

ON THE KINEMATICS OF A COROTATING RELATIVISTIC PLASMA STREAM IN THE PERPENDICULAR ROTATOR MODEL OF A PULSAR MAGNETOSPHERE

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(Received 20 April, 1995; accepted 5 March, 1996)

Abstract. An investigation of the kinematics of a rotating relativistic plasma stream in the perpendicular rotator model of the pulsar magnetosphere is presented. It is assumed that the plasma (ejected from the pulsar) moves along the pulsar magnetic field lines and also corotates with them. The field lines are considered to be radial straight lines, located in the plane which is perpendicular to the pulsar rotation axis. The necessity of taking particle inertia into account is discussed. It is argued that the “massless” (“force-free”) approximation cannot be used for the description of this problem. The frame selection is discussed and it is shown that it is convenient to discuss the problem in the noninertial frame of ZAMOs (Zero Angular Momentum Observers). The equation of motion and the exact set of equations describing the behaviour of a relativistic plasma stream in the pulsar magnetosphere is obtained. The possible relevance of this investigation for the understanding of the formation process of a pulsar magnetosphere is discussed.

1. Introduction

In connection with several astrophysical problems, e.g. the theoretical description of pulsar magnetospheres, the question of investigation of plasma ejection from a rotating source (e.g. a star) along its magnetic field lines is arising. It is most important to investigate the behaviour of this stream near the pulsar surface.

While investigating the pulsar wind, the change of particle momentum is usually neglected. The equations are then formulated in the “massless” approximation $m = 0$ (Mestel, 1973; Endean, 1974), in which the particles are moving along the pulsar magnetic field lines like photons moving along geodetic lines. It seems to us indubitable, that the inertia of the particles, disregarded in a large number of previous papers, must have a great influence on the pulsar magnetosphere formation process. The importance of taking into consideration the inertia of the particles was understood a long time ago and treated as a perturbation to the “massless” approximation (Scharlemann, 1974; Henriksen and Norton, 1975). It seems to us that giving up the “massless” approximation not only shows the essence of the problem more adequately, but also simplifies the problem from the mathematical point of view.

We use the perpendicular rotator model of the pulsar magnetosphere and treat only the polar cap. It is assumed that the pulsar magnetic field lines are the radial straight lines. In our paper is discussed the case, when the plasma particles corotate

with the pulsar magnetic field lines. We also assume that the $E_{\parallel} = 0$, i.e. the electric field component along the magnetic field lines is zero.

We will choose the simplified geometry, discuss the frame selection and the derivation of the equation of motion of a relativistic stream. We will argue that the “massless” (“force-free”) approximation cannot be used for the description of this problem and discuss the difference of our statement of the problem from those discussed before by a number of authors (Mestel, 1973; Endean, 1974; Henriksen and Norton, 1975; Michel, 1969; Kennel *et al.*, 1983 and others).

We will obtain the equation of motion of a relativistic plasma stream and the exact set of equations for the description of the behaviour of the stream.

2. Basic Considerations

Let us consider the simplified geometrical model, when the pulsar rotation axis is perpendicular to its magnetic momentum. The magnetic field lines are considered as radial straight lines, located in the plane which is perpendicular to the pulsar rotation axis. If we discuss the processes in the region which has the linear size $d \ll R_c$, where R_c is the curvature radius of the magnetic field lines, this approximation is justified. The ejected particles not only move along the radius r , but also corotate with the pulsar magnetosphere because the field lines are corotating with the plasma:

$$\mathbf{E} + [\mathbf{V}\mathbf{B}] = 0. \quad (1)$$

Here and below we use geometric units $c = G = 1$.

The condition (1) means, that because of the rotation of the magnetic field \mathbf{B} , a field \mathbf{E} is generated, which forces plasma particles to corotate with the magnetic field lines with linear velocity:

$$\mathbf{V} = [\boldsymbol{\Omega}\mathbf{r}], \quad (2)$$

where $\boldsymbol{\Omega}$ is the angular velocity vector of the star.

In the “massless” approximation, in an inertial frame there are no other forces in the equation of motion than the Lorentz force $e(\mathbf{E} + [\mathbf{V}\mathbf{B}])$. The centrifugal force is neglected. However, by condition (1) the Lorentz force vanishes and $\frac{d\mathbf{p}}{dt} = 0$. According to Rylov (1989) the equation $\frac{d\mathbf{p}}{dt} = 0$ in approximation $m = 0$ can be justified in two cases: $\mathbf{p} \neq 0$ and $\gamma \rightarrow \infty$, or finite γ and $\mathbf{p} = 0$. The first case is called by the Rylov “force-free” approximation and the second one the “massless” approximation. But in order to disregard the centrifugal force it must be shown under which conditions this is a valid approximation. Below we will show that disregarding the centrifugal force is equivalent to giving up corotation. In order to understand the physical reason for corotation, let us consider a well known example: let us make a conductor move across a magnetic field with a constant

velocity \mathbf{V} . Then, the Lorentz force $e[\mathbf{V}\mathbf{B}]$ will act in different directions on the charges of different signs of the conductor which will cause a charge separation, i.e. generation of the electrostatic field. Due to the separation of charges, some energy of the source making the conductor move across magnetic field lines with constant velocity (this velocity satisfies the condition (1)), must be expended. An analogous situation holds in the case when an observer moves with respect to the conductor with the velocity \mathbf{V} . The condition (1) allows also such an interpretation: the motion of the particles (of any sign) perpendicularly to the magnetic field is possible if an electric field $\mathbf{E}(\mathbf{E}\perp\mathbf{B})$ exists and the velocity of this motion is equal to the electric drift velocity $\mathbf{V} = \frac{[\mathbf{E}\mathbf{B}]}{B^2}$.

Now let us assume that the outer source makes the conductor move with an acceleration. Then it is evident, that the work of the source is expended not only on the charge separation, but also on the acceleration of the conductor. Therefore, except of the Lorentz force, there will be an additional force \mathbf{F} in the equation of motion, but due to the condition (1) it will take the following form:

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}.$$

Returning to our problem we conclude: the pulsar rotation makes the magnetic field lines rotate, this rotation generates the electric field, which is oriented perpendicularly to the magnetic field lines and because of the condition (1) makes the charged particles (ejected from the star) corotate with the magnetic field lines. During this motion, the particle velocity changes its direction, therefore a force is acting on the particles. This force in our case is directed along the magnetic field lines and is not compensated.

Considering analogous problems in classical mechanics, for example the motion of a bead located in a rotating tube, the reaction force appears in the equation of motion. This force makes the bead not only rotate together with the tube, but also accelerates it along the tube.

In our problem, according to the condition (1) the magnetic field lines and the generated electric field act like the rotating tube for the charged particles. It is more convenient to find the form of the reaction force in a noninertial frame, which rotates together with the particles. The situation described above is very similar to the problems connected with the description of particle motion in curved space-time, in particular in the gravitational field of the black hole – in the Schwarzschild or Kerr metric, because according to the Einstein principle of equivalence, locally we cannot distinguish gravitation from noninertiality. Using the principle of equivalence, the (3 + 1) formalism is suitable for our problem. The (3 + 1) formalism is described in Thorne *et al.* (1988). In this formalism instead of four-vectors, three spatial and one time coordinates are used.

The problem of the motion of particles in the pulsar magnetosphere can be solved in the local-inertial frame of the observers, who are measuring the physical quantities in the immediate vicinity of themselves. They are called the Zero Angular

Momentum Observers (ZAMOs). ZAMOs cover the whole space-time. Each of the observers uses its proper time τ , which is different from the universal time t . We must mention that the proper time of the observer who moves together with the particle, is different from the proper time of ZAMOs τ .

The transition from the inertial frame to the frame, which is connected with the pulsar magnetosphere rotation can be done by using following relation:

$$t = t', \quad \varphi = \Omega t, \quad r = r', \quad z = 0. \quad (3)$$

The metric, describing such a frame has the following form:

$$dS^2 = -(1 - \Omega^2 r^2) dt^2 + dr^2. \quad (4)$$

From (4) it follows, that the interval of the proper time of ZAMOs is connected with the universal time interval as

$$d\tau = \alpha dt, \quad (5)$$

where $\alpha = \sqrt{1 - \Omega^2 r^2}$ and it is called the “lapse function”.

A photon, propagating from the periphery to the center of rotation, from observer to observer would show “reddening”. The reason is that the rate of the time lapse for each of the ZAMOs differs from those for the others and also differs from the lapse of the universal (laboratory) time. The lapse function α not only connects the proper time of ZAMOs τ with the universal time t , but also expresses the gravitational potential, i.e. the inertial force. In the frame of ZAMOs the particle has the gravitational acceleration

$$\mathbf{g} = -\frac{\nabla\alpha}{\alpha}. \quad (6)$$

The equation of motion of the particle in the frame of ZAMOs takes the following form:

$$\frac{d\mathbf{p}}{d\tau} = \gamma \mathbf{g} + \frac{e}{m} (\mathbf{E} + [\mathbf{V}\mathbf{B}]), \quad (7)$$

where $\gamma = \frac{1}{\sqrt{1-V^2}}$ is the Lorentz-factor and $\mathbf{V} = \frac{1}{\alpha} \frac{d\mathbf{r}}{dt}$ is the velocity of the particle, \mathbf{p} is the dimensionless momentum, $\mathbf{p} \rightarrow \frac{\mathbf{p}}{m}$.

We must mention, that the equation of motion contains the acceleration \mathbf{g} from (6) only in the frame, for which $d\varphi' = 0$. In the generalized rotating frame, where $d\varphi' \neq 0$, according to the (3 + 1) formalism, the lapse function $\alpha = 1$, but instead of the acceleration (6) the “gravimagnetic” acceleration appears (Thorne *et al.*, 1988), which must be written in the equation of motion. This equation differs from (7) only by the redesignation of the angular velocities and, of course, describes the same process and gives the same results as (7).

While considering the hydrodynamical problem, according to the (3 + 1) formalism, instead of the equation of motion for a single particle, the energy-momentum

tensor conservation (the law of force balance) is used. Also, using the particle conservation law (the continuity equation) and neglecting the hydrodynamical stream pressure, the equation, which describes the stream motion takes the following form:

$$\frac{1}{\alpha} \frac{\partial \mathbf{p}}{\partial t} + (\mathbf{V} \nabla) \mathbf{p} = -\gamma \frac{\nabla \alpha}{\alpha} + \frac{e}{m} (\mathbf{E} + [\mathbf{V} \mathbf{B}]), \tag{8}$$

In the Equation (8) the derivative $\frac{d}{d\tau}$ is changed with the convectional derivative $\frac{d}{d\tau} \rightarrow \frac{1}{\alpha} \frac{\partial}{\partial t} + (\mathbf{V} \nabla)$ and the term proportional to the pressure gradient ∇P (P is the pressure) is neglected.

Let us discuss the difference between Equation (8) and the equation, usually used during the consideration of similar problems in the magnetohydrodynamical approximation. In previous papers, the acceleration $\frac{\nabla \alpha}{\alpha}$ was disregarded. The problem was formulated in the inertial frame. It is easy to show, that the momentum \mathbf{p} in the frame of ZAMOs coincides with the momentum \mathbf{p}' in inertial frame. Really, according to the definition $\mathbf{p} = \gamma \mathbf{V}$, where $\mathbf{V} = \frac{1}{\alpha} \frac{d\mathbf{r}}{dt}$, and

$$\gamma = \alpha \gamma', \tag{9}$$

we will find, that $\mathbf{p}' = \mathbf{p}$, because $\mathbf{V}' = \frac{d\mathbf{r}'}{dt} = \frac{d\mathbf{r}}{dt}$. It is easy then to rewrite the Equation (8) for the quantities defined in inertial frame. Omitting the prime for all quantities, we will obtain

$$\frac{\partial \mathbf{p}}{\partial t} + (\mathbf{V} \nabla) \mathbf{p} = -\gamma \alpha \nabla \alpha + \frac{e}{m} (\mathbf{E} + [\mathbf{V} \mathbf{B}]). \tag{10}$$

The Equation (10) contains the force $\mathbf{F} = -\gamma \alpha \nabla \alpha$, which was disregarded in previous papers. It is the analogy of the force, which causes the centrifugal effect and is equal to

$$\mathbf{F} = \frac{\Omega^2 \mathbf{r}}{\sqrt{1 - \Omega^2 r^2 - \left(\frac{dr}{dt}\right)^2}}. \tag{11}$$

This force exceeds ∇P (P is the pressure), which was taken into consideration in the paper by Kennel *et al.* (1983). To find the force \mathbf{F} , it is not necessary to write the equation in the noninertial frame. Staying in the inertial frame and taking into consideration that the axially symmetric magnetic field changes its direction in the space, i.e. $\mathbf{B} = \mathbf{B}(\mathbf{r}) = B_r \left(\frac{\mathbf{r}}{|\mathbf{r}|} \right)$, the equation of motion for the guiding center can be obtained. The relativistic generalization of this problem (which was stated and solved for the first time by Alfven) can be found in Sivukhin (1963). We must mention, that in the paper of Cohen and Rosenblum (1972) the problem was investigated in the rotating frame and the equations were written for the four-vectors, but then the exact equations for the motion of the particles were not used.

Besides, there are two principal differences of our consideration from previous ones (e.g. Michel, 1969; Kennel *et al.*, 1983). In the problem discussed here the

extremely large surface magnetic field ($B_0 \simeq 10^{12} - 10^{13}$ G) of the neutron star is essential. On the other hand, in the magnetosphere outside the star a magnetic field \mathbf{B}_1 can be generated. The source of the field \mathbf{B}_0 is located inside the star, but the source of the field \mathbf{B}_1 is a current \mathbf{j} , which is generated in the magnetosphere. The current \mathbf{j} has no influence on the source of the pulsar magnetic field \mathbf{B}_0 and we will consider \mathbf{B}_0 as an external magnetic field, in which the plasma is located.

Furthermore, in the previous papers another assumption was made, namely that the inertia of the particles, i.e. the term which contains the partial time derivative $\frac{\partial}{\partial t}$, was disregarded. In Henriksen and Norton (1975) the derivative $\frac{\partial}{\partial t}$ for the transversal components of the quantities was changed to the derivative with respect to the azimuthal coordinate, $\frac{\partial}{\partial t} \rightarrow \Omega \frac{\partial}{\partial \varphi}$ and the change of the quantities along the magnetic field was disregarded.

The quantities in Equation (10) can be written as:

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1, \quad \mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1, \quad \mathbf{p} = \mathbf{p}_0 + \mathbf{p}_1, \quad (12)$$

where \mathbf{E}_0 , \mathbf{B}_0 and \mathbf{p}_0 are the basic terms, and \mathbf{E}_1 , \mathbf{B}_1 and \mathbf{p}_1 are the perturbations in the first approximation of expansion over the parameter of weak turbulence $\left(\frac{E_1^2}{mn\gamma}\right)$.

From the condition (1) in the zeroth approximation we have:

$$\mathbf{E}_0 + [\mathbf{V}_0 \mathbf{B}_0] = 0$$

and from the Equation (10) for the radial acceleration we will obtain (Machabeli and Rogava, 1994):

$$\frac{d^2 r}{dt^2} = \frac{\Omega^2 r}{1 - \Omega^2 r^2} \left[1 - \Omega^2 r^2 - 2 \left(\frac{dr}{dt} \right)^2 \right]. \quad (13)$$

Here we discuss only the radial motion, because the azimuthal one is determined by corotation ($V_\varphi = \Omega r$). The solution of the Equation (13) can be presented in the form:

$$r(t) = \frac{V_0}{\Omega} \frac{Sn \Omega t}{dn \Omega t}, \quad (14)$$

where Sn and dn are the Jacobian elliptical sine and modulus respectively (Abramovitz and Stegun, 1964), V_0 is the initial velocity of the particle. From the Equation (13) it follows that if the radial velocity $V_r > \frac{1}{\sqrt{2}}$, the acceleration changes its sign and the particle is not accelerated, but braked (see also Machabeli and Rogava (1994)).

Using the asymptotic expression for the Jacobian function we find that when $V_0 \rightarrow 1$,

$$r(t) = \frac{V_0}{\Omega} \sin \Omega t. \quad (15)$$

For the radial velocity we obtain:

$$V_{0r} = V_0 \cos \Omega t, \tag{16}$$

from which it follows that

$$V_0^2 = (V_{0r})^2 + (V_{0\phi})^2 = \text{const.} \tag{17}$$

This fact is not quite evident. In the nonrelativistic case, as it is known, the particle kinetic energy increases exponentially at the expense of the external source, which keeps constancy of rotation. This follows from the general solution (14) too, when $V_r \ll 1$; expanding the Jacobian functions we obtain:

$$r(t) = \frac{V_0}{\Omega} sh \Omega t, \tag{18}$$

where sh is the hyperbolic sine.

The principle difference between nonrelativistic and relativistic motion can be easily understood if we rewrite the Equation (10) in a form different from (13):

$$\gamma_0 \frac{dV_{0r}}{dt} + \left(\gamma_0 \Omega^2 r + \gamma_0 \frac{dV_{0r}}{dt} \right) \gamma_0^2 V_{0r}^2 = \gamma_0 \Omega^2 r. \tag{19}$$

The second and third terms in the left side describe the change of mass (γ -factor) in time. It is evident (from (19)) that for small velocities $V_{0r} \rightarrow 0$ ($\gamma_0 \rightarrow 1$) we will obtain $\frac{dV_{0r}}{dt} - \Omega^2 r = 0$ and the solution (18). But if $V_{0r} \rightarrow 1$ ($\gamma_0 \rightarrow \infty$), we will obtain $\frac{dV_{0r}}{dt} + \Omega^2 r = 0$ and the solution (16). From the Equation (19) also follows that the term connected with the change of mass dominates over the other terms if

$$V_{0r} > \sqrt{\frac{1 - V_{0\phi}^2}{2}}. \tag{20}$$

When (20) is valid, the inertia of particles is such, that the centrifugal force changes its sign and the motion is braked. We can see, that when $V_{0r} \rightarrow 1$, the “massless” (“force-free”) approximation can be justified only if “force-free” means $\frac{d\gamma_0}{dt} = 0$. If we disregard the right side in the Equation (19) in comparison with the first term of the left side, this fact automatically means disregarding the first term in comparison with the second term in brackets on the left side of the equation. Then $p_{0r} = \text{const}$ – the particle motion is uniform and straightforward and we have no corotation. Therefore disregarding the force \mathbf{F} is possible only in the case $\Omega = 0$.

Thus the “massless” (“force-free”) approximation cannot be used for the “solid-body” type rotating ($\mathbf{V} = [\Omega \mathbf{r}]$) relativistic plasma stream. $\frac{dp_{0r}}{dt} \neq 0$, especially $\frac{dp_{1r}}{dt}$ must not be equal to zero.

If we add to the Equation (10) the continuity equation and Maxwell equations we shall have a set of equations, which completely describes the behaviour of a relativistic plasma stream corotating with the pulsar magnetosphere:

$$\frac{\partial \mathbf{p}}{\partial t} + (\mathbf{V} \nabla) \mathbf{p} = -\gamma \alpha \nabla \alpha + \frac{e}{m} (\mathbf{E} + [\mathbf{V} \mathbf{B}])$$

$$\frac{\partial \rho}{\partial t} = -\nabla(\alpha \mathbf{j})$$

$$(\nabla \mathbf{B}) = 0$$

$$(\nabla \mathbf{E}) = 4\pi \rho$$

$$\frac{\partial \mathbf{B}}{\partial t} = -[\nabla(\alpha \mathbf{E})]$$

$$\frac{\partial \mathbf{E}}{\partial t} = [\nabla(\alpha \mathbf{B})] - 4\pi \alpha \mathbf{j}$$

where ρ and \mathbf{j} are the charge and current densities respectively.

3. Conclusion

We have investigated the role of particle inertia and shown that the “massless” (“force-free”) approximation cannot be used for the “solid-body” type rotating ($\mathbf{V} = [\boldsymbol{\Omega} \mathbf{r}]$) relativistic plasma stream description. We must mention that our investigation was carried out under the assumption $E_{\parallel} = 0$ and if $E_{\parallel} \neq 0$, this would probably have the strong influence on our results. We also derived the equation of motion and a set of equations for the description of a relativistic plasma stream in the perpendicular rotator model of pulsar magnetospheres.

Acknowledgements

We would like to thank the anonymous referee for constructive suggestions and criticism.

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