Gravitational Radiation from Primordial Helical Magnetohydrodynamic Turbulence

Tina Kahniashvili,[1,2,3,4] Grigol Gogoberidze,[5,2,4] and Bharat Ratra[2]

[1]*CCPP, New York University, 4 Washington Plaza, New York, New York 10003, USA*

[2]*Department of Physics, Kansas State University, 116 Cardwell Hall, Manhattan, Kansas 66506, USA*

[3]*Department of Physics, Laurentian University, Ramsey Lake Road, Sudbury, ON P3E 2C6, Canada*

[4]*National Abastumani Astrophysical Observatory, 2A Kazbegi Ave, Tbilisi, GE-0160, Georgia*

[5]*Centre for Plasma Astrophysics, K.U. Leuven, Celestijnenlaan 200B, 3001 Leuven, Belgium*

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We consider gravitational waves (GWs) generated by primordial inverse-cascade helical magnetohydrodynamical (MHD) turbulence produced by bubble collisions at the electroweak phase transitions (EWPT). Compared to the unmagnetized EWPT case, the spectrum of MHD-turbulence-generated GWs peaks at lower frequency with larger amplitude and can be detected by the proposed Laser Interferometer Space Antenna.

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When detected, primordial cosmological GWs will provide a very valuable probe of the very early Universe [1]. Various mechanisms that generate such GWs have been discussed: quantum fluctuations [2], bubble wall motion and collisions during phase transitions [3,4], cosmic strings [5], cosmological magnetic fields [6–8], and plasma turbulence [9–12]. From the direct detection point of view, GWs generated during the EWPT are promising since their peak frequency lies in or near the Laser Interferometer Space Antenna (LISA) [13] frequency band [14]; however, to produce a detectable signal the EWPT must be strong enough [15–17]. Currently, discussed EWPT models do not predict an observable GW signal from bubble collisions [17], nor for GWs produced by unmagnetized turbulence [12].

Here, we study the generation of GWs during a first-order EWPT assuming that bubble collisions produce helical MHD turbulence. (Kinetic or magnetic helicity generation at the EWPT is studied in Refs. [18]. Previously, we studied generation of GWs by direct-cascade turbulence and found that, due to parity violation in the early Universe, the induced GWs are circularly polarized [11]. Polarized GWs are present in other models [19], and the polarization of the GW background is in principle observable, either directly [20] or through the CMB [7,21].) In the case of unmagnetized hydrodynamical turbulence, the peak frequency of the GW power spectrum is determined by the inverse turnover time of the largest eddy and the energy scale when the GW is generated. Recently, discussed modifications of the standard EWPT model place the transition at a higher energy scale [16]. As a result, the GW power spectrum peak frequency is shifted to higher frequency which, since the GW spectrum is sharply peaked, reduces the possibility of detection by LISA. On the other hand, in the case of MHD turbulence, the presence of an energy inverse cascade leads to an increase in the effective size of the largest eddy (now associated with an helical magnetic field), and can result in the GW power spectrum peaking in the LISA band, with amplitude large enough to be detected by LISA. We adapt the technique developed in Ref. [12] to study this case here. We model MHD turbulence and obtain the GW spectrum by using an analogy with the theory of sound wave production by hydrodynamical turbulence [22–25].

Since the turbulent fluctuations are stochastic, so are the generated GWs. The GW energy density is [14]

$$\rho_{\rm GW}(\mathbf{x}) = \frac{1}{32\pi G}\langle \partial_t h_{ij}(\mathbf{x},t)\partial_t h_{ij}(\mathbf{x},t)\rangle = \frac{G}{2\pi}\int d^3\mathbf{x}' d^3\mathbf{x}'' \frac{\langle \partial_t S_{ij}(\mathbf{x}',t')\partial_t S_{ij}(\mathbf{x}'',t'')\rangle}{|\mathbf{x}-\mathbf{x}'||\mathbf{x}-\mathbf{x}''|}. \tag{1}$$

Here, the times $t'^{(\prime\prime)} = t - |\mathbf{x}-\mathbf{x}'^{(\prime\prime)}|$, i, and j are spatial indices (repeated indices are summed), the source $S_{ij}(\mathbf{x},t) = T_{ij}(\mathbf{x},t) - \delta_{ij}T^k_k(\mathbf{x},t)/3$ is the traceless part of the stress-energy tensor T_{ij}, G is the gravitational constant, and we use natural units with $\hbar = 1 = c$. We assume that the turbulence exists for a time short enough to neglect the cosmological expansion during GW production. We consider metric perturbations in the far-field limit (i.e., for $x \gg d$, where d is a characteristic length scale of the source region), where GWs are the only metric perturbations [26], and replace $|\mathbf{x}-\mathbf{x}'|$ by $|\mathbf{x}|$ in Eq. (1). If the turbulence is stationary, then the GW spectral energy density $I(\mathbf{x},\omega)$ [$\rho_{\rm GW}(\mathbf{x}) = \int d\omega I(\mathbf{x},\omega)$ where ω is the angular frequency] is [12]

$$I(\mathbf{x},\omega) = \frac{4\pi^2\omega^2 G w^2}{|\mathbf{x}|^2}\int d^3\mathbf{x}' H_{ijij}\left(\mathbf{x}', \frac{\mathbf{x}}{|\mathbf{x}|}\omega, \omega\right). \tag{2}$$

Here, $H_{ijij}(\mathbf{x}',\mathbf{k},\omega)$ (where $\mathbf{k}$ is a proper wave vector) is the (double traced) four-dimensional Fourier transform of the two-point time-delayed fourth-order correlation tensor, $\langle S_{ij}(\mathbf{x}',t)S_{lm}(\mathbf{x}'',t+\tau)\rangle/w^2$, with respect to $\mathbf{x}''-\mathbf{x}'$ and τ, where $w = \rho + p$ is the enthalpy density and p and ρ the pressure and energy density of the plasma.

We assume that primordial MHD turbulence is generated at time $t_\star$ at proper length scale $l_0 = 2\pi/k_0$ with characteristic velocity perturbation v_0. (We assume that the usual and magnetic Reynolds numbers are much greater than unity on scales $\sim l_0$; otherwise, there is no turbulence. Throughout this Letter, the symbol $\sim$ represents equality to the accuracy of a dimensionless multiplicative constant of order unity.) The dynamics of MHD turbulence is dominated by Alfvén waves for which the magnetic and kinetic energy densities are in approximate equipartition [27]. In this case, $v_0 \sim b_0$, where $b_0 = B_0/\sqrt{4\pi w}$ is the characteristic magnetic field perturbation expressed in velocity units. While MHD turbulence is isotropic on large scales, it is locally anisotropic on small scales [28], resulting in small-scale anisotropy in the generated GW background. However, GWs are generated mainly by the largest eddies [12] so we adopt an isotropic turbulence model, and thus the magnetic field two-point correlation function is $\langle b_i^\star(\mathbf{k}, t) b_j(\mathbf{k}', t+\tau)\rangle = F_{ij}^M(\mathbf{k}, t) f(\eta(k), \tau)\delta(\mathbf{k} - \mathbf{k}')$, with [22]

$$F_{ij}^M(\mathbf{k}, \tau) = P_{ij}(\mathbf{k}) \frac{E^M(k,t)}{4\pi k^2} + i\varepsilon_{ijl} k_l \frac{H^M(k,t)}{8\pi k^2}. \quad (3)$$

Here, $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2$ and $E^M(k,t)$ and $H^M(k,t)$ are the magnetic field energy and helicity densities. The Schawarz inequality implies $|H^M(k,t)| \leq 2E^M(k,t)/k$ [27]. For the total magnetic energy $\mathcal{E}_M(t) = \int E^M(k,t)dk$ and helicity $\mathcal{H}_M(t) = \int H^M(k,t)dk$, we get $\mathcal{H}_M(t) \leq 2\xi_M(t)\mathcal{E}_M(t)$, where $\xi_M(t) \equiv \int E^M(k,t)k^{-1}dk/\mathcal{E}_M(t)$ is the magnetic-eddy correlation length. $\eta(k)$ is an autocorrelation function that determines the characteristic function $f(\eta(k), \tau)$ that describes the temporal decorrelation of turbulent fluctuations. In the following, we use $f(\eta(k), \tau) = \exp(-\pi\eta^2(k)\tau^2/4)$ [29].

After generation, primordial turbulence freely decays. We adopt the decaying MHD turbulence model of Refs. [30,31]. For nonzero initial magnetic helicity, turbulence decay is a two-stage process. First, decay stage dynamics is governed by a direct cascade of energy density lasting for a time $\tau_{s0} = s_0\tau_0$, several times ($s_0 \sim 3$–5) longer then the characteristic largest-eddy turnover time $\tau_0 = l_0/v_0 = 2\pi/k_0 v_0$. During the first stage, energy density flows from large to small scales and finally dissipates on scales $\sim l_d = 2\pi/k_d$ ($k_d \gg k_0$) where one of the Reynolds numbers becomes of order unity. Because of the selective decay effect [27], magnetic helicity is nearly conserved during this stage [31]. To compute the GWs generated by decaying MHD turbulence, we assume that decaying turbulence lasting for time τ_{s0} is equivalent to stationary turbulence lasting for time $\tau_{s0}/2$. This can be justified using the Proudman [22,25] argument for (unmagnetized) hydrodynamical turbulence. Consequently, when computing the emitted GWs, we ignore the time dependence of $E^M(k,t)$ and $H^M(k,t)$. We also assume small initial magnetic helicity, $\alpha_\star \equiv \mathcal{H}_M(t_\star)/[2\xi_M(t_\star)\mathcal{E}_M(t_\star)] \ll 1$. For $E^M(k,t)$ and $\eta(k)$, we use the Kolmogorov model,

$$E^M(k,t) \sim \varepsilon^{2/3}k^{-5/3}, \qquad \eta(k) = \varepsilon^{1/3}k^{2/3}/\sqrt{2\pi}, \quad (4)$$

for $k_0 < k < k_d$. Here, $\varepsilon \sim k_0 v_0^3$ is the energy dissipation rate per unit enthalpy.

At the end of the first stage, turbulence relaxes to a maximally helical state, $\alpha_{s0} \sim 1$ [31,32]. Accounting for conservation of magnetic helicity, the characteristic velocity and magnetic field perturbations at this stage are $v_1 \sim \alpha_\star^{1/2} v_0$ and $b_1 \sim \alpha_\star^{1/2} b_0$. Second stage dynamics is governed by a magnetic helicity inverse cascade. If both Reynolds numbers are large at the end of the first stage, magnetic helicity is conserved during the second stage. The magnetic eddy correlation length evolves as $\xi_M(t) \sim l_0\sqrt{1 + t/\tau_1}$ [30,31] where $\tau_1 \sim l_0/v_1 = \tau_0/\sqrt{\alpha_\star}$ is the characteristic energy containing eddy turnover time at the beginning of the second stage. The magnetic $\mathcal{E}_M(t)$ and kinetic $\mathcal{E}_K(t)$ energy densities evolve as [30,31]

$$\mathcal{E}_M(t) \propto (1 + t/\tau_1)^{-1/2} w b_1^2, \qquad \mathcal{E}_K(t) \propto (1 + t/\tau_1)^{-1} w v_1^2. \quad (5)$$

These imply that the characteristic turnover ($\tau_{\rm to}$) and cascade ($\tau_{\rm cas}$) time scales evolve as

$$\tau_{\rm to} \sim \tau_{\rm cas} \sim \tau_1(1 + t/\tau_1). \quad (6)$$

To compute the GWs emitted during the second stage, we use the stationary turbulence model that has the same GW output. Introducing the characteristic wave number $k_\xi(t) = 2\pi/\xi_M(t)$ and using Eqs. (5), we find $\mathcal{E}_M \sim w v_1^2 k_\xi(t)/k_0$ and $\mathcal{E}_K \sim w v_1^2 [k_\xi(t)/k_0]^2$ since $b_1 \sim v_1$. The time when turbulence is present on scale $\xi_M(t)$ is determined by Eq. (6) which can be rewritten as $\tau_{\rm cas} \sim \tau_1[k_0/k_\xi(t)]^2$. So instead of considering decaying turbulence, we consider stationary turbulence with a scale-dependent duration time (time during which the magnetic energy is present at the scale), $\tau_{s1} \sim \tau_1[k_0/k]^2$ (for $k = k_\xi$ this coincides with $\tau_{\rm cas}$).

The expression for $\mathcal{E}_M$ yields the time-independent

$$E^M(k,t) = C_1 v_1^2/k_0 = kH^M(k,t)/2, \qquad k_S < k < k_0. \quad (7)$$

Here, C_1 is a constant of order unity, k_S is the smallest wave number where the inverse cascade stops, and the second equation follows from saturating the causality condition. For the second stage autocorrelation function, which is inversely proportional to the turnover time (6), we assume $\eta(k) = (k/k_0)^2/\sqrt{2\pi}\tau_1$. At the largest scales, there is no efficient dissipation mechanism, so the inverse cascade will be stopped at scale $l_S(t) = 2\pi/k_S$ where either the cascade time scale $\tau_{\rm cas}$ reaches the expansion time scale $H_\star^{-1} = H^{-1}(t_\star)$, or when the characteristic length scale $\xi_M(t) \sim l_S$ reaches the Hubble radius. These conditions are $\alpha_\star^{-1/2} l_S^2/v_0 l_0 \leq H_\star^{-1}$ or $l_S \leq H_\star^{-1}$ (the cascade time is scale dependent and maximal at $k = k_S$). Defining $\gamma = l_0/H_\star^{-1}$ ($\gamma \leq 1$), it is easy to see that the first condition is fulfilled first and consequently $k_0/k_S \leq (v_0/\gamma)^{1/2}\alpha_\star^{1/4}$. To have an inverse cascade requires $k_0/k_S \geq 1$, leading to a constraint on initial helicity, $\gamma \leq M\alpha_\star^{1/2}$ (where $M = v_0$ is the turbulence Mach number).

The magnetic field perturbation stress-energy tensor is $T_{ij}^M(\mathbf{x}, t) = wb_i(\mathbf{x}, t)b_j(\mathbf{x}, t)$. For the first decay stage, we compute for this magnetic part and then double the result to account for approximate magnetic and kinetic energy equipartition for Alfvén waves. During the second stage, according to Eqs. (5), kinetic energy can be neglected compared to magnetic energy. To compute $H_{ijij}(\mathbf{k}, \omega)$, we assume Millionshchikov quasinormality [22] and use the convolution theorem (for details see Sec. III of Ref. [12]). Using the ($k \to 0$) aero-acoustic approximation, which is accurate for low Mach number ($M \leq 1$), and slightly overestimates GWs amplitude for the Mach number approaching unity ($M \to 1$) [12]), we find

$$H_{ijij}(\mathbf{k}, \omega) \approx H_{ijij}(0, \omega) = \frac{7C_1^2 M^3 \alpha_\star^{3/2}}{6\pi^{3/2} k_0} \int_{k_S}^{k_0} \frac{dk}{k^4} \times \exp\left(-\frac{\omega^2 k_0^2}{\alpha_\star M^2 k^4}\right) \mathrm{erfc}\left(-\frac{\omega k_0}{\sqrt{\alpha_\star} M k^2}\right). \quad (8)$$

The integral is dominated by the contribution of large scale ($k \simeq k_S$) perturbations and is maximal at $\omega_{\max}^{(II)} \sim \alpha_\star^{1/2} M k_S^2/k_0 = 2\pi H_\star$. For the first-stage direct-cascade turbulence, the peak frequency is $\omega_{\max}^{(I)} \sim k_0 M$ [12]. To determine the peak frequency at the current epoch, we need to account for the cosmological expansion which decreases the GW amplitude and frequency by the factor $a_\star/a_0$, where $a_\star$ and a_0 are the values of the cosmological scale factor at the GW generation and current epochs.

The total GW energy spectrum at a given space-time event is obtained by integrating over all source regions with a lightlike separation from that event, and includes contributions from GW generated during the first and second stages. For the first stage (with duration time $\tau_T^{(I)} = s_0\tau_0$), $\rho_{\mathrm{GW}}^{(I)}(\omega)$ is given by Eqs. (21) and (A3) of Ref. [12]. For the second stage contribution, we must account for the scale dependence of the cascade time. The total GW fractional energy density parameter at the moment of emission $\Omega_{\mathrm{GW},\star}$ is $105 H_\star^4 \omega^3 \sum_m \tau_T^{(m)} H_{ijij}^{(m)}(0, \omega_\star)/H_0^2$ [12]. Here, the index m runs over I and II for the first and second decay stages, H_0 $\omega_\star$ is an angular frequency at the moment of emission. The current GW amplitude is related to the current fractional energy density parameter through $h_C(f) = 1.26 \times 10^{-18}(1\ \mathrm{Hz}/f)[h_0^2\Omega_{\mathrm{GW}}(f)]^{1/2}$ (where h_0 is the current Hubble parameter H_0 in units of $100\ \mathrm{km\,sec^{-1}\,Mpc^{-1}}$) [14], and

$$h_C(f) \simeq 2 \times 10^{-14}\left(\frac{100\ \mathrm{GeV}}{T_*}\right) \times \left(\frac{100}{g_*}\right)^{1/3} \sum_m [\tau_T^{(m)} \omega_\star H_\star^4 H_{ijij}^{(m)}(0, \omega_\star)]^{1/2}. \quad (9)$$

Here, the linear frequency $f = (a_\star/a_0)f_\star$ with $f_\star = \omega_\star/2\pi$, $T_\star$ and $g_\star$ are the temperature and effective number of relativistic (all fields) degrees of freedom at scale factor $a_\star$.

Figure 1 shows $h_C(f)$ for a few initial magnetic helicity values. GWs emitted during direct-cascade unmagnetized turbulence peak at current $f_{\max}^{(I)} \simeq M\nu_\star$ [12]. We find that the MHD-inverse-cascade generated GW (current epoch) peak frequency is determined by cosmology parameters, $f_{\max}^{(II)} = H_\star a_0/a_\star = 1.6 \times 10^{-5}\ \mathrm{Hz}(g_*/100)^{1/6} \times (T_*/100\ \mathrm{GeV})$ and is independent of turbulence parameters. On the other hand, $f_{\max}^{(II)} = \gamma f_{\max}^{(I)}/M$ is shifted to lower frequency compared to the unmagnetized case. From Eq. (9), the amplitude of MHD-turbulence-generated GWs at the peak is a factor $\sim\alpha_\star^{9/8}\gamma^{-3/4}M^{3/4}$ larger than that in the unmagnetized case.

When modeling turbulence, we used the Biskamp and Muller model, [30,31]. If we adopt the helical MHD turbulence model of Banerjee and Jedamzik [32] (also see Refs. [33]), the GW peak frequency remains the same while the amplitude of the signal doubles.

Figure 1 shows that even for small values of magnetic helicity, the main contribution to the GW energy density is from the second, inverse-cascade stage. The GWs will be strongly polarized since magnetic helicity is maximal at the end of the first stage [11]. LISA should be able to detect such GW polarization [20]. Unlike the unmagnetized case due to the second (inverse-cascade) stage contribution, the GW amplitude is large enough at 10^{-4} Hz to be detectable by LISA. If the EWPT occurs at higher energies ($T_\star > 100$ GeV), the peak is shifted to higher frequency, closer to LISA sensitivity peak, which leads to a stronger signal. Our formalism is applicable for GW production at an earlier

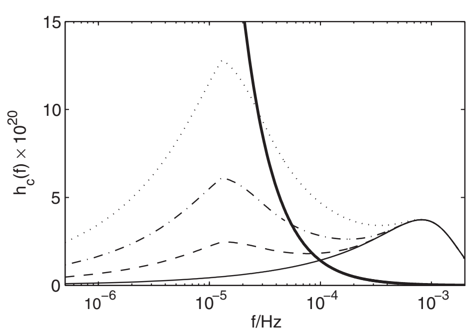

FIG. 1. The spectrum of gravitational radiation from MHD turbulence for $g_* = 100$, $T_* = 100$ GeV, $\gamma = 0.01$, and $M = 1/\sqrt{3}$, for four different initial magnetic helicity values, $\alpha_\star = 0$ (solid line), $\alpha_\star = 0.02$ (dashed line), $\alpha_\star = 0.05$ (dash-dotted line), and $\alpha_\star = 0.1$ (dotted line). The bold line is the LISA design sensitivity curve.

QCD phase transition, assuming the presence of colored magnetic fields [34], or for any other phase transitions [3]; the peak frequency will be shifted according to the changes in $T_\star$ and $g_\star$. The GW signal estimated here exceeds that from bubble collisions [4,15,16] or from hydrodynamical, unmagnetized turbulence [9,10,12]. Of course, this strong signal assumes initial nonzero (although small) magnetic helicity, so detection of polarized GWs by LISA will indicate parity violation during the EWPT as proposed in Refs. [18].

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