Spectrum of gravitational radiation from primordial turbulence

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Energy injection into the early universe can induce turbulent motions of the primordial plasma, which in turn act as a source for gravitational radiation. Earlier work computed the amplitude and characteristic frequency of the relic gravitational wave background, as a function of the total energy injected and the stirring scale of the turbulence. This paper computes the frequency spectrum of relic gravitational radiation from a turbulent source of the stationary Kolmogoroff form which acts for a given duration, making no other approximations. We also show that the limit of long source wavelengths, commonly employed in aeroacoustic problems, is an excellent approximation. The gravitational waves from cosmological turbulence around the electroweak energy scale will be detectable by future space-based laser interferometers for a substantial range of turbulence parameters.

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I. INTRODUCTION

Direct detection of a relic gravitational wave background is a subject of considerable current interest (see [1–5] for recent reviews), motivated by planned satellite detection missions in the near future [6]. Gravitational wave detection could probe directly the physical conditions in the early universe at the epoch of radiation generation [7], since after been generated, gravitational radiation freely propagates throughout the entire evolution of the universe. Once generated, any gravitational wave spectrum retains its shape, with all wavelengths simply scaling with the expansion of the universe. Various possibilities for early-universe physics leading to detectable cosmological gravitational wave backgrounds include quantum fluctuations during inflation [8] and subsequent oscillating classical fields during reheating [9]; cosmological defects [10]; bubble wall motions and collisions during phase transitions [11–13]; plasma turbulence [13–16]; and cosmological magnetic fields [16,17]. Depending on wavelength, the resulting gravitational waves might be detected either directly or through their imprint on the polarization of the cosmic microwave background [18]. If detected, gravitational radiation generated in the early universe would provide a remarkable new window into physics beyond the standard model of particle physics (e.g., [19,20]).

In this paper we revisit the generation of a cosmological gravitational wave background from turbulent motion of the primordial plasma. We employ methods similar to those originally developed in aeroacoustics for calculating sound generation by turbulent flows [21–24]. This allows us to incorporate the influence of the temporal characteristics of turbulent fluctuations on the gravitational wave generation process, and thus to determine the spectrum of the emitted gravitational waves at all frequencies. For simplicity, we assume isotropic nonhelical turbulence, ignoring all possibilities for generating polarized gravitational waves [25]. (Polarized radiation might be generated through anisotropic stress of the helical primordial magnetic field [26], or from other parity-violating sources in the early universe such as Chern-Simons coupling [27–29] or an axion field coupling with gravity [30]. Detection of these polarized backgrounds is discussed in Ref. [31].)

As is well known, gravitational waves are sourced by the transverse and traceless part of the stress-energy tensor (see, e.g., [32]). In our case the stress-energy tensor results from turbulent plasma motions:

$$T_{ij}(\mathbf{x}) \propto \mathrm{w} v_i(\mathbf{x}) v_j(\mathbf{x}), \tag{1}$$

where $\mathbf{v}(\mathbf{x})$ is the velocity vector field of the fluid and $\mathrm{w} = p + \epsilon$ is the enthalpy density with p and ϵ the pressure and the energy density of plasma, which is assumed to be constant throughout space [14]. To model a period of cosmological turbulence, we assume that at time t_* in the early universe, a vacuum energy density $\rho_{\rm vac}$ is converted into (turbulent) kinetic energy of the cosmological plasma via stirring on a characteristic source length scale L_S, over a time scale $\tau_{\rm stir}$ [12]. The characteristic length scale L_S of the generated fluctuations is directly related to the Hubble length $H_*^{-1} = H^{-1}(t_*)$ at the time of energy injection. We consider only a forward cascade: after being generated on the length scale L_S, the turbulence kinetic energy cascades from larger to smaller scales. The cascade stops at some damping scale L_D, when the energy of turbulence thermalizes due to some dissipation mechanism, such as viscosity or plasma resistivity. In this paper we consider w, $\rho_{\rm vac}$, $\tau_{\rm stir}$, H_*, L_S, and L_D as phenomenological parameters which

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can approximately describe any period of cosmological turbulence, and derive the dependence of the gravitational wave spectrum on these parameters. As expected from the universal nature of turbulence, the shape of the spectrum scales with the characteristic amplitude and frequency of the gravitational radiation.

We perform the computation of the gravitational wave spectrum in real space, instead of using conventional Fourier space techniques as in Ref. [14]. This makes the physical interpretation of all quantities straightforward. The spatial structure of the turbulence is taken to be isotropic with a Kolmogoroff spectrum [33], and the time dependence of the turbulence is described by the Kraichnan time autocorrelation function [34]. While relativistic turbulence in the early universe might depart somewhat from these scalings, these assumptions are based on observed properties of laboratory turbulence and will give the correct qualitative features of the resulting radiation spectrum. Generalization to alternative turbulence models is straightforward. We use natural units $\hbar = c = k_B \equiv 1$ throughout.

II. GENERAL FORMALISM

We assume the duration of the turbulence, τ_T, is much less than the Hubble time $H_\star^{-1}$ [14,15], so the effects of the expansion of the universe may be neglected in the generation of gravitational radiation. This adiabatic assumption will be valid for any turbulence which is produced in a realistic cosmological phase transition [35]. (Note that the duration of the turbulence τ_T can be substantially longer than the stirring time $t_{\rm stir}$; see the detailed discussion in [14].) Then the radiation equation in real space can be written as [32]

$$\nabla^2 h_{ij}(\mathbf{x}, t) - \frac{\partial^2}{\partial t^2} h_{ij}(\mathbf{x}, t) = -16\pi G S_{ij}(\mathbf{x}, t), \quad (2)$$

where $h_{ij}(\mathbf{x}, t)$ is the tensor metric perturbation, the traceless part of the stress-energy tensor $T_{ij}(\mathbf{x}, t)$ is [36]

$$S_{ij}(\mathbf{x}, t) = T_{ij}(\mathbf{x}, t) - \tfrac{1}{3}\delta_{ij} T_k^k(\mathbf{x}, t), \quad (3)$$

and t is physical time. During the period of turbulence, the stress tensor takes the form of Eq. (1) [33].

The general solution of Eq. (2) is [32,36]

$$h_{ij}(\mathbf{x}, t) = 4G \int \mathrm{d}^3\mathbf{x}' \frac{S_{ij}(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|)}{|\mathbf{x} - \mathbf{x}'|}. \quad (4)$$

Because of the stochastic character of the turbulent stress tensor S_{ij}, the generated metric perturbations h_{ij} also are stochastic. We aim to derive the energy density spectrum of these perturbations at the end of the turbulent phase; after that the amplitude and wavelength of the gravitational radiation scales simply with the expansion of the universe. The energy density of gravitational waves is defined as [1]

$$\rho_{GW}(\mathbf{x}, t) = \frac{1}{32\pi G}\langle \partial_t h_{ij}(\mathbf{x}, t)\partial_t h_{ij}(\mathbf{x}, t)\rangle = \frac{G}{2\pi}\int \mathrm{d}^3\mathbf{x}'\mathrm{d}^3\mathbf{x}'' \frac{\langle \partial_t S_{ij}(\mathbf{x}', t')\partial_t S_{ij}(\mathbf{x}'', t'')\rangle}{|\mathbf{x} - \mathbf{x}'||\mathbf{x} - \mathbf{x}''|}, \quad (5)$$

where the brackets denote an ensemble average over realizations of the stochastic source, $t' = t - |\mathbf{x} - \mathbf{x}'|$ and $t'' = t - |\mathbf{x} - \mathbf{x}''|$.

A. Localized source

We will first consider turbulence in a bounded region of space centered around $\mathbf{x} = 0$. In this case, the energy density flux $\mathbf{P}(\mathbf{x}, t)$ of the radiation propagating outward in the direction $\hat{\mathbf{n}}$ is just

$$\mathbf{P}(\mathbf{x}) = \hat{\mathbf{n}}\rho(\mathbf{x}, t). \quad (6)$$

At large distances from the turbulent source, the far-field approximation is justified [32,36]. This assumption replaces $|\mathbf{x} - \mathbf{x}'|$ by $|\mathbf{x}|$ in Eq. (5), yielding for the gravitational wave energy density flux

$$\mathbf{P}(\mathbf{x}) = \frac{G\hat{\mathbf{n}}}{2\pi|\mathbf{x}|^2}\int \mathrm{d}^3\mathbf{x}'\mathrm{d}^3\mathbf{x}''\langle \partial_t S_{ij}(\mathbf{x}', t')\partial_t S_{ij}(\mathbf{x}'', t'')\rangle. \quad (7)$$

The flux from a spatially bounded source drops as the inverse square of the distance from the radiation source, as expected.

The autocorrelation function of the tensor metric perturbations is defined as

$$L(\mathbf{x}, \tau) \equiv \frac{1}{32\pi G}\langle \partial_t h_{ij}(\mathbf{x}, t)\partial_t h_{ij}(\mathbf{x}, t + \tau)\rangle, \quad (8)$$

with $\tau = t' - t$, such that $\rho_{GW}(\mathbf{x}) = L(\mathbf{x}, 0)$. Defining the usual Fourier transform of $L(\mathbf{x}, \tau)$ as

$$I(\mathbf{x}, \omega) = \frac{1}{2\pi}\int \mathrm{d}\tau e^{i\omega\tau} L(\mathbf{x}, \tau), \quad (9)$$

with ω as the angular frequency, it readily follows that

$$\rho_{GW}(\mathbf{x}) = \int \mathrm{d}\omega I(\mathbf{x}, \omega), \quad (10)$$

and therefore $I(\mathbf{x}, \omega)$ represents the spectral energy density of induced gravitational waves [1,32].

Substituting Eq. (4) into Eq. (8) gives

$$L(\mathbf{x}, \tau) = \frac{G}{2\pi|\mathbf{x}|^2}\int \mathrm{d}^3\mathbf{x}'\mathrm{d}^3\mathbf{x}''\langle \partial_t S_{ij}(\mathbf{x}', t')\partial_t S_{ij}(\mathbf{x}'', t'')\rangle. \quad (11)$$

For the case of stationary turbulence, it can be proven that [23]

$$\langle \partial_t S_{ij}(\mathbf{x}', t')\partial_t S_{ij}(\mathbf{x}'', t'')\rangle = -\partial_\tau^2\langle S_{ij}(\mathbf{x}', t')S_{ij}(\mathbf{x}'', t'')\rangle. \quad (12)$$

Using Eq. (12) with the far-field approximation $|\mathbf{x} - \mathbf{x}'| = |\mathbf{x}| - \mathbf{x} \cdot \mathbf{x}'/|\mathbf{x}|$, and using the fact that the cross correlation of a stationary random function is independent of time translation, Eq. (11) reduces to

$$L(\mathbf{x}, \tau) = \frac{-G}{2\pi|\mathbf{x}|^2} \partial_\tau^2 \int \mathrm{d}^3\mathbf{x}' \mathrm{d}^3\mathbf{x}'' \langle S_{ij}(\mathbf{x}', t) S_{ij}(\mathbf{x}'', \tau')\rangle, \tag{13}$$

where

$$\tau' = t + \tau + \frac{\mathbf{x}}{|\mathbf{x}|} \cdot (\mathbf{x}'' - \mathbf{x}'). \tag{14}$$

Defining the two-point time-delayed forth order correlation tensor by

$$R_{ijkl}(\mathbf{x}', \boldsymbol{\xi}, \tau) = \frac{1}{\mathrm{w}^2} \langle S_{ij}(\mathbf{x}', t) S_{kl}(\mathbf{x}'', t + \tau)\rangle, \tag{15}$$

where $\boldsymbol{\xi} = \mathbf{x}'' - \mathbf{x}'$ and $\mathrm{w} = \rho + p$ is the enthalpy density of the plasma, Eq. (13) yields

$$L(\mathbf{x}, \tau) = \frac{-G\mathrm{w}^2}{2\pi|\mathbf{x}|^2} \partial_\tau^2 \int \mathrm{d}^3\mathbf{x}' \mathrm{d}^3\xi R_{ijij}\left(\mathbf{x}', \boldsymbol{\xi}, \tau + \frac{\mathbf{x}}{|\mathbf{x}|} \cdot \xi\right). \tag{16}$$

Fourier transforming this equation gives

$$I(\mathbf{x}, \omega) = \frac{4\pi^2 \omega^2 G \mathrm{w}^2}{|\mathbf{x}|^2} \int \mathrm{d}^3\mathbf{x}' H_{ijij}\left(\mathbf{x}', \frac{\mathbf{x}}{|\mathbf{x}|}\omega, \omega\right) \tag{17}$$

(summation on i and j assumed), where the four-dimensional power spectral energy density tensor of stationary turbulence is defined as

$$H_{ijkl}(\mathbf{x}', \mathbf{k}, \omega) \equiv \frac{1}{(2\pi)^4} \int \mathrm{d}^3\boldsymbol{\xi} \mathrm{d}\tau e^{i(\omega\tau - \mathbf{k}\cdot\boldsymbol{\xi})} R_{ijkl}(\mathbf{x}', \boldsymbol{\xi}, \tau). \tag{18}$$

Equation (17) allows us to calculate the spectral energy density of gravitational waves from a localized source, if the real-space statistical properties of the turbulent source are known.

B. Spatially homogeneous source of finite duration

For a cosmological source of stochastic gravitational radiation, we assume that the source is statistically homogeneous, so that the averaged correlators of the stress tensor have no spatial dependence, and isotropic, so that the correlator between two spatial points depends only on the distance between the points and not on the direction. We can also simply account for the expansion of the universe by a simple rescaling of the frequency of all radiation after its production, so we compute the radiation spectrum in a nonexpanding spacetime and include the expansion effect at the end.

With these assumptions, Eq. (18) simplifies to

$$H_{ijkl}(\mathbf{x}', \mathbf{k}, \omega) = H_{ijkl}(\mathbf{k}, \omega) = \frac{1}{(2\pi)^4} \int d^3\boldsymbol{\xi} d\tau e^{i\omega\tau} e^{-i\mathbf{k}\cdot\boldsymbol{\xi}} R_{ijkl}(\boldsymbol{\xi}, \tau) \tag{19}$$

$$= \frac{1}{4\pi^3} \int d\tau d\xi \xi^2 e^{i\omega\tau} j_0(k\xi) R_{ijkl}(\xi, \tau), \tag{20}$$

so $H_{ijij}(\hat{\mathbf{x}}\omega, \omega) = H_{ijij}(\omega, \omega)$ independent of the observation direction $\hat{\mathbf{x}}$, as expected on physical grounds. Now consider a stochastic source lasting for a finite duration τ_T, the duration of the turbulent source. The total radiation energy spectrum at some point and time is obtained by integrating over all source regions with a lightlike separation from the observer, which comprises a spherical shell around the observer with a thickness corresponding to the duration of the phase transition, and a radius equal to the proper distance along any lightlike path from the observer to the source. Because of statistical isotropy and homogeneity, the integral is trivial, contributing only a volume factor, giving for the total energy spectrum

$$\rho_{GW}(\omega) \equiv \frac{d\rho_{GW}}{d\ln\omega} = 16\pi^3 \omega^3 G \mathrm{w}^2 \tau_T H_{ijij}(\omega, \omega). \tag{21}$$

This spectrum is of course independent of the position of the observer, as it should be for a stochastic background. In the absence of the expansion of the universe, a stochastic source generates a spectrum of radiation which then remains constant for all later times.

III. STATISTICS OF STATIONARY KOLMOGOROFF TURBULENCE

For a particular model of turbulent motion, the correlations needed for computing gravitational radiation can be estimated. Here we consider the simplest turbulence model, the original Kolmogoroff picture. The spectral function $F_{ij}(\mathbf{k}, \tau)$ for stationary, isotropic, and homogenous turbulence is defined as a spatial Fourier transform of the two-point velocity correlation function

$$R_{ij}(\mathbf{r}, \tau) \equiv \langle v_i(\mathbf{x}, t) v_j(\mathbf{x} + \mathbf{r}, t + \tau)\rangle. \tag{22}$$

This function can be expressed in the form [22]

$$F_{ij}(\mathbf{k}, \tau) = \frac{E_k}{4\pi k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) f(\eta_k, \tau), \tag{23}$$

where E_k is the one-dimensional turbulent spectrum of energy density, η_k is the autocorrelation function [34], and the function $f(\eta_k, \tau)$ characterizes temporal decorrelation of turbulent fluctuations, such that it becomes negligibly small for $\tau \gg 1/\eta_k$.

Here we consider Kolmogoroff turbulence for which the energy density spectrum is given by the power law [33]

$$E_k = C_K \varepsilon^{2/3} k^{-5/3}, \qquad k_0 < k < k_d, \tag{24}$$

defined over the range of wave numbers from k_0, determined by the stirring length scale $L_S \equiv 2\pi/k_0$ on which the energy is injected into turbulent motions, to k_d, determined by the dissipation length scale $L_D \equiv 2\pi/k_d$ on which the plasma kinetic energy is thermalized. Here C_K is a constant of order unity; for simplicity we set $C_K = 1$. The parameter ε is the energy dissipation rate per unit enthalpy, $\varepsilon \simeq \rho_{\rm vac}/(\tau_T {\rm w})$. The corresponding autocorrelation function is [22]

$$\eta_k = \frac{1}{\sqrt{2\pi}} \varepsilon^{1/3} k^{2/3}. \tag{25}$$

We assume that the stirring and dissipation scales are well separated, i.e., $k_0 \ll k_d$, which corresponds to the turbulence having high Reynolds number. This will be an excellent approximation in any early-universe phase transition with the stirring scale related to the Hubble length. For simplicity, we adopt Kraichnan's square exponential time dependence [34] to model the temporal decorrelation,

$$f(\eta_k, \tau) = \exp\left(-\frac{\pi}{4}\eta_k^2 \tau^2\right). \tag{26}$$

While other forms of $f(\eta_k, \tau)$ are also frequently used (see, e.g., [37]), neither total power of generated waves nor the spectrum are very sensitive to the specific form of the temporal decorrelation [21].

To compute the fourth-order velocity correlation tensors Eq. (15) needed in the gravitational wave formula Eq. (17), we invoke the Millionshchikov quasinormal hypothesis [22]:

$$\langle v_i^a v_j^a v_k^b v_l^b \rangle = \langle v_i^a v_j^a \rangle \langle v_k^b v_l^b \rangle + \langle v_i^a v_k^b \rangle \langle v_j^a v_l^b \rangle + \langle v_i^a v_l^b \rangle \langle v_j^a v_k^b \rangle, \tag{27}$$

where $v_i^a \equiv v_i(\mathbf{x}, t)$ and $v_i^b \equiv v_i(\mathbf{x} + \mathbf{r}, t + \tau)$. Using Eqs. (3), (15), and (27) we obtain

$$R_{ijij}(\mathbf{x}', \mathbf{x}' + \mathbf{r}, \tau) = R_{ii}(\mathbf{r}, \tau) R_{jj}(\mathbf{r}, \tau) + \tfrac{1}{3} R_{ij}(\mathbf{r}, \tau) R_{ij}(\mathbf{r}, \tau). \tag{28}$$

Then Eq. (19) can be evaluated using Eqs. (23)–(25) and the convolution theorem to give

$$H_{ijij}(\mathbf{k}, \omega) = \frac{1}{6} \int \mathrm{d}\mathbf{k}_1 \mathrm{d}\omega_1 g(\mathbf{k}_1, \omega_1) g(\mathbf{k} - \mathbf{k}_1, \omega - \omega_1) \times \left[27 - \frac{k^2}{k_1^2} + \frac{k^4}{2k_1^2 u^2} + \frac{k_1^2}{2u^2} - \frac{k^2}{u^2} + \frac{u^2}{2k_1^2}\right], \tag{29}$$

where we have defined $u \equiv |\mathbf{k} - \mathbf{k}_1|$ and

$$g(\mathbf{k}, \omega) \equiv \frac{E_k}{4\pi^2 k^2 \eta_k} \exp\left(-\frac{\omega^2}{\pi \eta_k^2}\right). \tag{30}$$

Choose the vector $\hat{\mathbf{k}}$ as the axis for spherical coordinates (θ_1, ϕ_1) of the $\mathbf{k}_1$ integral. The azimuthal angular integral over ϕ_1 is trivial. The dependence on the direction of $\mathbf{k}_1$ is clearly only through $\mathbf{k} \cdot \mathbf{k}_1$, so $H_{ijij}(\mathbf{k}, \omega) = H_{ijij}(k, \omega)$. The other angular integral can be simplified by changing variables from θ_1 to u, giving

$$H_{ijij}(k, \omega) = \frac{\pi}{3} \int dk_1 d\omega_1 g(k_1, \omega_1) \left(\frac{27k_1}{k} - \frac{k}{k_1}\right) \int_{|k_1 - k|}^{k_1 + k} du u g(u, \omega - \omega_1) + \frac{\pi}{3} \int dk_1 d\omega_1 g(k_1, \omega_1) \left(\frac{k^3}{2k_1} + \frac{k_1^3}{2k} - k k_1\right) \times \int_{|k_1 - k|}^{k_1 + k} du \frac{1}{u} g(u, \omega - \omega_1) + \frac{\pi}{6} \int dk_1 d\omega_1 g(k_1, \omega_1) \frac{1}{k k_1} \int_{|k_1 - k|}^{k_1 + k} du u^3 g(u, \omega - \omega_1). \tag{31}$$

We need to integrate this expression numerically. The ω_1 integral can be done analytically in terms of the error function; the entire expression is reduced to an integral over two dimensionless quantities in Appendix A, Eq. (A4). The result scales with the stirring scale k_0, and depends on the Mach number $\mathrm{M} = (\varepsilon/k_0)^{1/3}$ of the turbulence. Its dependence on the dissipation scale k_d is through the Reynolds number $\mathrm{R} = (k_d/k_0)^{4/3}$; as expected from physical considerations, the radiated power is almost completely independent of R. Numerical results are displayed in the next section.

IV. RELIC GRAVITATIONAL WAVES

The previous section and the Appendix have given an analytic expression for the gravitational wave energy spectrum resulting from a period of turbulence lasting a time τ_T, stirred on a scale k_0, with Reynolds number R and Mach number M. The only significant approximation made is that the turbulence is stationary and acts as a source of gravitational waves for a finite time interval; the error made through this idealization is discussed below. In order to improve on this approximation, it would be necessary to create a detailed numerical model of the turbulent source, including incorporating an actual stirring mechanism, such as colliding bubbles in a phase transition. We have also assumed that the expansion of the universe can be ignored during the turbulence; this should be a good approximation for any realistic early-universe phase transition. The main effect of expansion would be only to damp the total energy in the turbulence by a modest fraction, assuming the turbulence does not last much longer than a Hubble time.

A. The spectrum at the present epoch

To obtain the present spectrum, the gravitational waves generated by the turbulent source must be propagated through the expanding universe until today. The wavelengths of the gravitational waves simply scale with the scale factor a of the universe, while their total energy density evolves like a^{-4} and their amplitude decays like a^{-1}. From $\rho_{GW}(\omega)$, Eq. (21), we can form $\Omega_G(\omega) \equiv \rho_{GW}(\omega)/\rho_c$, with the critical density $\rho_c = 3H_0^2/8\pi G$. Then, changing to linear frequency $f = \omega/2\pi$, a characteristic strain amplitude is conventionally defined as

$$h_c(f) = 1.263 \times 10^{-18}\left(\frac{1\ \mathrm{Hz}}{f}\right)[h_0^2\Omega_G(f)]^{1/2}, \qquad (32)$$

where h_0 is the current Hubble parameter H_0 in units of 100 km sec^{-1} Mpc^{-1}. From the computed $h_c(f)$ at the epoch of the turbulence, given by a scale factor a_*, the factor by which the amplitude is reduced and the frequency is increased is

$$\frac{a_*}{a_0} = 8.0 \times 10^{-16}\left(\frac{100}{g_*}\right)^{1/3}\left(\frac{100\ \mathrm{GeV}}{T_*}\right), \qquad (33)$$

where T_* is the temperature of the universe with scale factor a_*, and g_* is the effective number of relativistic degrees of freedom the universe has at this time. To give expressions which are physically transparent, we write the turbulence stirring scale and the turbulence duration as fractions of the Hubble length during the turbulence:

$$\gamma H_*^{-1} = 2\pi/k_0, \qquad \zeta H_*^{-1} = \tau_T; \qquad (34)$$

in other words, γ is the stirring scale's fraction of the Hubble length and ζ is the turbulence duration's fraction of the Hubble length. For any particular angular frequency ω_* of the radiation at the time of the phase transition, we can then convert ω_* and $h_c(\omega_*)$ to the amplitude $h_c(f)$ and frequency f of the relic gravitational wave background today using the useful expressions for a radiation-dominated universe,

$$\mathrm{w} = \frac{4\rho_*}{3} = \frac{2\pi^2}{45}g_*T_*^4, \qquad H_* = 1.66 g_*^{1/2}\frac{T_*^2}{m_{\mathrm{Pl}}}, \qquad (35)$$

to get

$$f = 1.55 \times 10^{-3}\ \mathrm{Hz}\left(\frac{\omega_*}{k_0}\right)\left(\frac{g_*}{100}\right)^{1/6}\left(\frac{\gamma}{0.01}\right)^{-1}\left(\frac{T_*}{100\ \mathrm{GeV}}\right), \qquad (36)$$

$$h_c(f) = 1.62 \times 10^{-18}\left(\frac{T_*}{100\ \mathrm{GeV}}\right)\left(\frac{g_*}{100}\right)^{-5/6}\left(\frac{\gamma}{0.01}\right)^{3/2} \times \left(\frac{\zeta}{0.01}\right)^{1/2}[k_0^3 f H_{ijij}(2\pi f, 2\pi f)]^{1/2}. \qquad (37)$$

The characteristic strain spectrum $h_c(f)$ is plotted in Fig. 1. The solid lines show three different values for the Mach number, M = 0.01, M = 0.1, and M = 1, from lowest to highest amplitude. This dependence on M is in addition to the explicit M^3 scaling in Eq. (A4), which is accounted for in the y-axis units. The peak frequency of the spectrum scales inversely with the stirring scale and linearly with the characteristic fluid velocity, which is proportional to the Mach number. The peak frequency is thus proportional to the inverse of the circulation time on the stirring scale of the turbulence. This is the usual result for radiation generation: the characteristic frequency of radiation is determined by the characteristic time scale of the source.

The characteristic parameter values to which the numbers in the plot are scaled ($T_* = 100$ GeV, $g_* = 100$, $\gamma = \zeta = 0.01$) are values consistent with turbulence arising from a strongly first-order phase transition at the electroweak scale; see [13] for a detailed discussion of the appropriate parameters.

B. The aeroacoustic limit

Also plotted in Fig. 1 is an approximation common in aeroacoustics [23], which replaces $H_{ijij}(k = \omega, \omega)$ with $H_{ijij}(k = 0, \omega)$. It is clear that this simplifying approximation is very good for $\mathrm{M} \leq 0.1$, and overestimates the maximum amplitude of $h_c(f)$ by around 30% for M = 1. In this limit, the argument of the Bessel function in Eq. (20)

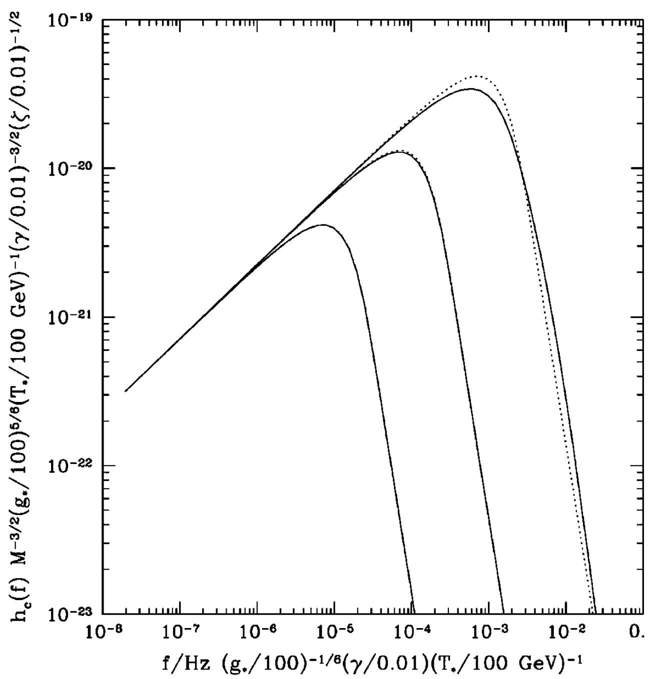

FIG. 1. The spectrum of gravitational radiation from turbulence. The three solid lines are for different Mach numbers, with M = 0.01, M = 0.1, and M = 1 from lowest to highest amplitude. Note that these three cases have also been scaled by a factor of $\mathrm{M}^{-3/2}$ for display, since this is how the low-frequency tail scales with M. The dotted lines, which are virtually indistinguishable from the solid lines except for the M = 1 case, show the $k = 0$ approximation to the gravitational wave source.

becomes small. Substituting Eq. (20) into Eq. (21) gives

$$\rho_{GW}(\omega) = 4\omega^3 G \mathrm{w}^2 \tau_T \int d\tau d\xi \xi^2 e^{i\omega\tau} j_0(\omega\xi) R_{ijij}(\xi, \tau). \tag{38}$$

Thus if $\omega\xi$ is small compared to unity, the aeroacoustic limit $k \to 0$ is guaranteed to be valid.

In the case of aeroacoustics, this approximation works because the fluid velocity is always assumed to be small compared to the velocity of the radiated acoustic waves (low Mach number). In the cosmological regime, the interesting case is for plasma with a relativistic amount of kinetic energy (otherwise there is not substantial gravitational radiation produced). This will occur only when the plasma is at a high enough temperature that it is fully relativistic: otherwise, the amount of energy injected into plasma motions would have to be a substantial fraction of the particle mass scale rather than the cosmological temperature scale, and this is unlikely on general grounds. A relativistic plasma has sound speed $1/\sqrt{3}$, and the Mach number of the turbulent plasma can never be much larger than 1; it will also not be too much smaller than 1. In this case, the fluid velocities will be roughly the sound speed, but this is close to the propagation speed of the emitted radiation. Therefore, we do not automatically have $\xi\omega \ll 1$ in Eq. (38), and the validity of the aeroacoustic approximation must be ascertained by explicit calculation. As we see in Fig. 1, the approximation still gives the right order of magnitude for the spectrum amplitude even for Mach number $\mathrm{M} = 1$, corresponding to a fluid velocity equal to the sound speed.

C. Asymptotic limits

The validity of the aeroacoustic approximation $k \to 0$ simplifies finding asymptotic forms for the spectrum. Consider various frequency regimes of Eq. (31) with $k = 0$; this is facilitated by Eq. (A5). Note that in this limit, the dependence on the Mach number M simply scales with the frequency. We assume $\mathrm{R} \gg 1$, or else fully developed turbulence cannot exist; this is an excellent approximation for early-universe plasma stirred on scales near the Hubble length. In the low-frequency regime, simply take the limit $\omega \to 0$ to get

$$H_{ijij}(0, \omega) \sim \frac{28\mathrm{M}^3}{15k_0^4(2\pi)^{5/2}}, \qquad \omega \to 0. \tag{39}$$

Physically, these frequencies are lower than the lowest characteristic frequency in the problem, corresponding to the eddy turnover time on the stirring scale. This result of a constant H_{ijij} is universal and does not depend on either the spectrum or temporal characteristics of the turbulence (see, e.g., Refs. [12–15] and also [38]). It translates to $h_c(f)$ scaling as $f^{1/2}$ at low f.

At high frequencies $\omega \gg k_0\mathrm{M}\mathrm{R}^{1/2}$, the integral in Eq. (A5) is dominated by the contribution from its lower limit. After using the asymptotic form $\mathrm{erfc}(x) \sim x^{-1}\pi^{-1/2}\exp(-x^2)$, $x \to \infty$, an integration by parts gives the leading-order asymptotic behavior as

$$H_{ijij}(0, \omega) \sim \frac{7\mathrm{M}^3}{2^{7/2}\pi^3 k_0^4 \mathrm{R}^{7/4}} \frac{k_0^2\mathrm{M}^2}{\omega^2} \exp(-2\omega^2/(k_0^2\mathrm{M}^2\mathrm{R})), \qquad \omega \gg k_0\mathrm{M}\mathrm{R}^{1/2}. \tag{40}$$

This exponential suppression is evident in Fig. 1; the dependence on R is negligible as expected from physical considerations. The functional form of the high-frequency suppression is determined by the specific form of the time autocorrelation function of the turbulence, Eq. (26), but for any autocorrelation the amplitude of the emitted waves should be very small in this regime. Physically, this limit corresponds to radiation frequencies which are larger than any frequencies in the turbulent motions; consequently, no scale of turbulent fluctuations generates these radiation frequencies directly, and the resulting small radiation amplitude is due to the sum of small contributions from many lower-frequency source modes. Since the integral is dominated by the lower integration limit, the highest-frequency source fluctuations (which contain very little of the total turbulent energy) contribute most to this high-frequency radiation tail.

In the intermediate frequency regime, for frequencies $k_0\mathrm{M} < \omega < k_0\mathrm{M}\mathrm{R}^{1/2}$, the integral in Eq. (A5) is dominated by the contribution around $k_0^2\mathrm{M}^2/\omega^2$ due to the exponential factor in the integrand, with a width of the same order. Physically, this implies that radiation emission at some frequency in this range is dominated by the turbulent vortices of the same frequency. Consequently, we have the rough estimate

$$H_{ijij}(0, \omega) \simeq \frac{7\mathrm{M}^3}{k_0^4(2\pi)^{5/2}}\left(\frac{k_0\mathrm{M}}{\omega}\right)^{15/2}. \tag{41}$$

This yields $h_c(f) \propto f^{-13/4}$, compared to $h_c(f) \propto f_{\mathrm{stir}}^{-1/2} f^{-11/4}$ in Ref. [14], where f_{stir} is the turbulence circulation frequency at the stirring scale. The slight discrepancy from the spectrum shape in Ref. [14] comes about because we have treated the time correlations of the turbulence in a more realistic way. Here we distinguish two time scales, the decorrelation time which describes how the fluid velocities in a given size eddy are correlated with each other after a given time interval, and the largest eddy turnover time. In practice, the dropoff with frequency in this regime is strong enough that the high-frequency behavior in Eq. (40) only holds when the spectrum is many orders of magnitude below the peak amplitude.

The intermediate frequency regime scaling with frequency depends on the specific model of the turbulence power spectrum. The Kolmogoroff model is not the only possibility, especially in the presence of magnetic fields. Any model of turbulence which includes the local transfer of energy in the wave number space will satisfy $E_k^2/\eta_k \propto k^{-4}$ [39]. In the $k = 0$ limit, it is straightforward to derive that, in general, $H_{ijij}(0, \bar{q})$ scales as $1/\bar{q}^{5/n}$, where n is the scaling exponent of the turbulence autocorrelation function. For Kolmogoroff turbulence, $n = 2/3$ [Eq. (25)]. But for Iroshnikov-Kraichnan turbulence (for example), $n = 1$, and consequently the frequency dependence is somewhat softer, $H_{ijij}(0, \bar{q}) \propto 1/\bar{q}^5$. In practical terms, this modified turbulence spectrum produces radiation with very similar detectability properties to that from the Kolmogoroff turbulence spectrum considered here.

V. DISCUSSION

We have calculated for the first time the spectrum of relic gravitational radiation resulting from a period of stationary turbulence in the early universe, in terms of the turbulence duration, stirring scale, Reynolds and Mach numbers, and the temperature of the universe when the turbulence occurs. This is probably the best that can be done without a detailed simulation of actual turbulent motions. The most likely source of energy injection leading to turbulence is an early-universe phase transition; the connection between the phenomenological parameters describing a phase transition and the parameters describing the turbulence are given explicitly in Ref. [14].

The calculation we present here is conceptually simple. The only assumptions made are that the turbulence lasts for a finite duration which is at least a turnover time on the stirring scale, and that during this time the turbulence can be characterized as stationary. The spatial power spectrum is taken to be the Kolmogoroff form, Eq. (24), with temporal correlations of the Kraichnan form, Eq. (26). These scalings are appropriate for nonrelativistic turbulence with large Reynolds number. While the cosmological case will have large Reynolds numbers, the turbulence will be relativistic in the most interesting cases for gravitational radiation generation. As argued in Ref. [14], a nonrelativistic approximation to relativistic turbulence likely underestimates the resulting radiation: relativistic turbulence contains more kinetic energy for a given fluid velocity. We expect the same general results to hold, except the expression for the Mach number $\mathrm{M}^3 = \epsilon/k_0$ will clearly be modified, giving smaller Mach numbers than this nonrelativistic expression.

Cosmological turbulence will never be precisely stationary, since the universe is expanding. Turbulence from a phase transition will also not be stationary because the duration of the source is comparable to the eddy turnover time on the stirring scale [14], so the turbulence will decay with time. Even so, as long as the eddies on a given length scale can be treated as uncorrelated sources of turbulence, Ref. [14] argues that the resulting radiation spectrum will be close to that from a stationary source, simply due to the inevitable cascade of energy from the stirring scale down to the diffusion scale. This point can be made somewhat more formally, using an argument similar to that given by Proudman [21,22]. In the case of stationary turbulence, the time derivatives in Eq. (7) lead to factors of $1/\tau_0$ when computing the radiation spectrum. If the turbulence is decaying, then additional terms proportional to time derivatives of the correlation functions also will appear. But the characteristic time scale of the turbulence decay τ_d is at least several times greater than the turnover time on the stirring scale, and consequently, these additional terms which are proportional to $1/\tau_d$ can be neglected compared to the stationary term.

It has been claimed that turbulence in the early universe is a source of such short duration that it cannot be treated as stationary at all, and that the resulting radiation spectrum should be imprinted with the characteristic wave number of the turbulent source instead of its characteristic frequency [38]. We fully agree that a gravitational wave source lasting for a sufficiently short duration will not be well described by a short piece of a stationary source; clearly in the limit that the turbulence duration τ goes to zero, the resulting radiation spectrum should peak at a frequency corresponding to the characteristic wave number of the source (the inverse stirring scale), and our formalism will not be valid in that limit. However, it is straightforward to see that a turbulent source in the early universe will actually last long enough so that our calculations are valid. Reference [38] uses a simple toy model for a stochastic cosmological gravity wave source to argue that the condition $\tau_T \omega_s \approx 1$ represents the dividing line between sources that imprint their characteristic frequency on the radiation and could be described using our formalism, and sources that imprint their wave number; here we write the duration of the source as τ_T and the characteristic source angular frequency as ω_s. If we write the turbulence turnover time on the stirring scale as t_{stir}, the associated angular frequency is $\omega_s = 2\pi/t_{\mathrm{stir}}$; also write the duration of the turbulent source as $\tau_T = N t_{\mathrm{stir}}$, so that it lasts N times the stirring-scale turnover time. We can thus write $\omega_s \tau_T = 2\pi N$, and even for the unrealistically short duration $N = 1$, $\omega_s \tau_T$ is significantly larger than unity. For realistic turbulence, we expect the dissipation time to be multiple turnover times. Inspecting the exact gravitational wave solution for the toy model source in Ref. [38] confirms that, even for N as small as 1, the radiation spectrum will be peaked at ω_s.

We also assume that turbulence is nonmagnetic and nonhelical. Either of these complications can modify the power law in Eq. (24) or the form of the time correlation Eq. (26) [25]. The main effect of any modification is to change the rate at which the radiation spectrum falls off at high frequencies, but since this dependence is quite steep, even substantial changes to the asymptotic behavior of the spectrum lead to little qualitative difference in the spectrum. As mentioned above, the low-frequency behavior is independent of any details of the turbulence, and the peak frequency is determined by the eddy turnover time on the stirring scale where the energy density peaks, which will also be independent of any details of the turbulent cascade.

The proposed Laser Interferometer Space Antenna (LISA) satellite mission has a 5σ strain sensitivity to stochastic backgrounds of below $h_c = 10^{-23}$ between frequencies 10^{-3} and 10^{-2} Hz, and decreasing to around $h_c = 10^{-20}$ at 10^{-4} Hz, for one year of integration (see, e.g., [40]). Comparing with Fig. 1, turbulence with a Mach number $\mathrm{M} = 1$ would be a factor of 1000 larger than the LISA detection threshold at the peak frequency around 10^{-3} Hz. For a Mach number $\mathrm{M} = 0.1$, the peak amplitude decreases by a factor of 100 due to the $\mathrm{M}^{-3/2}$ scaling and the different signal spectrum. However, the peak frequency also shifts to 10^{-4} Hz, at which point LISA's sensitivity has declined greatly; the steep high-frequency tail of the gravitational wave spectrum makes detection with LISA marginal in this case. Detectors consisting of two or more correlated LISA detectors or enhanced versions of LISA optimized for detecting stochastic backgrounds have been discussed [41], such as the envisioned GREAT mission [42]; future space-based interferometers could be configured to give strain sensitivities comparable to LISA, but with a frequency window between 10^{-4} and 10^{-6} Hz. Such an experiment would easily detect turbulence at the electroweak scale with a Mach number $\mathrm{M} = 0.1$, and would even flirt with a detection at $\mathrm{M} = 0.01$. Turbulence generated at somewhat higher energy scales shifts to higher frequencies and easier detection with LISA.

As is widely appreciated, detecting cosmological backgrounds of gravitational radiation is not only an issue of detector sensitivity, but also of foreground discrimination. The galactic population of short-period binaries of compact objects, mostly white dwarfs, is known to produce a confusion-limited stochastic background at frequencies below 10^{-3} Hz [43]. At low frequencies, separating this galactic source from a cosmological source is essential, likely by exploiting the nonuniform directional distribution of an galactic source [44–46]. A uniform stochastic source arising from the confusion limit of numerous extragalactic binaries provides a further complication [47], which can only be distinguished from a primordial background via differing spectra. We also note that the source of the turbulence itself may produce a gravitational wave spectrum, and that the characteristic peak frequency may scale differently from the turbulent spectrum; see, e.g., the spectra for first-order phase transitions in Ref. [13]. A distinctive two-peaked shape to the gravitational wave spectrum in certain regions of parameter space will also aid in its detection.

We have no guarantees of violent events in the early universe. However, turbulence is a completely generic result of energy injection on a characteristic length scale, and we have shown in this paper that the resulting relic gravitational waves are within the realm of detectability, even for turbulence with Mach numbers as low as 0.01, corresponding to an energy input into the early universe of 10^{-4} of the total energy density. Many scenarios for the electroweak phase transition [48] and other physics [20] will result in releases of energy that are interestingly large. The remarkable possibility of probing high-energy physics via the detection of vanishingly small spacetime distortions left from when the universe was a trillionth of a second old impels us to look.

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APPENDIX A: NUMERICAL EVALUATION OF $H_{ijij}(k, \omega)$

We need to evaluate Eq. (31) explicitly, with $g(k, \omega)$ given by Eq. (30). The integral over ω_1 can be evaluated analytically using the identity

$$\int_0^\infty dy \exp(-Ay^2)\exp(-B(x-y)^2) = \left(\frac{2}{\pi(A+B)}\right)^{1/2} \exp\left(-\frac{ABx^2}{A+B}\right) \mathrm{erfc}\left(\frac{Bx}{\sqrt{A+B}}\right). \quad \text{(A1)}$$

This expression is simple to derive by writing the integrand as a single exponential and completing the square in the argument of the exponential, followed by a linear change of variables to give the error function. Then Eq. (31) becomes

$$H_{ijij}(k, \omega) = \frac{\varepsilon}{24\pi^{5/2}k} \int_{k_0}^{k_d} dk_1 k_1^{-10/3} \int du u^{-10/3} \times (k_1^{-4/3} + u^{-4/3})^{-1/2}\left[27 - \frac{k^2}{k_1^2} - \frac{k^2}{u^2} + \frac{k^4}{2k_1^2u^2} + \frac{k_1^2}{2u^2} + \frac{u^2}{2k_1^2}\right] \exp\left(-\frac{2\varepsilon^{-2/3}\omega^2}{k_1^{4/3} + u^{4/3}}\right) \times \mathrm{erfc}\left(\frac{2^{1/2}\varepsilon^{-1/3}\omega}{(k_1^{-4/3} + u^{-4/3})^{1/2}}\right). \quad \text{(A2)}$$

The lower limit on the u integral is $\max[|k_1 - k|, k_0]$ and the upper limit is $\min[k_1 + k, k_d]$, provided the lower limit is less than the upper limit; otherwise the integral over u is zero. These conditions on the limits arise due to the limited range of k over which the function E_k has support. Note that Eq. (A2) is regular as $k \to 0$, with the limit

$$H_{ijij}(0, \omega) = \frac{7\varepsilon}{3\pi^2(2\pi)^{1/2}} \int_{k_0}^{k_d} dk_1 k_1^{-6} \times \exp\left(-\frac{\omega^2}{\varepsilon^{2/3}k_1^{4/3}}\right) \mathrm{erfc}\left(\frac{\omega}{\varepsilon^{1/3}k_1^{2/3}}\right). \quad \text{(A3)}$$

Now rescale all dimensionful quantities by powers of k_0 to make them dimensionless; we abbreviate $\varepsilon/k_0 = \mathrm{M}^3$, where M is the Mach number of the turbulence, $k_d/k_0 = \mathrm{R}^{3/4}$, where R is the Reynolds number of the turbulence, $p \equiv k/k_0$, and $q = \omega/k_0$. The change of variables $x = (k_1/k_0)^{-4/3}$, $y = (u/k_0)^{-4/3}$ simplifies the remaining integrals, giving

$$H_{ijij}(p, q) = \frac{3\mathrm{M}^3 k_0^{-4}}{256\pi^{5/2}p} \int_{\mathrm{R}^{-1}}^{1} dx x^{3/4} \int dy y^{3/4}(x+y)^{-1/2} \times \exp\left(-\frac{2xy}{x+y}\frac{q^2}{\mathrm{M}^2}\right) \mathrm{erfc}\left(\frac{2^{1/2}y}{(x+y)^{1/2}}\frac{q}{\mathrm{M}}\right) \times \left[54 - 2p^2x^{3/2} - 2p^2y^{3/2} + p^4x^{3/2}y^{3/2} + \frac{x^{3/2}}{y^{3/2}} + \frac{y^{3/2}}{x^{3/2}}\right]; \quad \text{(A4)}$$

the lower limit of the y integral is $\max[(x^{-3/4} + p)^{-4/3}, \mathrm{R}^{-1}]$ and the upper limit is $\min[|x^{-3/4} - p|^{-4/3}, 1]$, provided the lower limit is less than the upper limit. In the limit $p \to 0$, both of these limits are x, so the integral has a leading-order behavior proportional to p and thus $H_{ijij}(p, q)$ is regular, with the limit

$$H_{ijij}(0, q) \simeq \frac{7\mathrm{M}^3 k_0^{-4}}{(2\pi)^{5/2}} \int_{\mathrm{R}^{-1}}^{1} dx x^{11/4} \exp(-\bar{q}^2 x)\, \mathrm{erfc}(\bar{q}x^{1/2}), \quad \text{(A5)}$$

where we have abbreviated $\bar{q} \equiv q/\mathrm{M}$ since in this limit the integral depends only on $\bar{q}$ and not on either q or M separately, aside from the constant prefactor. Note that $\mathrm{R} \gg 1$ for a medium which supports turbulence; we expect $\mathrm{R} > 2000$ during the cosmological epochs of relevance. The integrals converge as $R \to \infty$, and the lower limit of the x-integrals in Eqs. (A4) and (A5) can be replaced by zero. In Eq. (A4), the terms with factors of $x^{-3/4}$ and $y^{-3/4}$ in the integrand converge somewhat slowly but have small prefactors compared to the first term, giving a negligible dependence of the integral on the diffusion scale. Numerically, it is convenient to take R as some large but finite value; then the integrand in Eq. (A4) is smooth and regular over the full range of integration, and can now be easily performed for any values of p and q.

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