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The mixed BVP for second order nonlinear ordinary differential equation at resonance

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Efficient sufficient conditions are established for the solvability of the mixed problem

$$u''(t) = p(t)u(t) + f(t, u(t)) + h(t), \qquad u(a) = 0, \quad u'(b) = 0,$$

where $h, p \in L([a, b]; R)$ and $f \in K([a, b] \times R; R)$, in the case where the homogeneous linear problem $w''(t) = p(t)w(t)$, $w(a) = 0$, $w'(b) = 0$ has nontrivial solutions.

1 Introduction

Consider on the set $I = [a, b]$ the second order nonlinear ordinary differential equation

$$u''(t) = p(t)u(t) + f(t, u(t)) + h(t) \quad \text{for} \quad t \in I \tag{1.1}$$

with the boundary conditions

$$u(a) = 0, \quad u'(b) = 0, \tag{1.2}$$

where $h, p \in L(I; R)$ and $f \in K(I \times R; R)$. By a solution of problem (1.1), (1.2) we understand a function $u \in \widetilde{C}'(I, R)$, which satisfies Equation (1.1) almost everywhere on I and satisfies conditions (1.2). Along with (1.1), (1.2) we consider the homogeneous problem

$$w''(t) = p(t)w(t) \quad \text{for} \quad t \in I, \tag{1.3}$$

$$w(a) = 0, \quad w'(b) = 0. \tag{1.4}$$

At present, the foundations of the general theory of two-point boundary value problems are already laid and problems of this type are studied by many authors and investigated in detail (see, for instance, [3], [4], [6], [14]–[15]). On the other hand, in all of these works, only the non resonance case is considered. If we study literature, we'll see that the case where the problem (1.3), (1.4) has the nontrivial solution has not been practically investigated in difference with the Dirichlet BVP, which has been considered in a number of articles in resonance case. Even the Dirichlet BVP for the second order ODE at resonance in the majority of articles, is studied in the case when the first coefficient of homogeneous linear problem is a constant (see, for instance, [1], [2], [4], [9], [11], [16]). In the article [17] we developed the technique which enabled us to established the Landesman–Lazer's type conditions for the solvability of Dirichlet BVP for second order ODE at resonance in the case when the first coefficient of homogeneous equation is not necessarily constant. The theorems proved in this study significantly generalize and improve other authors results (see, [1], [2], [4], [7], [16]). In the present paper we generalize the

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method developed in the article [17], and prove the Landesman–Lazer's type conditions for solvability of our problem at the resonance case, when the function $p \in L(I;R)$ is not necessarily constant.

Throughout the paper we use the following notations: N is the set of all natural numbers. R is the set of all real numbers, $R_+ = [0, +\infty[$. $C(I;R)$ is the Banach space of continuous functions $u : I \to R$ with the norm $\|u\|_C = \max\{|u(t)| : t \in I\}$. $\widetilde{C}'(I;R)$ is the set of functions $u : I \to R$ which are absolutely continuous together with their first derivatives. $L(I;R)$ is the Banach space of the Lebesgue integrable functions $p : I \to R$ with the norm $\|p\|_L = \int_a^b |p(s)|\,ds$. $K(I \times R;R)$ is the set of the functions $f : I \times R \to R$ satisfying the Carathéodory conditions. Having $w : I \to R$, we put: $N_w \overset{def}{=} \{t \in]a,b] : w(t) = 0\}$.

2 Main results

Theorem 2.1 *Let w be a nonzero solution of problem* (1.3), (1.4),

$$N_w = \emptyset, \tag{2.1}$$

let there exist a constant $r > 0$, functions $f^-, f^+ \in L(I;R_+)$ and $g, h_0 \in L(I;]0,+\infty[\,)$ such that

$$f(t,x)\mathrm{sgn}x \le g(t)|x| + h_0(t) \quad \text{for} \quad |x| \ge r \tag{2.2}$$

and

$$f(t,x) \le -f^-(t) \ \text{ for } \ x \le -r, \qquad f^+(t) \le f(t,x) \ \text{ for } \ x \ge r \tag{2.3}$$

on I. Let, moreover, there exist $\varepsilon > 0$ such that

$$\begin{aligned} -\int_a^b f^-(s)|w(s)|\,ds + \varepsilon||\gamma_r||_L &\le -\int_a^b h(s)|w(s)|\,ds \\ &\le \int_a^b f^+(s)|w(s)|ds - \varepsilon||\gamma_r||_L, \end{aligned} \tag{2.4$_1$}$$

where $\gamma_r(t) = \sup\{|f(t,x)| : |x| \le r\}$. Then problem (1.1), (1.2) *has at least one solution.*

Theorem 2.2 *Let w be a nonzero solution of problem* (1.3), (1.4), *condition* (2.1) *hold, there exist a constant $r > 0$, functions $f^-, f^+ \in L(I;R_+)$ and $q \in K(I \times R;R_+)$ such that q is non-decreasing in the second argument,*

$$|f(t,x)| \le q(t,x) \quad \text{for} \quad |x| \ge r, \tag{2.5}$$

$$f^-(t) \le f(t,x) \ \text{ for } \ x \le -r, \qquad f(t,x) \le -f^+(t) \ \text{ for } \ x \ge r \tag{2.6}$$

on I, and

$$\lim_{|x|\to+\infty} \frac{1}{x}\int_a^b q(s,x)\,ds = 0. \tag{2.7}$$

Let, moreover, there exist $\varepsilon > 0$ such that

$$\begin{aligned} -\int_a^b f^-(s)|w(s)|\,ds + \varepsilon||\gamma_r||_L &\le \int_a^b h(s)|w(s)|\,ds \le \\ &\le \int_a^b f^+(s)|w(s)|\,ds - \varepsilon||\gamma_r||_L, \end{aligned} \tag{2.4$_2$}$$

where γ_r is the function defined in Theorem 2.1. Then problem (1.1), (1.2) *has at least one solution.*

Corollary 2.3 *Let a nonzero solution w of problem* (1.3), (1.4), *and a constant $r > 0$, be such that conditions* (2.1) *and* (2.2) *hold. Let, moreover, the equality*

$$\liminf_{x\to\pm\infty} f(s,\ x)\mathrm{sgn}x = +\infty \qquad \textit{uniformly on} \quad I \tag{2.8}$$

be fulfilled. Then problem (1.1)*,* (1.2) *has at least one solution for arbitrary* $h \in L(I, R)$.

Corollary 2.4 *Let a nonzero solution* w *of problem* (1.3), (1.4)*, a constant* $r > 0$*, and the non-decreasing in the second argument function* $q \in K(I \times R; R_+)$ *be such that conditions* (2.1)*,* (2.5) *and* (2.7) *hold. Let, moreover, the equality*

$$\liminf_{x \to \pm\infty} f(s,\ x)\mathrm{sgn}x = -\infty \qquad \textit{uniformly on} \quad I \tag{2.9}$$

be fulfilled. Then problem (1.1)*,* (1.2) *has at least one solution for arbitrary* $h \in L(I, R)$.

Example 2.5 It follows from Corollary 2.3 that the equation

$$u''(t) = -u(t) + \sigma|u(t)|^{\alpha}\mathrm{sgn}\, u(t) + h(t) \quad \text{for} \quad a \le t \le b \tag{2.10}$$

where $\sigma = 1,\ \alpha \in]0, 1]$, and $a = 0, b = \pi/2$ with conditions (1.2) has at least one solution for arbitrary $h \in L([0, \pi/2], R)$.

Example 2.6 It follows from Corollary 2.4 that problem (2.10), (1.2) with $\sigma = -1, \alpha \in]0, 1[$ and $a = 0$, $b = \pi/2$ has at least one solution for arbitrary $h \in L([0, \pi/2]; R)$.

Remark 2.7 If $f \not\equiv 0$ the condition (1.4_i) of Theorem 2.i $(i = 1, 2)$ can be replaced by

$$-\int_a^b f^-(s)|w(s)|\,ds < (-1)^i \int_a^b h(s)|w(s)|\,ds < \int_a^b f^+(s)|w(s)|\,ds, \tag{2.11_i}$$

because, from 2.11_i there follows the existence of a constant $\varepsilon > 0$ such that condition (1.4_i) is satisfied.

3 Auxiliary propositions

Let $u_n \in \widetilde{C}'(I; R)$, $\|u_n\|_C \ne 0$ $(n \in N)$, w be an arbitrary solution of problem (1.3), (1.4), and $r > 0$. Then, for every $n \in N$, we define the sets $A_{n,1} \overset{def}{=} \{t \in I : |u_n(t)| \le r\}$, $A_{n,2} \overset{def}{=} \{t \in I : |u_n(t)| > r\}$, for which it is clear that

$$A_{n,1} \cap A_{n,2} = \emptyset, \quad A_{n,1} \cup A_{n,2} = I. \tag{3.1}$$

Lemma 3.1 *Let* $u_n \in \widetilde{C}'(I; R)(n \in N)$, $r > 0$, w *be an arbitrary nonzero solution of problem* (1.3)*,* (1.4)*,* $N_w = \emptyset$*, and*

$$||u_n||_C \ge 2rn \quad \text{for} \quad n \in N, \tag{3.2}$$

$$||v_n - w||_C \le 1/2n \quad \text{for} \quad n \in N, \tag{3.3}$$

where $v_n(t) = u_n(t)||u_n||_C^{-1}$*. Then*

$$\lim_{n\to+\infty} \mathrm{mes}\, A_{n,1} = 0, \qquad \lim_{n\to+\infty} \mathrm{mes}\, A_{n,2} = \mathrm{mes}\, I. \tag{3.4}$$

P r o o f. First we define the set $T_n \overset{def}{=} [a, a + 1/n]$ and show that, for every $n_0 \in N$, there exists $n_1 > n_0$ such that

$$A_{n,1} \subseteq T_{n_0} \quad \text{for} \quad n \ge n_1. \tag{3.5}$$

Suppose on the contrary that, for some $n_0 \in N$, there exists the sequence $t_{n_j} \in A_{n_j,1}(j \in N)$ with $n_j < n_{j+1}$, such that $t_{n_j} \notin T_{n_0}$ for $j \in N$. Without loss of generality we can assume that $\lim_{j\to+\infty} t_{n_j} = t_0$. Then from conditions (3.2), (3.3), the definition of the set $A_{n,1}$ and the equality $w(t_0) = (w(t_0) - w(t_{n_j})) + (w(t_{n_j}) - v_{n_j}(t_{n_j})) + v_{n_j}(t_{n_j})$, we get $w(t_0) = 0$, i.e., $t_0 = a$. But this contradicts the condition $t_{n_j} \notin T_{n_0}$ and thus (3.5) is true. Since $\lim_{n\to+\infty} \mathrm{mes}\, T_n = 0$, it follows from (3.1) and (3.5) that (3.4) is valid. □

Lemma 3.2 *Let all the conditions of Lemma 3.1 be fulfilled and there exist* $r > 0$ *such that the condition*

$$0 \le f_1(t, x)\mathrm{sgn}x \quad \text{for} \quad t \in I,\ |x| \ge r \tag{3.6}$$

holds, where $f_1 \in K(I \times R; R)$. *Then*

$$\lim_{n\to+\infty} \inf \int_s^t f_1(\xi, u_n(\xi))\mathrm{sgn} u_n(\xi)\, d\xi \geq 0 \quad \text{for} \quad a \leq s < t \leq b. \tag{3.7}$$

P r o o f . Let $\gamma_r^*(t) \overset{def}{=} \sup\{|f_1(t,x)| : |x| \leq r\}$ for $t \in I$. Then, according to (3.1) and (3.6), we obtain the estimate

$$\int_s^t f_1(\xi, u_n(\xi))\mathrm{sgn} u_n(\xi)\, d\xi \geq -\int_{[s,t]\cap A_{n,1}} \gamma_r^*(\xi)\, d\xi + \int_{[s,t]\cap A_{n,2}} \left| f_1(\xi, u_n(\xi))\right| d\xi$$

for $a \leq s < t \leq b$, $n \in N$. This estimate and (3.4) imply (3.7). □

Lemma 3.3 *Let* $r > 0$, *the functions* $f_1 \in K(I \times R; R)$, $h_1 \in L(I; R)$, $f^+, f^- \in L(I; R_+)$ *be such that*

$$f_1(t,x) \leq -f^-(t) \text{ for } x \leq -r, \qquad f^+(t) \leq f_1(t,x) \text{ for } x \geq r \tag{3.8}$$

on I, *and there exist* $\varepsilon > 0$, *and nonzero solution* w_0 *of problem* (1.3), (1.4), *such that* $N_{w_0} = \emptyset$ *and*

$$\begin{aligned} -\int_a^b f^-(s)|w_0(s)|ds + \varepsilon||\gamma_r^*||_L &\leq -\int_a^b h_1(s)|w_0(s)|\, ds \\ &\leq \int_a^b f^+(s)|w_0(s)|ds - \varepsilon||\gamma_r^*||_L, \end{aligned} \tag{3.9}$$

where γ_r^* *is the function defined in the proof of Lemma 3.2. Then, for every nonzero solution* w *of problem* (1.3), (1.4), *and functions* $u_n \in \widetilde{C}'(I; R)(n \in N)$ *such that conditions* (3.2),

$$\left| v_n^{(i)}(t) - w^{(i)}(t)\right| \leq 1/2n \quad \text{for} \quad t \in I, n \in N, (i = 0, 1) \tag{3.10}$$

where $v_n(t) = u_n(t)||u_n||_C^{-1}$ *for* $t \in I$ *and*

$$u_n(a) = 0, u_n'(b) = 0 \tag{3.11}$$

are fulfilled, there exists $n_1 \in N$ *such that*

$$\mathbf{M}_n(w) \overset{def}{\equiv} \int_a^b (h_1(s) + f_1(s, u_n(s)))w(s)\, ds \geq 0 \quad \text{for} \quad n \geq n_1. \tag{3.12}$$

P r o o f . First note that, for any nonzero solution w of problem (1.3), (1.4), there exists $\beta \neq 0$ such that $w(t) = \beta w_0(t)$. Also, it is not difficult to verify that all the assumptions of Lemma 3.1 are satisfied for the function $w(t) = \beta w_0(t)$. From the unique solvability of the Cauchy problem for Equation (1.3) and conditions (1.4) we conclude that $w'(a) \neq 0$ and $w(b) \neq 0$. Therefore, in view of (3.10) and (3.11), there exists $n_2 \in N$ such that

$$u_n(t)\, \mathrm{sgn}\, \beta\, w_0(t) > 0 \qquad \text{for} \quad n \geq n_2, a < t \leq b. \tag{3.13}$$

Moreover, if $\sigma = \mathrm{sgn}\beta$, by (3.1) we get the estimate

$$\begin{aligned} \frac{\mathbf{M}_n(w)}{|\beta|} \geq &-\int_{A_{n,1}} \gamma_r^*(s)|w_0(s)|\, ds + \sigma \int_a^b h_1(s)w_0(s)\, ds \\ &+ \sigma \int_{A_{n,2}} f_1(s, u_n(s))w_0(s)\, ds. \end{aligned} \tag{3.14}$$

Now note that $f^- \equiv 0$, $f^+ \equiv 0$ if $f_1(t,x) \equiv 0$. Then by virtue of (3.4), we see that there exist $\varepsilon > 0$ and $n_1 \in N(n_1 \geq n_2)$ such that $\int_a^b f^{\pm}(s)|w_0(s)|\, ds - \frac{\varepsilon}{2}||\gamma_r^*||_L \leq \int_{A_{n,2}} f^{\pm}(s)|w_0(s)|\, ds$ and $\frac{\varepsilon}{2}||\gamma_r^*||_L \geq \int_{A_{n,1}} \gamma_r^*(s)|w_0(s)|\, ds$ for $n \geq n_1$. By these inequalities, (3.2), (3.8) and (3.13), from (3.14) we obtain

$$\frac{\mathbf{M}_n(w)}{|\beta|} \geq -\varepsilon||\gamma_r^*||_L + \int_a^b h_1(s)|w_0(s)|\, ds + \int_a^b f^+(s)|w_0(s)|\, ds$$

if $n \geq n_1,\ \sigma w_0(t) \geq 0$, and

$$\frac{\mathbf{M}_n(w)}{|\beta|} \geq -\varepsilon||\gamma_r^*||_L - \int_a^b h_1(s)|w_0(s)|\,ds + \int_a^b f^-(s)|w_0(s)|\,ds$$

if $n \geq n_1,\ \sigma w_0(t) \leq 0$. From the last two estimates in view of (3.9) it follows that (3.12) is valid. □

Now we consider the definitions of the set $V_{2\,0}((a,b))$ introduced and described in [14] (see [Definition 1.3, p. 2350]).

Definition 3.4 We say that the function $p \in L([a,b])$ belongs to the set $V_{2\,0}((a,b))$ if for any function p^* satisfying the inequality $p^*(t) \geq p(t)$ for $t \in I$ the unique solution of the initial value problem

$$v''(t) = p^*(t)v(t) \quad \text{for} \quad t \in I, \quad v(a) = 0, v'(a) = 1, \tag{3.15}$$

has no zeros in the interval $]a,b[$ and $v'(b) > 0$.

Lemma 3.5 *Let w be a solution of the problem* (1.3), (1.4)*, and $N_w = \emptyset$. Then for every $n \in N$*

$$p + 1/n \in V_{2\,0}((a,b)). \tag{3.16}$$

P r o o f . Assume that there exists $n_0 \in N$ such that $p + 1/n_0 \notin V_{2\,0}((a,b))$. Then in view of Definition 3.4 there exists $p^*(t) \geq p(t) + 1/n_0$ such that the solution v of problem (3.15) has a zero point $t^* \in]a,\ b]$. Then from the Sturm's comparision theorem it follows the existence of $t_0 \in]a, t^*[$ such that $w(t_0) = 0$, which contradicts the condition $N_w = \emptyset$, i.e., our assumption is invalid and $p + 1/n \in V_{2\,0}((a,b))$ for every $n \in N$. □

Lemma 3.6 *Let problem* (1.3)*,* (1.4) *has the nontrivial solution. Then there exists $\varepsilon > 0$ such that the equation*

$$w''(t) = \lambda p(t)w \quad \text{for} \quad t \in I, \tag{3.17}$$

under boundari conditions (1.4) *has only the trivial solution if $\lambda \in]1, 1+\varepsilon]$.*

P r o o f . Let G be the Green's function of the boundary value problem $u''(t) = 0,\quad u(a) = 0,\ u'(b) = 0$, then problem (3.17), (1.4) is equivalent to the equation $w(t) = \lambda\Gamma(w)(t)$, where the operator $\Gamma : C(I;R) \to C(I;R)$ by the equality $\Gamma(x)(t) = \int_a^b G(t,s)p(s)x(s)\,ds$ is defined. As it is well-known $\Gamma : C(I;R) \to C(I;R)$ is the compact operator, and then for every $r > 0$ the disc $|\lambda| \leq r$, contains at most finite number of characteristic values [see. [10], Capitel XIII, §3, Theorem 1]. From this fact the existence of $\varepsilon > 0$ such that the set $]1, 1+\varepsilon]$ does not contain the characteristic values of the equation $w(t) = \lambda\Gamma(w)(t)$, it follows. Consequently this equation, i.e., problem (3.17), (1.4) has only the trivial solution if $\lambda \in]1, 1+\varepsilon]$. □

4 Proof of the main results

P r o o f o f T h e o r e m 2.1 Let $p_n(t) = p(t) + 1/n$ and, for any $n \in N$, consider the equation

$$u_n''(t) = p_n(t)u_n(t) + f(t, u_n(t)) + h(t) \quad \text{for} \quad t \in I, \tag{4.1}$$

under boundary conditions (3.11). In view of condition (2.1) and Lemma 3.5, the inclusion (3.16) holds for every $n \in N$. On the other hand, from conditions (2.2) and (2.3) we find

$$0 \leq f(t,x)\,\mathrm{sgn}\,x \leq g(t)|x| + h_0(t) \quad \text{for } t \in I,\ |x| \geq r. \tag{4.2}$$

Then the inclusion (3.16), as is well-known (see [14, Theorem 2.2, p. 2367]), guarantees that problem (4.1), (3.11) has at least one solution, suppose u_n. In view of condition (2.2), without loss of generality we can assume that there exists $\varepsilon^* > 0$ such that $h_0(t) \geq \varepsilon^*$ on I. Then it is not difficult to verify that u_n is also a solution of the equation

$$u_n''(t) = (p_n(t) + p_0(t, u_n(t))\,\mathrm{sgn}\,u_n(t))u_n(t) + p_1(t, u_n(t)) \tag{4.3}$$

with $p_0(t,x) = \frac{f(t,x)g(t)}{g(t)|x|+h_0(t)}$, $\quad p_1(t,x) = h(t) + \frac{f(t,x)h_0(t)}{g(t)|x|+h_0(t)}$. Now assume that

$$\lim_{n\to+\infty} ||u_n||_C = +\infty \tag{4.4}$$

and $v_n(t) = u_n(t)||u_n||_C^{-1}$. Then

$$v_n''(t) = (p_n(t) + p_0(t,u_n(t)) \operatorname{sgn} u_n(t))v_n(t) + ||u_n||_C^{-1} p_1(t,u_n(t)), \tag{4.5}$$

$$v_n(a) = 0, \qquad v_n'(b) = 0, \tag{4.6}$$

and

$$||v_n||_C = 1 \tag{4.7}$$

for any $n \in N$. In view of the condition (4.2), the functions $p_0,\, p_1 \in K(I\times R; R)$ are bounded respectively by the functions $g(t)$ and $h(t) + h_0(t)$. Therefore, from (4.5), by virtue of (4.4), (4.6) and (4.7), we see that there exists $r_0 > 0$ such that $||v_n'||_C \le r_0$. Consequently in view of (4.7), by the Arzela–Ascoli lemma, without loss of generality we can assume that there exists $w \in \widetilde{C}'(I,R)$ such that $\lim_{n\to+\infty} v_n^{(i)}(t) = w^{(i)}(t)(i = 0,1)$ uniformly on I. From the last equality and (4.4) there follows the existence of an increasing sequence $\{\alpha_k\}_{k=1}^{+\infty}$ of a natural numbers, such that $||u_{\alpha_k}||_C \ge 2rk$ and $\big|\big|v_{\alpha_k}^{(i)} - w^{(i)}\big|\big|_C \le 1/2k$ for $k \in N$. Without loss of generality we can suppose that $u_n \equiv u_{\alpha_n}$ and $v_n \equiv v_{\alpha_n}$. In this case we see that u_n and v_n are the solutions of problems (4.1), (3.11) and (4.5), (4.6) respectively with $p_n(t) = p(t) + 1/\alpha_n$ for $t \in I,\ n \in N$, and that the inequalities

$$||u_n||_C \ge 2rn, \quad \big|\big|v_n^{(i)} - w^{(i)}\big|\big|_C \le 1/2n \quad \text{for} \quad n \in N \tag{4.8}$$

are fulfilled. Analogously, since the functions $p_0,\, p_1 \in K(I\times R; R)$ are bounded, in view of (4.4), we can assume without loss of generality that there exists a function $\widetilde{p} \in L(I;R)$ such that

$$\lim_{n\to+\infty} ||u_n||_C^{-j} \int_a^t p_j(s,u_n(s)) \operatorname{sgn} u_n(s)\, ds = (1-j)\int_a^t \widetilde{p}(s)ds \tag{4.9$_j$}$$

uniformly on I for $j = 0, 1$. By virtue of (4.7)–4.9$_j$ $(j = 0,1)$, from (4.5) we obtain

$$w''(t) = (p(t) + \widetilde{p}(t))w(t), \qquad w(a) = 0, \quad w'(b) = 0, \tag{4.10}$$

and

$$||w||_C = 1. \tag{4.11}$$

From conditions (2.3) and (4.8) it is clear that all the assumptions of Lemma 3.2 with $f_1(t,x) = f(t,x)$ are satisfied, and thus we obtain from (4.9) $(j = 0)$ the relation $\int_s^t \widetilde{p}(\xi)\,d\xi \ge 0$ for $a \le s < t \le b$, i.e., $\widetilde{p}(t) \ge 0$ on I. Now assume that $\widetilde{p} \not\equiv 0$ and w_0 is a solution of problem (1.3), (1.4). Then using Sturm's comparison theorem for Equations (1.3) and (4.10), from the inequality $\widetilde{p}(t) \ge 0$ we see that there exists a point $t_0 \in\,]a,b[$ such that $w_0(t_0) = 0$, which contradicts the condition $N_w = \emptyset$. This contradiction proves that $\widetilde{p} \equiv 0$ and w is a solution of problem (1.3), (1.4). Multiplying Equations (4.1) and (1.3) respectively by w and $-u_n$, and therefore integrating their sum from a to b, in view of conditions (3.11) and (1.4), we obtain

$$-\frac{||u_n||_C}{\alpha_n}\int_a^b w(s)v_n(s)\,ds = \int_a^b (h(s) + f(s,u_n(s)))w(s)\,ds \tag{4.12}$$

for $n \ge n_0$. Therefore by virtue of (4.8) we get

$$\int_a^b (h(s) + f(s,u_n(s)))w(s)\,ds < 0 \quad \text{for} \quad n \ge n_0. \tag{4.13}$$

On the other hand, in view conditions (2.1)–(2.4$_1$), (3.11), and (4.8) it is clear that all the assumption of Lemma 3.3 with $f_1(t,x) = f(t,x)$, $h_1(t) = h(t)$ are fulfilled. Therefore, inequality (3.12) is true, which contradicts (4.13). This contradiction proves that (4.4) does not hold and thus there exists $r_1 > 0$ such that $||u_n||_C \le r_1$ for $n \in N$. Consequently, from (4.1) and (3.11) it is clear that there exists $r_1' > 0$ such that $||u_n'||_C \le r_1'$ and $|u_n''(t)| \le \sigma(t)$ for $t \in I, n \in N$, where $\sigma(t) = (1 + |p(t)|)r_1 + |h(t)| + \gamma_{r_1}(t)$. Hence, by the Arzela–Ascoli lemma, without loss of generality we can assume that there exists a function $u_0 \in \widetilde{C}'(I;R)$ such that $\lim_{n\to+\infty} u_n^{(i)}(t) = u_0^{(i)}(t)(i = 0, 1)$ uniformly on I. Therefore, it follows from (4.1) and (3.11) that u_0 is a solution of problem (1.1), (1.2). □

Proof of Corollary 2.3 Let $h \in L(I,]0, +\infty[)$ is an arbytrary function and $\beta \stackrel{def}{=} -\int_a^b h(s)|w(s)|\,ds$. On the other hand, from condition (2.8) it follows that for our β there exists numbers $r > 0$ such that $f(t, x)\operatorname{sgn} x > (|\beta| + 1)||w||_L^{-1}$ almost everywhere on I for $|x| > r$. Consequently all the conditions of Theorem 2.1 except (2.4_1), are fulfilled with $f^-(t) = f^+(t) = (|\beta| + 1)||w||_L^{-1}$, and it is easy to verify that instead (2.4_1) condition (2.11_1) holds. Then from Remark 2.7 it follows validity of our corollary. □

Proof of Theorem 2.2 Let $p_n(t) = (1 + 1/n)p(t)$ for any $n \in N$, and $n_0 \in N$ be such that $\lambda \stackrel{def}{=} 1 + 1/n \in]1, 1 + \varepsilon]$ for $n \geq n_0$, where ε is the number defined in Lemma 3.6. Now consider the problem (4.1), (3.11), and the corresponding homogeneous problem

$$w''(t) = p_n(t)w(t), \quad w(a) = 0, \ w'(b) = 0. \tag{4.14}$$

In view of Lemma 3.6, problem (4.14) has only the zero solution for every $n \geq n_0$. Therefore, as is well-known (see [13, Corollary 2.1, p. 2271]), from conditions (2.5), (2.7) it follows that problem (4.1), (3.11) has at least one solution, suppose u_n. Assume that (4.4) holds and put $v_n(t) = u_n(t)||u_n||_C^{-1}$. Then conditions (4.6) and (4.7) are fulfilled, and

$$v_n''(t) = p_n(t)v_n(t) + ||u_n||_C^{-1}(f(t, u_n(t))) + h(t)). \tag{4.15}$$

In view conditions (2.5) and (2.7), from (4.15) there follows the existence of $r_0 > 0$ such that $||v_n'||_C \leq r_0$. Consequently, in view (4.7) by the Arzela–Ascoli lemma, without loss of generality we can assume that there exists a function $w \in \widetilde{C}'(I, R)$ such that $\lim_{n\to+\infty} v_n^{(i)}(t) = w^{(i)}(t)(i = 0, 1)$ uniformly on I. Analogously as in the proof of Theorem 2.1, we can find an increasing sequence $\{\alpha_k\}_{n=1}^{+\infty}$ of natural numbers such that, if we suppose $u_n = u_{\alpha_n}$ then the conditions (4.8) will by true when the functions u_n and v_n are the solutions of problems (4.1), (3.11) and (4.15), (4.6) respectively with $p_n(t) = (1 + 1/\alpha_n)p(t)$ for $t \in I, \ n \in N$. From (4.15), by virtue of (4.6), (4.8) and (2.7), we obtain that w is a solution of problem (1.3), (1.4) and $||w||_C = 1$. In a similar manner as condition (4.12) in the proof of Theorem 2.1, we show that

$$\int_a^b (h(s) + f(s, u_n(s)))w(s)\,ds = -\frac{||u_n||_C}{\alpha_n}\int_a^b p(s)w(s)v_n(s)\,ds$$

for $n \geq n_0$. On the other hand by (4.8), (1.3), and (1.4) we get

$$\lim_{n\to+\infty}\int_a^b p(s)w(s)v_n(s)\,ds = \int_a^b p(s)w^2(s)\,ds = -\int_a^b w'^2(s)\,ds < 0,$$

and then $\int_a^b (h(s) + f(s, u_n(s)))w(s)\,ds > 0$. Now note that, in view of conditions (2.1), (2.6), (2.4_2), (3.11), and (4.8), all the assumptions of Lemma 3.3 with $f_1(t, x) = -f(t, x), h_1(t) = -h(t)$ are satisfied. Hence, analogously as in the proof of Theorem 2.1, from the last inequality follows the solvability of problem (1.1), (1.2). □

Proof of Corollary 2.4 Is similar to proof of Corollary 2.3. □

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