angle can be expressed simply as an average of the rotated polarization angle from each infinitesimal piece of path length through the surface of last scattering, neglecting depolarization effects. We make the simplifying approximation that any magnetic field component with a wavelength shorter than the thickness of the surface of last scattering is neglected. For these components, the rotation of polarization generated at different optical depths will tend to cancel, leaving little net rotation. [This assumption breaks down if the magnetic field strength is dominated by components on small scales. In this case, a more realistic power-law dropoff in $P_B(k)$ should be used instead of a sharp cutoff.] This is equivalent to imposing a cutoff scale on the magnetic field at the photon damping scale at last scattering. Then we can treat the magnetic field as constant throughout the rotation region, so that the total rotation, which is the sum over the rotations of each infinitesimal piece of generated polarization, can be expressed as the total rotation incurred by the polarization generated at

some particular effective optical depth, which we denote by $\eta_{\rm rot}$.

Assuming that $\eta_{\rm rot}$ is the conformal time corresponding to $\tau = 1$ (the actual value is a little lower since the polarization visibility function peaks at a lower redshift than the temperature visibility function), we can derive a simple expression for the approximate Faraday rotation in a given observation direction. Writing $\mathbf{x} = \mathbf{n}(\eta_0 - \eta)$ (i.e., putting the observer at the origin of the coordinate system) and Fourier expanding the magnetic field, Eq. (14) can be written as

$$\alpha(\mathbf{n}, \eta_0) \simeq \frac{3}{4(2\pi)^5 \nu_0^2 q} \int d^3k \mathbf{B}(\mathbf{k}) \cdot \mathbf{n} e^{-i\mathbf{k}\cdot\mathbf{n}\Delta\eta}, \quad (15)$$

where $\Delta\eta = \eta_0 - \eta_{\rm rot}$. These approximations have split the polarization generation and rotation problems; we can treat the full problem as equivalent to the generation of polarization in the usual way, followed by rotation using an effective rotation screen just prior to the radiation reaching the observer. The rest of this paper is devoted to computing the properties of the effective rotation screen and the resulting power spectrum of microwave background polarization.

III. ROTATION POWER SPECTRUM

It is conventional to introduce a wavelength-independent measure of the rotation of the polarization orientation, the rotation measure $R(\mathbf{n}) \equiv \alpha(\mathbf{n})\nu_0^2$. The two-point correlation function of the effective rotation measure for a stochastic magnetic field based on the approximate solution in Eq. (15) is

$$\langle R(\mathbf{n})R(\mathbf{n}')\rangle \simeq \frac{9}{16(2\pi)^{10}q^2} \times \int d^3k \int d^3k' e^{i\mathbf{k}\cdot\mathbf{n}\Delta\eta} e^{-i\mathbf{k}'\cdot\mathbf{n}'\Delta\eta} \times n_i n_j' \langle B_i^*(\mathbf{k}) B_j(\mathbf{k}')\rangle \quad (16)$$

where $\Delta\eta \equiv \eta_0 - \eta_{\rm rot} \approx \eta_0$.

Prior to the ensemble averaging, it is convenient to decompose the vector plane wave into vector spherical harmonics [36]

$$\mathbf{B}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{n}\eta_0} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{\lambda=-1}^{1} A_{lm}^{(\lambda)}(\mathbf{k}) \mathbf{Y}_{lm}^{(\lambda)}(\mathbf{n}) \quad (17)$$

where λ labels the three orthonormal vector spherical harmonics defined by

$$\mathbf{Y}_{lm}^{(1)}(\mathbf{n}) = \frac{1}{\sqrt{l(l+1)}} \boldsymbol{\nabla}_{\mathbf{n}} Y_{lm}(\mathbf{n}),$$
$$\mathbf{Y}_{lm}^{(0)}(\mathbf{n}) = \frac{-i}{\sqrt{l(l+1)}} [\mathbf{n} \times \boldsymbol{\nabla}_{\mathbf{n}}] Y_{lm}(\mathbf{n}), \quad (18)$$
$$\mathbf{Y}_{lm}^{(-1)}(\mathbf{n}) = \mathbf{n} Y_{lm}(\mathbf{n}).$$

Here $\boldsymbol{\nabla}_{\mathbf{n}}$ is the two-dimensional covariant derivative orthogonal to the unit vector $\mathbf{n}$, so the harmonics $\lambda = 1$ and $\lambda = 0$ are transverse to $\mathbf{n}$, while the $\lambda = -1$ harmonic is parallel to $\mathbf{n}$. When the expansion (17) is substituted into Eq. (16), only the $\lambda = -1$ terms contribute since the magnetic fields only appear through the contraction $\mathbf{n} \cdot \mathbf{B}$. Explicit forms for the coefficients $A_{lm}^{(\lambda)}(\mathbf{k})$ are given in Ref. [36]; the one needed here is

$$A_{lm}^{(-1)} = \frac{4\pi}{2l+1} i^{l-1} \{\sqrt{l(l+1)}[j_{l+1}(k\eta_0) + j_{l-1}(k\eta_0)] \times \mathbf{B}(\mathbf{k}) \cdot \mathbf{Y}_{lm}^{(1)*}(\hat{\mathbf{k}}) - [(l+1)j_{l+1}(k\eta_0) - l j_{l-1}(k\eta_0)] \mathbf{B}(\mathbf{k}) \cdot \mathbf{Y}_{lm}^{(-1)*}(\hat{\mathbf{k}})\}. \quad (19)$$

The second term proportional to $\mathbf{Y}_{lm}^{(-1)}$ is zero since the magnetic field is divergenceless, $\mathbf{k} \cdot \mathbf{B}(\mathbf{k}) = 0$, leaving

$$A_{lm}^{(-1)} = 4\pi i^{l-1} \sqrt{l(l+1)} \frac{j_l(k\eta_0)}{k\eta_0} \mathbf{B}(\mathbf{k}) \cdot \mathbf{Y}_{lm}^{(1)*}(\hat{\mathbf{k}}). \quad (20)$$

With these definitions the rotation angle becomes

$$\alpha(\mathbf{n}, \eta_0) \simeq \frac{3}{32\pi^4 \nu_0^2 q} \int d^3k \sum_{lm} i^{-l+1} \sqrt{l(l+1)} \frac{j_l(k\eta_0)}{k\eta_0} \times \mathbf{B}(\mathbf{k}) \cdot \mathbf{Y}_{lm}^{(1)\star}(\hat{\mathbf{k}}) Y_{lm}(\mathbf{n}), \quad (21)$$

and for the magnetic field spectrum of Eq. (1) we obtain

$$\langle R(\mathbf{n})R(\mathbf{n}')\rangle \simeq \frac{9}{128\pi^5 q^2} \int dk k^2 P_B(k) \int d\Omega_{\hat{\mathbf{k}}} \times \sum_{lm} \sum_{l'm'} i^{l'-l} \sqrt{l(l+1)l'(l'+1)} \times \frac{j_l(k\eta_0) j_{l'}(k\eta_0)}{(k\eta_0)^2} Y_{lm}^*(\mathbf{n}) Y_{l'm'}(\mathbf{n}') \times \mathbf{Y}_{lm}^{(1)}(\hat{\mathbf{k}}) \cdot \mathbf{Y}_{l'm'}^{(1)*}(\hat{\mathbf{k}}). \quad (22)$$

In this result, the term proportional to $\hat{k}_i \hat{k}_j$ in the projector P_{ij} in the magnetic field power spectrum in Eq. (1) vanishes because $\hat{\mathbf{k}} \cdot \mathbf{Y}_{lm}^{(1)} = 0$. The term proportional to the magnetic field helicity spectrum $P_H(k)$ is also identically zero because $[\hat{\mathbf{k}} \times \mathbf{Y}_{lm}^{(1)}(\hat{\mathbf{k}})] \cdot \mathbf{Y}_{l'm'}^{(1)*}(\hat{\mathbf{k}}) = \mathbf{Y}_{lm}^{(0)}(\hat{\mathbf{k}}) \cdot \mathbf{Y}_{l'm'}^{(1)*}(\hat{\mathbf{k}}) = 0$. This result is in agreement with Ensslin and Vogt [43] and Campanelli *et al.* [34], who conclude that Faraday rotation cannot be used to reconstruct the helical part of the magnetic field spectrum. The opposite conclusion of Ref. [14] is erroneous. Physically, in real space, nonzero helicity of the magnetic field only affects the off-diagonal elements in the correlation matrix of the magnetic field components, while the Faraday rotation is due only to the diagonal component corresponding to the propagation direction; see [43] for a more detailed discussion.

Using the orthogonality of vector spherical harmonics,

$$\int d\Omega_{\hat{\mathbf{k}}} \mathbf{Y}_{lm}^{(\lambda)*}(\hat{\mathbf{k}}) \cdot \mathbf{Y}_{l'm'}^{(\lambda')}(\hat{\mathbf{k}}) = \delta_{\lambda\lambda'}\delta_{ll'}\delta_{mm'}, \tag{23}$$

and the usual spherical harmonic summation formula

$$P_l(\mathbf{n}\cdot\mathbf{n}') = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}^*(\mathbf{n})Y_{lm}(\mathbf{n}'), \tag{24}$$

Eq. (22) simplifies to

$$\langle R(\mathbf{n})R(\mathbf{n}')\rangle \simeq \frac{9}{128\pi^5 q^2}\sum_l \frac{2l+1}{4\pi} l(l+1)P_l(\mathbf{n}\cdot\mathbf{n}') \times \int dk k^2 P_B(k)\left(\frac{j_l(k\eta_0)}{k\eta_0}\right)^2, \tag{25}$$

with corresponding multipole moments

$$C_l^R \simeq \frac{9l(l+1)}{(4\pi)^3 q^2}\frac{B_\lambda^2}{\Gamma(n_B/2+3/2)}\left(\frac{\lambda}{\eta_0}\right)^{n_B+3}\int_0^{x_D} dx x^{n_B} j_l^2(x), \tag{26}$$

where $x_D = k_D\eta_0$ and the multipole moments are defined via

$$\langle R(\mathbf{n})R(\mathbf{n}')\rangle = \sum_l \frac{2l+1}{4\pi} C_l^R P_l(\mathbf{n}\cdot\mathbf{n}'). \tag{27}$$

The rotation angle power spectrum is simply given by the rescaling

$$C_l^\alpha = \nu_0^{-4} C_l^R. \tag{28}$$

Note that Refs. [14,44] have incorrect prefactors in their corresponding expressions for the rotation multipoles C_l^R, due to the non-self-consistent choice of the units.

This rotation multipole expression contains a sharp short-wavelength cutoff k_D; in reality, the effective cutoff will be smoothly spread over a range of scales. To prevent unphysical oscillations in the integral of Eq. (26), and to simplify the numerical evaluation of the integral, we replace the oscillatory function $j_l^2(x)$ by half of its envelope, $1/(2x^2)$, for all x larger than the second zero of $j_l(x)$. Then the tail of the integrand is just a trivial power law, and the total integral will not show oscillations, as would be expected from a more realistic cutoff function.

IV. *C* POLARIZATION FROM FARADAY ROTATION

The preceding section gives an expression for the power spectrum of an approximation to the polarization orientation Faraday rotation field $\alpha(\mathbf{n})$ which can be applied to the unrotated polarization field of the microwave background radiation. If the unrotated polarization field arises only from scalar perturbations it will only have a nonzero G-polarization component. Faraday rotation will induce nonzero C polarization; here we compute the C-polarization power spectrum. We use the differential geometry formalism of Ref. [27] for this computation, and work with tensors defined on the two-dimensional spherical manifold representing the sky.

We represent the unrotated polarized CMB with the tensor field $P_{ab}(\hat{\mathbf{n}})$ on the two-dimensional sphere orthogonal to the direction vector $\mathbf{n}$, and decompose this in terms of tensor spherical harmonics in the usual way

$$P_{ab} = \sum_{lm}[a_{lm}^G Y_{(lm)ab}^G + a_{lm}^C Y_{(lm)ab}^C]. \tag{29}$$

Here the usual lm indices are enclosed in parentheses to distinguish them from the ab tensor indices. Explicit forms for the orthonormal tensor spherical harmonics are

$$Y_{(lm)ab}^G(\mathbf{n}) = N_l\left[Y_{(lm):ab}(\mathbf{n}) + \frac{1}{2}g_{ab}l(l+1)Y_{(lm)}(\mathbf{n})\right],$$
$$Y_{(lm)ab}^C(\mathbf{n}) = \frac{N_l}{2}(Y_{(lm):ac}(\mathbf{n})\epsilon_b^c + Y_{(lm):bc}(\mathbf{n})\epsilon_a^c), \tag{30}$$

with the normalization factor $N_l = (2(l-2)!/(l+2)!)^{1/2}$. Here g_{ab} is the metric on the sphere, ϵ_{ab} is the antisymmetric Levi-Cività tensor, and the covariant derivative on the sphere is $A(\mathbf{n})_{:a} \equiv \nabla_{\mathbf{n}}A(\mathbf{n})$, using the notation of the previous section. For polarization from scalar perturbations, $a_{lm}^C = 0$ [29,30]. Primordial tensor perturbations, or the vector and tensor perturbations induced by magnetic fields, will give nonzero a_{lm}^C.

A rotation of the polarization orientation by an angle α is represented by the transformation

$$P'_{ab} = R_a{}^c P_{cb} \tag{31}$$

with the rotation operator given by

$$R_a{}^c = \cos(2\alpha)g_a{}^c + \sin(2\alpha)\epsilon_a{}^c. \tag{32}$$

In the limit of small rotation angle, $\alpha \ll 1$, the rotation operator reduces to $R_a{}^c \simeq g_a{}^c + 2\alpha\epsilon_a{}^c$, which is linear in α. In our case the rotation field varies with direction $\alpha = \alpha(\mathbf{n})$, so we decompose it into spherical harmonics

$$\alpha(\mathbf{n}) = \sum_{lm} b_{lm}Y_{(lm)}(\mathbf{n}). \tag{33}$$

We now use Eq. (29) to expand the resulting polarization field in tensor spherical harmonics. Inverting for the coefficients of the rotated C polarization we have

$$\begin{aligned} a_{lm}^C &= \int d\Omega_{\mathbf{n}} P'_{ab}(\mathbf{n})Y_{(lm)}^{Cab*}(\mathbf{n}) \\ &= 2\int d\Omega_{\mathbf{n}}\alpha(\mathbf{n})\epsilon_a{}^c P_{cb}(\mathbf{n})Y_{(lm)}^{Cab*}(\mathbf{n}) \\ &= 2\sum_{l_1m_1}\sum_{l_2m_2} b_{l_1m_1}a_{l_2m_2}^G \\ &\quad\times\int d\Omega_{\mathbf{n}} Y_{(l_1m_1)}(\mathbf{n})\epsilon_a{}^c Y_{(l_2m_2)cb}^G(\mathbf{n})Y_{(lm)}^{Cab*}(\mathbf{n}), \end{aligned} \tag{34}$$

and likewise for the G-polarization multipole coeffecients,

$$a^G_{lm} = 2\sum_{l_1m_1}\sum_{l_2m_2} b_{l_1m_1}a^C_{l_2m_2} \times \int d\Omega_{\mathbf{n}} Y_{(l_1m_1)}(\mathbf{n})\epsilon_a{}^c Y^C_{(l_2m_2)cb}(\mathbf{n})Y^{Gab*}_{(lm)}(\mathbf{n}). \quad (35)$$

The integrals over three spherical harmonics can be evaluated through repeated integrations by parts, after making use of the form of the tensor spherical harmonics and the identities

$$\begin{aligned} \epsilon_{ac}\epsilon_{bd} &= g_{ab}g_{cd} - g_{ad}g_{cb}, \\ \epsilon_{ca}\epsilon^c{}_b &= g_{ab} = -\epsilon_{ac}\epsilon^c{}_b, \qquad \epsilon_{ab:c} = 0 \end{aligned} \quad (36)$$

which eliminates the product of antisymmetric tensors in the integrand. We also need the eigenvalue equation $Y_{(lm):a}{}^{:a}(\mathbf{n}) = -l(l+1)Y_{(lm)}(\mathbf{n})$ and use the integration by parts formula

$$\int d\Omega_{\mathbf{n}} A_{:a}B = -\int d\Omega_{\mathbf{n}} AB_{:a}, \quad (37)$$

which has no surface term contribution as long as the integration is over the entire sphere. Using the explicit forms for $Y^G_{(lm)ab}(\mathbf{n})$ and $Y^C_{(lm)ab}(\mathbf{n})$ in Eqs. (30) and using Eq. (36) results in

$$\begin{aligned} &\int d\Omega_{\mathbf{n}} Y_{(l_1m_1)}(\mathbf{n})\epsilon_a{}^c Y^G_{(l_2m_2)cb}(\mathbf{n})Y^{Cab*}_{(lm)}(\mathbf{n}) \\ &= \frac{N_{l_2}N_l}{2}\int d\Omega_{\mathbf{n}} Y_{(l_1m_1)}(\mathbf{n})Y_{(l_2m_2):b}{}^{:b}(\mathbf{n})Y^*_{(lm):a}(\mathbf{n}) \\ &\quad - N_{l_2}N_l\int d\Omega_{\mathbf{n}} Y_{(l_1m_1)}(\mathbf{n})Y_{(l_2m_2):ab}(\mathbf{n})Y^*_{(lm)}(\mathbf{n}). \end{aligned} \quad (38)$$

For $G \leftrightarrow C$ in the integrand the result is the same except for an overall sign change. The derivatives in the first integral just add $l(l+1)$ factors.

The second integral is more involved, and is computed in Appendix A. Substituting Eqs. (A1) and (A7) into Eq. (38) and then into Eq. (34) gives finally

$$a^C_{lm} = N_l\sum_{l_1m_1}\sum_{l_2m_2} N_{l_2}K(l, l_1, l_2)b_{l_1m_1}a^G_{l_2m_2} \times \int d\Omega Y_{(l_1m_1)}Y_{(l_2m_2)}Y^*_{(lm)}, \quad (39)$$

where

$$K(l, l_1, l_2) \equiv -\frac{1}{2}(L^2 + L_1^2 + L_2^2 - 2L_1L_2 - 2L_1L + 2L_1 - 2L_2 - 2L), \quad (40)$$

with $L = l(l+1)$, $L_1 = l_1(l_1+1)$, and $L_2 = l_2(l_2+1)$. The integral over three spherical harmonics is a well-known expression in terms of Clebsch-Gordan coefficients, Eq. (A1). The coefficient a^G_{lm} is given by the same formula with the replacements $a^G_{l_2m_2} \to a^C_{l_2m_2}$ and $N_l \to -N_l$.

To derive the power spectrum, we need to evaluate

$$\langle a^C_{l'm'}a^C_{lm}\rangle \equiv C^C_l\delta_{ll'}\delta_{mm'}, \quad (41)$$

where the definition of C^C_l holds for any statistically isotropic distribution; here we assume that all relevant cosmological quantities satisfy this condition. Assuming that the rotation field and the original temperature polarization field are uncorrelated, we have

$$\langle b^*_{l_1m_1}b_{l_2m_2}a^{G*}_{l_3m_3}a^G_{l_4m_4}\rangle = \delta_{l_1l_2}\delta_{m_1m_2}\delta_{l_3l_4}\delta_{m_3m_4}C^\alpha_{l_1}C^G_{l_3}, \quad (42)$$

where C^α_l is the angular power spectrum of the rotation field and C^G_l is the G-polarization power spectrum of the unrotated polarization field just after decoupling. Using the expressions given in Eqs. (39) and (42) in Eq. (41) results in a long expression with sums over l_1m_1 and l_2m_2. This can be further simplified using the Clebsch-Gordan identity

$$\sum_{m_1m_2} C^{lm}_{l_1m_1l_2m_2}C^{l'm'}_{l_1m_1l_2m_2} = \delta_{ll'}\delta_{mm'}, \quad (43)$$

and we find

$$C^C_l = N_l^2\sum_{l_1l_2}N_{l_2}^2K(l, l_1, l_2)^2C^G_{l_2}C^\alpha_{l_1}\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)} \times (C^{l0}_{l_10l_20})^2. \quad (44)$$

This is our basic result for the C-polarization power spectrum induced by Faraday rotation of the primordial G polarization; we numerically evaluate it in the following section. (Numerical techniques for evaluating the remaining Clebsch-Gordan coefficients are given in Appendix B.) Note that the more familiar multipole coefficients C^E_l and C^B_l [28] are just a factor of 2 larger than C^G_l and C^C_l, respectively.

We note that, for a stochastic magnetic field, the cross correlations between the intrinsic temperature and G-polarization fluctuations from nonmagnetic perturbations and the C polarization from Faraday rotation result in $C^{TC}_l = 0$. These multipoles are linear in the magnetic field so the ensemble average vanishes. Nonzero correlations result only from the intrinsic T fluctuations arising from the magnetic field itself and the C polarization from Faraday rotation, but this case is not considered here. The C^{GC}_l correlation will also be zero unless the magnetic field has nonzero helicity, which follows from the invariance of $\langle QU\rangle$ under Faraday rotation by a stochastic magnetic field [45].[1]

V. NUMERICAL RESULTS

The rotation angle power spectrum, Eq. (28), is shown in Fig. 1 for values of the magnetic field spectral index

[1]Nonzero off-diagonal C^{GC}_l correlation will appear in the case of homogeneous magnetic field Ref. [33].

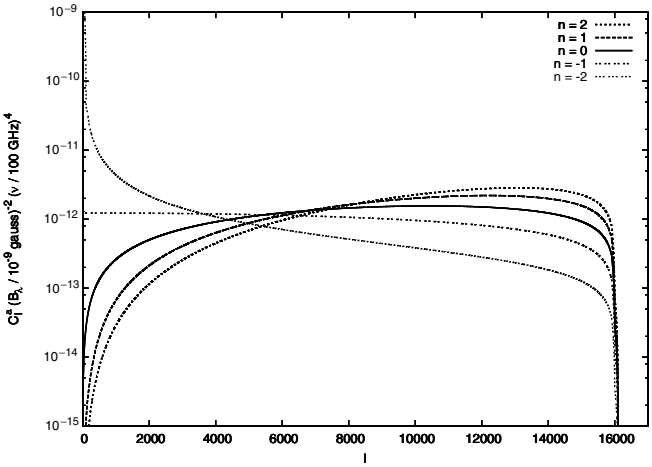

FIG. 1. The angular power spectrum of the Faraday rotation angle, C_l^α, induced by a stochastic magnetic field. The curves in order of decreasing amplitude on the right side of the plot correspond to magnetic field power spectral indices $n_B = 2$, 1, 0, -1, and -2. The magnetic fields have been normalized to a nanogauss at the smoothing scale $\lambda = 1$ Mpc.

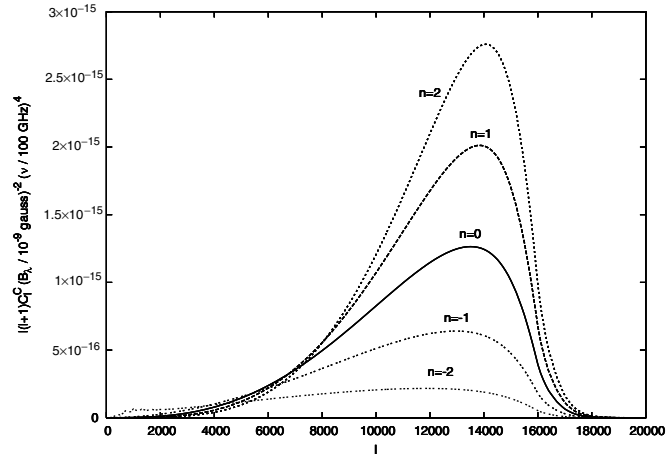

FIG. 2. The C-polarization power spectrum of the microwave background induced by the Faraday rotation field in Fig. 1, again with the magnetic field normalization scale $\lambda = 1$ Mpc.

ranging from $n_B = -2$ to $n_B = 2$. The plotted rotation power spectra are for a magnetic field with $B_\lambda = 10^{-9}$ G and an observation frequency of $\nu = 100$ GHz; the rotation power scales like $B_\lambda^2 \nu^{-4}$. We have assumed a cutoff scale $k_D = 2.0\ \mathrm{Mpc}^{-1}$ approximately corresponding to the Silk damping scale. The polarization has a root-mean-square rotation angle given by

$$\bar{\alpha} \equiv \langle \alpha^2 \rangle^{1/2} = \left[\sum_l \frac{2l+1}{4\pi} C_l^\alpha \right]^{1/2}. \tag{45}$$

At $\nu = 100$ GHz and $B_\lambda = 10^{-9}$ G with $\lambda = 1$ Mpc, $\bar{\alpha} \approx 0.3°$ for all considered values of n_B. This is consistent with $\bar{\alpha}$ found in Ref. [23] for a constant magnetic field of strength B_λ (around 1.6° at $\nu = 30$ GHz). For negative values of $n_B + 1$, the magnetic field power spectrum grows with the length scale, so the largest rotations are seen at large scales corresponding to small l values. The opposite is true for positive values of $n_B + 1$, which increase in power at smaller scales and larger l values.

The power spectrum of C polarization induced by this Faraday rotation power spectrum is displayed in Fig. 2. This power spectrum is also for $B_\lambda = 10^{-9}$ G and $\nu = 100$ GHz, scaling like $B_\lambda^2 \nu^{-4}$. The most notable feature of the C-polarization spectrum is its peak at small angular scales at $l > 10^4$. The cutoff at $l = 16\,000$ is sharper than in a more realistic damping model where it occurs over a range of scales somewhat past k_D, but the peak position and amplitude would be largely unchanged by a different damping prescription. We also assume a standard ΛCDM cosmological model and compute the C_l^G power spectrum appearing in Eq. (44) to $l = 5000$ using CMBFAST [46].

The peak angular scale is readily understood: at the last scattering surface Faraday rotation induces polarization fluctuations on angular scales corresponding to the characteristic wavelengths of the stochastic magnetic field; just subsequent to recombination the polarization is imprinted with fluctuations on scales much smaller than the quadrupole scale which induced the polarization fluctuations in the first place. Free streaming then shifts these fluctuations to even smaller scales. So the Faraday rotation power spectrum provides a distinctive signature of primordial magnetic fields.

This computation only applies to small rotation angles, which for a given magnetic field translates into a limiting lower observation frequency. The root-mean-square rotation $\bar{\alpha}$, Eq. (45), scales like $B_\lambda \nu_0^{-2}$. For a field strength of 10^{-9} G, observations at $\nu = 30$ GHz yield a mean rotation of around 3°, well within the small rotation limit, but observations at $\nu = 10$ GHz would give mean rotation angles of around 30°, for which the small-angle approximation employed in Eq. (34) clearly is not valid. If the rotation angle becomes large, depolarization effects become important and the mean polarization amplitude is reduced [44].

The other sources of cosmological C polarization have been widely discussed. The first is tensor (or vector) primordial perturbations. In particular, an epoch of inflation in the early universe necessarily produces nearly scale-invariant tensor as well as scalar perturbations, and for many inflation models these tensor modes are observable through the C polarization they produce [47]. This class of tensor perturbation signals all have the property that their C-polarization power peaks at large scales, $l \lesssim 100$. The other generic source of C polarization is gravitational lensing of the primordial G polarization from scalar perturbations [31], which peaks at angular scales in the region of $l \approx 1000$. The C polarization from Faraday rotation peaks at significantly smaller angular scales, well sepa-

rated from either the tensor or lensing signals, and would thus be easy to separate.

VI. DISCUSSION AND CONCLUSIONS

Primordial magnetic fields produce C-polarization fluctuations directly via the vector and tensor perturbations they induce in the primordial plasma [13]. For a field with $B_\lambda = 10^{-9}$ G, the peak polarization amplitude is, for example, $l^2 C_l^C \approx 10^{-13}$ at $l = 1000$ for $n = -2$ [17]. The Faraday rotation signal from the same magnetic field gives a peak polarization amplitude of around 10^{-14} for $\nu =$ 30 GHz and (neglecting the depolarization effect at significant rotation angles) 10^{-12} for $\nu = 10$ GHz. The direct polarization fluctuations peak at significantly larger scales than those from Faraday rotation: primordial magnetic fields have a double signature, with two polarization peaks. In comparison, the G-polarization power spectrum from primordial density perturbations peaks at a comparable amplitude of around $l^2 C_l^G \approx 3 \times 10^{-12}$ at an angular scale of around $l = 1000$, while the C polarization from gravitational lensing has an amplitude on the order of $l^2 C_l^C \approx 10^{-14}$ at $l = 1000$. The C polarization from Faraday rotation for realistic primordial magnetic fields is small but potentially measurable, in the same ballpark as detecting the lensing polarization signal.

Several claims have been made that primordial magnetic fields of the amplitudes considered here are ruled out. These claims are not all compelling; we review some here. Caprini and Durrer have asserted remarkably stringent limits on primordial magnetic fields through their conversion into gravitational waves, which then are constrained by the expansion rate of the universe at nucleosynthesis [48]. But the expansion rate of the universe is the same whether energy density is converted from magnetic fields to gravitational waves or not, since the energy density of both scale the same way with the expansion of the universe. So the actual constraint is on the total radiation energy density in the magnetic field, which is constrained to be about 1% of the total energy density in the usual manner that nucleosynthesis limits extra neutrino species. The corresponding limit on the total comoving mean magnetic field strength is around 10^{-8} G, not the 10^{-27} G claimed in [48].

Potentially more interesting limits come from magnetic fields in galaxy clusters. It is believed that clusters contain magnetic fields with μG field strengths, from Faraday rotation of background polarized radio sources observed through the cluster (e.g., [49,50]). Vogt and Ensslin [51] have recently analyzed several cluster rotation maps, concluding that fields of at least a few μG are present, with a lower limit on the steepness of the magnetic field power spectrum in the clusters. On the other hand, simulations starting from constant comoving magnetic fields of 10^{-11} G show clusters generating fields sufficiently large to explain Faraday rotation measurements [20,52]. Banerjee and Jedamzik [19] have claimed that these results rule out primordial fields with comoving amplitudes as large as 10^{-9} G, arguing that they would overproduce fields in galaxy clusters today. We feel this is a potentially powerful argument, but premature. First, simulations have shown that primordial fields which are 1% of those considered in this paper may lead to observed cluster fields, but magnetic field generation is a nonlinear process and it is not obvious that the final cluster fields will scale linearly with the initial fields; we know they will definitely not as equipartition field strength is approached in the cluster. While it seems likely that cluster fields are generally below equipartition value [53], simulations over a wide range of initial field strengths and configurations must be performed to rule out certain initial field configurations. Second, we do not fully understand galaxy clusters, and it is not immediately clear whether the cluster simulations to date include all of the relevant physics. Third, the Faraday rotation signal in galaxy clusters depends on not only the field strength but also its configuration: it is conceivable that larger initial fields might lead to larger cluster fields, but that much of the field strength would be concentrated at small scales due to more efficient turbulent cascades, and small-scale fields contribute much less to observed Faraday rotation measurements. It is likely that further cluster simulations will resolve these issues; until then it is important to examine other techniques for probing magnetic fields like the one outlined in this paper.

Another constraint on primordial fields comes from field dissipation on small scales [9] inducing spectral distortions in the microwave background radiation. Field strengths greater than 3×10^{-8} G on scales of 400 pc are ruled out. This limit constrains primordial magnetic fields which are blue, $n > 0$, with most field strength on small scales. For the fields considered in this paper, on scales larger than 1 Mpc, such constraints are important if the magnetic field's power spectrum remains a power law over a range of 10^4 in length scale. As this may not be the case, we present results in this paper for various values of n, but the reader should note that pure power-law fields with $n > 0.5$ must have amplitudes smaller than 10^{-9} G on Mpc scales.

A signal we have not computed here explicitly, but which follows simply from the formulas derived above, is the G-polarization signal arising from the Faraday rotation of a primordial C-polarization signal. Of course, scalar perturbations provide no such primordial signal, but any magnetic field does [13]. It is likely that the magnetic field polarization signal is small compared to the G polarization from primordial density perturbations, but the rotated G polarization appears, like the C polarization, at small angular scales of $l \approx 100\,00$. At these small scales, any direct primordial signals will be suppressed due to diffusion damping, so this G-polarization signal would provide confirmation of the corresponding C-polarization rotation signal in Fig. 2.

Additionally, comparison of microwave background imprints from Faraday rotation and intrinsic fluctuations can probe the helicity of the primordial field. Here we have demonstrated explicitly that any helical part of the magnetic field gives zero Faraday rotation. Helical stochastic magnetic fields do, however, produce nonzero temperature and polarization fluctuations directly [14,16]. By comparing the microwave background power spectra at scales of $20'$ with the polarization power spectra at scales of $1'$, the helical power spectrum $P_H(k)$ can be probed separately from the nonhelical power spectrum $P_B(k)$.

In this paper, we have modeled the Faraday rotation signal by separating the polarization generation and rotation processes through a simple approximation. A more accurate calculation demands a full numerical evolution of the coupled Boltzmann hierarchy of equations describing photons, electrons, and magnetic and gravitational fields. Such a code has been developed for the direct temperature and polarization fluctuations from magnetic fields [17], and at first glance it would appear to be a simple matter to modify such a code to include Faraday rotation, including a more accurate treatment than presented here of combined polarization generation and rotation, and damping at small scales or large magnetic field strengths. However, currently used microwave background Boltzmann codes use polarization basis functions corresponding directly to G- and C-polarization modes. While this provides formal simplicity and conceptual clarity, Faraday rotation, described by the rotation operator Eq. (32), does *not* correspond to a simple rotation of G polarization into C polarization, but is naturally expressed as a rotation mixing the Q and U Stokes parameters. A straightforward numerical implementation of Faraday rotation requires using a code employing the Stokes parameters as variables. Techniques for doing this and reconstructing the G and C power spectra are known (see, e.g., [41]) and were used in a number of older microwave background polarization codes.

The polarization signals discussed in this paper have an amplitude of μK or smaller, on angular scales of $1'$ or smaller. The combination of small amplitude and angular scale make detection of this potential signal challenging. For a given magnetic field strength, the Faraday rotation power spectrum scales like ν^{-4} with observing frequency, so going to a low enough frequency will compensate for smaller magnetic field strengths. But lower-frequency observations at the same angular resolution require larger experiments, and to obtain arc minute resolution at frequencies of 30 GHz or below over a reasonable field of view likely requires interferometric experiments (the corresponding diffraction limit would require a single-dish diameter of 50 m). No polarized interferometers acting at these frequencies and angular scales are currently envisioned, but polarization experiments have only just begun to detect any signals at all [54–57], and we can expect rapid advances over the coming decade. The new generation of millimeter-wave frequency high-resolution experiments for temperature fluctuations (e.g., [58]) will help lay the technical basis for complementary polarization observations.

The Faraday rotation signal in Fig. 2 is a demanding experimental target, and its amplitude is highly uncertain (and could certainly be zero). But we will not be obliged to go directly from ignorance to probes of these small scales, because cosmological magnetic fields strong enough to induce detectable Faraday rotation will also cause temperature and polarization fluctuations at larger scales. If primordial magnetic fields are present, we will suspect their existence long before the Faraday rotation power spectrum is probed directly, and the existence of detectable Faraday rotation can be verified through multifrequency polarization observations along individual lines of sight. If the coming five years strengthen current suspicions of a possible primordial magnetic field into more solid evidence, then the Faraday rotation signal calculated in this paper will become a compelling experimental opportunity to characterize a new and poorly understood relic of the early universe.

ACKNOWLEDGMENTS

We thank Ruth Durrer, Grigol Gogoberidze, Levon Pogosian, and Tanmay Vachaspati for helpful discussions. T. K. and B. R. acknowledge support from NSF CAREER Grant No. AST-9875031 and DOE EPSCoR Grant No. DE-FG02-00ER45824. T. K., A. K., and G. L. are partially supported by the U.S.–Georgia Bilateral Program of the Civilian Research and Development Foundation, Grant No. 3316. A. K. is supported by the Research Corporation.

APPENDIX A: INTEGRALS INVOLVING THREE TENSOR SPHERICAL HARMONICS

The evaluation of the needed integrals over three tensor spherical harmonics can be built on the integral over three usual (scalar) spherical harmonics using repeated integration by parts. The integral over three spherical harmonics is well known [36] ,

$$\int d\Omega Y_{(l_1m_1)}Y_{(l_2m_2)}Y^*_{(lm)} = \left(\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}\right)^{1/2} \times C^{l0}_{l_10l_20}C^{lm}_{l_1m_1l_2m_2}, \tag{A1}$$

where $C^{lm}_{l_1m_1l_2m_2}$ is a Clebsch-Gordan coefficient.

Now we need to compute the second integral in Eq. (38),

$$I_1 \equiv \int d\Omega_{\mathbf{n}} Y_{(l_1m_1)}(\mathbf{n})Y_{(l_2m_2):ab}(\mathbf{n})Y^*_{(lm)}(\mathbf{n}). \tag{A2}$$

First consider the simpler integral

$$I_2 \equiv \int d\Omega_{\mathbf{n}} Y_{(l_1m_1)}(\mathbf{n})Y_{(l_2m_2):a}(\mathbf{n})Y^*_{(l_3m_3)}(\mathbf{n}). \tag{A3}$$

This can be evaluated by integrating by parts 3 times: first on the derivative of $Y_{(l_3 m_3)}$, which gives two terms, for one of which the derivatives change into $l_2(l_2+1)$. Now for the other term, integrate by parts a second time on the derivative of $Y_{(l_2 m_2)}$, and then follow by a third integration by parts on the derivative of $Y_{(l_1 m_1)}$. The integral which is left is the negative of the original integral; solving for this integral results in

$$I_2 = \frac{1}{2}(-L_1 + L_2 + L_3) \times \int d\Omega_{\mathbf{n}} Y_{(l_1 m_1)}(\mathbf{n}) Y_{(l_2 m_2)}(\mathbf{n}) Y^*_{(l_3 m_3)}(\mathbf{n}) \tag{A4}$$

with the abbreviation $L_i \equiv l_i(l_i+1)$. The remaining integral is just Eq. (A1).

Using Eq. (A4), the integral (38) can be evaluated using the same general strategy, with three integrations by parts. In this case, we also need to compute $Y_{:ab}{}^{:b} = Y_{:ba}{}^{:b}$. To simplify this, we need to commute the two outer derivatives. The curvature tensor for this 2-dimensional maximally symmetric space is just

$$R_{abcd} = g_{ac} g_{bd} - g_{ad} g_{bc}. \tag{A5}$$

Thus we have

$$\begin{aligned} Y_{:ab}{}^{:b} &= Y_{:ba}{}^{:b} = Y^{:b}{}_{:ba} + R_{dba}{}^{b} Y^{:d} \\ &= -[l(l+1) - 1] Y_{:a}. \end{aligned} \tag{A6}$$

Now we have all the pieces needed to evaluate Eq. (A2): after three integrations by parts, we are again left with the original integral, plus surface terms, like in the case of Eq. (A3). The surface terms can all be evaluated in closed form with the help of Eqs. (A4) and (A6), giving

$$I_1 = \frac{1}{4}(L_2 + L - L_1 - 2)(L_2 + L - L_1) \times \int d\Omega_{\mathbf{n}} Y_{(l_1 m_1)} Y_{(l_2 m_2)} Y^*_{(lm)}. \tag{A7}$$

Again, the remaining integral is given by Eq. (A1).

APPENDIX B: CLEBSCH-GORDAN COEFFICIENT NUMERICS

The Clebsch-Gordan coefficients in Eq. (44) can be evaluated in closed form as [36]

$$C^{c0}_{a0b0} = \frac{(-1)^{g-c}\sqrt{2c+1}\, g!}{(g-a)!(g-b)!(g-c)!} \times \left[\frac{(2g-2a)!(2g-2b)!(2g-2c)!}{(2g+1)!}\right]^{1/2}, \tag{B1}$$

which is valid for $a+b+c = 2g$ with g a positive integer; a, b, and c must also satisfy the triangle inequalities. If $a+b+c$ is odd or if a, b, and c cannot form the sides of a (possibly degenerate) triangle, the coefficient vanishes. This coefficient can be directly evaluated numerically, as long as the various factors are taken in an order to prevent overflow and/or underflow errors. However, direct evaluation is not fast enough for efficient evaluation of Eq. (44), since performing the sums up to l values of several thousand demands millions of Clebsch-Gordan coefficients, most with very large indices. But the two-term Stirling asymptotic expansion

$$n! \sim n^n e^{-n}\sqrt{2\pi n}\left(1 + \frac{1}{12n}\right), \qquad n \to \infty, \tag{B2}$$

is accurate to 0.1% even for $n = 1$. Applying this expansion to all of the factorial factors in Eq. (B1) gives the approximation

$$\begin{aligned} \frac{(C^{c0}_{a0b0})^2}{2c+1} \approx{}& \frac{e}{2\pi}\left(1 + \frac{1}{2g}\right)^{-2g-3/2} \\ &\times \exp\left(\frac{1}{8g} - \frac{1}{8(g-a)} - \frac{1}{8(g-b)} - \frac{1}{8(g-c)}\right) \\ &\times [g(g-a)(g-b)(g-c)]^{-1/2}, \end{aligned} \tag{B3}$$

when $g-a$, $g-b$, and $g-c$ are all nonzero. This approximation is accurate to just over 1% for the case $a = b = c = 2$, and better than 1% for all other needed terms. For the degenerate case $a+b=c$,

$$\begin{aligned} \frac{(C^{c0}_{a0b0})^2}{2c+1} \approx{}& \frac{e}{2\sqrt{\pi}}\left(1 + \frac{1}{2g}\right)^{-2g-3/2} \\ &\times \exp\left(\frac{1}{8g} - \frac{1}{8a} - \frac{1}{8b}\right)(gab)^{-1/2}, \end{aligned} \tag{B4}$$

along with cyclic permutations for the other two cases; this approximation is accurate to better than a part in 10^{-4}.

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