ON ONE TWO POINT BVP FOR THE FOURTH ORDER LINEAR ORDINARY DIFFERENTIAL EQUATION

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Abstract. In this article we consider the two-point boundary value problem

$$u^{(4)}(t) = p(t)u(t) + h(t) \quad \text{for} \quad a \le t \le b,$$

$$u^{(i)}(a) = c_{1i}, \quad u^{(i)}(b) = c_{2i} \ \ (i = 0, 1),$$

where $c_{1i}, c_{2i} \in R$, $h, p \in L([a, b]; R)$. Here we study the question of dimension of the space of nonzero solutions and oscillatory behaviors of nonzero solutions on the interval $[a, b]$ for the corresponding homogeneous problem, and establish efficient sufficient conditions of solvability for nonhomogeneous problem.

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Introduction

Consider on the interval $[a, b]$ the fourth order linear ordinary differential equation

$$u^{(4)}(t) = p(t)u(t) + h(t) \tag{0.1}$$

with the boundary conditions

$$u^{(i)}(a) = c_{0i}, \quad u^{(i)}(b) = c_{1i} \ \ (i = 0, 1) \tag{0.2}$$

and corresponding homogeneous problem

$$v^{(4)}(t) = p(t)v(t) \tag{0.3}$$

$$v^{(i)}(a) = 0, \quad v^{(i)}(b) = 0 \ \ (i = 0, 1) \tag{0.4}$$

where $c_{0i}, c_{1i} \in R$, h, $p \in L([a, b]; R)$.

By a solution of the problem (0.1), (0.2) we understand a function $u \in \widetilde{C}^3(I, R)$, which satisfies the equation (0.1) almost everywhere on I and satisfies the conditions (0.2).

The bibliography on two-point problems studied specifically for fourth order ordinary differential equations is currently not extensive. The few papers devoted to this topic are focused mainly on beam equations under the boundary conditions $u(a) = u(b) = 0, u''(a) = u''(b) = 0$

(see [1], [2], [3], [4], [6]), although there are works where these problems are studied in more general cases (see [5], [9], [10], [11]). As regards the oscillatory properties on unbounded intervals for fourth order linear ordinary differential equations, one can mention, e.g., the work [7], where detailed results are obtained.

Our aim here is to fill this gap in a certain sense. In the first section, we study the problem on the dimension of the space of the nonzero solutions and oscillatory behavior of the nonzero solutions of problem (0.3), (0.4) on the bounded interval $[a,\, b]$. The second section is devoted to the unique solvability of problem (0.1), (0.2), where some results from the first section are essentially used.

We recall that the problem problem (0.1),(0.2) has the Fredholm's property (see [8]).

The following notations are used throughout the paper:

N is the set of all natural numbers. R is the set of all real numbers. $C(I;R)$ is the Banach space of continuous functions $u : I \to R$ with the norm $\|u\|_C = \max\{|u(t)| : t \in I\}$. $\widetilde{C}(I;R)$ is the set of functions $u : I \to R$ which are absolutely continuous. $\widetilde{C}^3(I;R)$ is the set of functions $u : I \to R$ which are absolutely continuous together with their third derivatives. $L(I;R)$ is the Banach space of Lebesgue integrable functions $p : I \to R$ with the norm $\|p\|_L = \int_a^b |p(s)|ds$. $L_\infty(I;\, R)$, is the Banach space of essentially bounded functions $p : I \to R$ with the norm $||p||_\infty = \operatorname{vrai\,sup}\limits_{t\in I} |p(t)|$. S_p is the space of the nonzero solutions of problem (0.3), (0.4).

Remark 0.1. If we assume existence of the nonzero solution v of problem (0.3), (0.4) and multiply equation (0.3) by this solution, then by the integration by parts in view of conditions (0.4) we get $\int_b^a v''^2(s)ds = \int_b^a p(s)v^2(s)ds$. Than it is obvious that we get the contradiction if $p(t) \le 0$ for $t \in [a,\, b]$. Therefore, this is the trivial case when problem (0.3), (0.4) has only the zero solution and then the problem (0.1), (0.2) is uniquely solvable for any $c_{0i}, c_{1i} \in R,\ h,\, p \in L([a,b];R)$. For this reason, we study problem (0.1), (0.2) under assumption that there exists such set $A_p \subset [a,\, b]$ of the positive measure, that

$$p(t) > 0 \quad \text{for} \quad t \in A_p.$$

1. Main Results

1.1. **Problem** (0.3), (0.4)**.** First of all we introduce the propositions about the dimension of the space of the nontrivial solutions and oscillatory behaviors of the nontrivial solutions of the homogeneous linear problem (0.3), (0.4) on the finite interval $[a,\, b]$.

Theorem 1.1. *Let $p \in L([a,\, b];R)$, than:*

a) $\dim S_p \le 2$*;*

b) $\dim S_p \leq 1$ *if*
$$\int_a^b |p(s)|ds \leq \frac{110}{(b-a)^3}. \tag{1.1}$$

Theorem 1.2. *Let* $p \in L([a,\, b]; R),\ n \in N,$ *and*
$$p(s) \geq 0 \quad for \quad t \in [a,\, b], \tag{1.2}$$
then:

a) $\dim S_p \leq 1$;

b) Any nontrivial solution of problem $(0.3), (0.4)$ *has less than* n *zeros if*
$$\int_a^b p(s)ds \leq \frac{8 \cdot 2^{n/(n+2)}(n+2)^4}{(b-a)^3}. \tag{1.3}$$

Theorem 1.3. *Let* $p \in L_\infty([a,\, b]; R),\ n \in N,$ *and condition* (1.2) *holds. Than any nontrivial solution of problem* $(0.3), (0.4)$ *has less than* n *zeros if*
$$||p||_\infty \leq \frac{16 \cdot 4^{n/(n+2)}(n+2)^4}{(b-a)^4}.$$

1.2. **Problem** (0.1),(0.2)**.**

Theorem 1.4. *Let* $p, h \in L([a,\, b]; R)$, $c_{ij} \in R\,(i, j = 0, 1)$ *and*
$$\int_a^b |p(s)|ds \leq \frac{32}{(b-a)^3}. \tag{1.4}$$
Then problem (0.1),(0.2) *is uniquely solvable.*

Theorem 1.5. *Let* $p, h \in L([a,\, b]; R)$, $c_{ij} \in R\,(i, j = 0, 1)$, *condition* (1.2) *holds, and*
$$\int_a^b p(s)ds \leq \frac{128}{(b-a)^3}. \tag{1.5}$$
Then problem (0.1),(0.2) *is uniquely solvable.*

Theorem 1.6. *Let* $p \in L_\infty([a,\, b]; R), h \in L([a,\, b];\, R)$, $c_{ij} \in R\,(i, j = 0, 1)$, *condition* (1.2) *holds, and*
$$\int_a^b p(s)ds \leq \frac{256}{(b-a)^3}.$$
Then problem (0.1),(0.2) *is uniquely solvable.*

2. Auxiliary Propositions

Definition 2.1. Let $\sigma \in \{-1,\, 1\},\ n \in \{0, 1, 2, ...\}, y \in L([a,\, b]; R)$ and I be a subinterval of $[a,\, b]$. Then we sad that $N(y, I) = n$ if exists the system of intervals $\{I_i\}_{i=1}^{n+1}$ such that:
$$\bigcup_{i=1}^{n+1} I_i = I, \quad \sup I_i = \inf I_{i+1}\ (i = \overline{1, n}),$$