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The mixed BVP for second order nonlinear ordinary differential equation at resonance

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Efficient sufficient conditions are established for the solvability of the mixed problem

$$u''(t) = p(t)u(t) + f(t, u(t)) + h(t), \qquad u(a) = 0, \quad u'(b) = 0,$$

where $h, p \in L([a, b]; R)$ and $f \in K([a, b] \times R; R)$, in the case where the homogeneous linear problem $w''(t) = p(t)w(t)$, $w(a) = 0$, $w'(b) = 0$ has nontrivial solutions.

1 Introduction

Consider on the set $I = [a, b]$ the second order nonlinear ordinary differential equation

$$u''(t) = p(t)u(t) + f(t, u(t)) + h(t) \quad \text{for} \quad t \in I \tag{1.1}$$

with the boundary conditions

$$u(a) = 0, \quad u'(b) = 0, \tag{1.2}$$

where $h, p \in L(I; R)$ and $f \in K(I \times R; R)$. By a solution of problem (1.1), (1.2) we understand a function $u \in \widetilde{C}'(I, R)$, which satisfies Equation (1.1) almost everywhere on I and satisfies conditions (1.2). Along with (1.1), (1.2) we consider the homogeneous problem

$$w''(t) = p(t)w(t) \quad \text{for} \quad t \in I, \tag{1.3}$$

$$w(a) = 0, \quad w'(b) = 0. \tag{1.4}$$

At present, the foundations of the general theory of two-point boundary value problems are already laid and problems of this type are studied by many authors and investigated in detail (see, for instance, [3], [4], [6], [14]–[15]). On the other hand, in all of these works, only the non resonance case is considered. If we study literature, we'll see that the case where the problem (1.3), (1.4) has the nontrivial solution has not been practically investigated in difference with the Dirichlet BVP, which has been considered in a number of articles in resonance case. Even the Dirichlet BVP for the second order ODE at resonance in the majority of articles, is studied in the case when the first coefficient of homogeneous linear problem is a constant (see, for instance, [1], [2], [4], [9], [11], [16]). In the article [17] we developed the technique which enabled us to established the Landesman–Lazer's type conditions for the solvability of Dirichlet BVP for second order ODE at resonance in the case when the first coefficient of homogeneous equation is not necessarily constant. The theorems proved in this study significantly generalize and improve other authors results (see, [1], [2], [4], [7], [16]). In the present paper we generalize the

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method developed in the article [17], and prove the Landesman–Lazer's type conditions for solvability of our problem at the resonance case, when the function $p \in L(I; R)$ is not necessarily constant.

Throughout the paper we use the following notations: N is the set of all natural numbers. R is the set of all real numbers, $R_+ = [0, +\infty[$. $C(I; R)$ is the Banach space of continuous functions $u : I \to R$ with the norm $\|u\|_C = \max\{|u(t)| : t \in I\}$. $\widetilde{C}'(I; R)$ is the set of functions $u : I \to R$ which are absolutely continuous together with their first derivatives. $L(I; R)$ is the Banach space of the Lebesgue integrable functions $p : I \to R$ with the norm $\|p\|_L = \int_a^b |p(s)|\, ds$. $K(I \times R; R)$ is the set of the functions $f : I \times R \to R$ satisfying the Carathéodory conditions. Having $w : I \to R$, we put: $N_w \overset{def}{=} \{t \in]a, b] : w(t) = 0\}$.

2 Main results

Theorem 2.1 *Let w be a nonzero solution of problem* (1.3), (1.4),

$$N_w = \emptyset, \tag{2.1}$$

let there exist a constant $r > 0$, functions $f^-, f^+ \in L(I; R_+)$ and $g, h_0 \in L(I;]0, +\infty[\,)$ such that

$$f(t, x)\mathrm{sgn} x \le g(t)|x| + h_0(t) \quad \text{for} \quad |x| \ge r \tag{2.2}$$

and

$$f(t, x) \le -f^-(t) \ \text{ for } \ x \le -r, \qquad f^+(t) \le f(t, x) \ \text{ for } \ x \ge r \tag{2.3}$$

on I. Let, moreover, there exist $\varepsilon > 0$ such that

$$-\int_a^b f^-(s)|w(s)|\, ds + \varepsilon||\gamma_r||_L \le -\int_a^b h(s)|w(s)|\, ds$$

$$\le \int_a^b f^+(s)|w(s)| ds - \varepsilon||\gamma_r||_L, \tag{2.4$_1$}$$

where $\gamma_r(t) = \sup\{|f(t, x)| : |x| \le r\}$. Then problem (1.1), (1.2) *has at least one solution.*

Theorem 2.2 *Let w be a nonzero solution of problem* (1.3), (1.4), *condition* (2.1) *hold, there exist a constant $r > 0$, functions $f^-, f^+ \in L(I; R_+)$ and $q \in K(I \times R; R_+)$ such that q is non-decreasing in the second argument,*

$$|f(t, x)| \le q(t, x) \quad \text{for} \quad |x| \ge r, \tag{2.5}$$

$$f^-(t) \le f(t, x) \ \text{ for } \ x \le -r, \qquad f(t, x) \le -f^+(t) \ \text{ for } \ x \ge r \tag{2.6}$$

on I, and

$$\lim_{|x| \to +\infty} \frac{1}{x} \int_a^b q(s, x)\, ds = 0. \tag{2.7}$$

Let, moreover, there exist $\varepsilon > 0$ such that

$$-\int_a^b f^-(s)|w(s)|\, ds + \varepsilon||\gamma_r||_L \le \int_a^b h(s)|w(s)|\, ds \le$$

$$\le \int_a^b f^+(s)|w(s)|\, ds - \varepsilon||\gamma_r||_L, \tag{2.4$_2$}$$

where γ_r is the function defined in Theorem 2.1. Then problem (1.1), (1.2) *has at least one solution.*

Corollary 2.3 *Let a nonzero solution w of problem* (1.3), (1.4), *and a constant $r > 0$, be such that conditions* (2.1) *and* (2.2) *hold. Let, moreover, the equality*

$$\liminf_{x \to \pm\infty} f(s,\ x)\mathrm{sgn} x = +\infty \qquad \textit{uniformly on} \quad I \tag{2.8}$$

