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The focal boundary value problem for strongly singular higher-order nonlinear functional-differential equations

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Abstract

The *a priori* boundedness principle is proved for the two-point right-focal boundary value problems for strongly singular higher-order nonlinear functional-differential equations. Several sufficient conditions of solvability of the two-point right-focal problem under consideration are derived from the *a priori* boundedness principle. The proof of the *a priori* boundedness principle is based on Agarwal-Kiguradze type theorems, which guarantee the existence of the Fredholm property for strongly singular higher-order linear differential equations with argument deviations under the two-point right-focal boundary conditions.

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1 Statement of the main results

1.1 Statement of the problem and the literature survey

Consider the functional differential equation

$$u^{(n)}(t) = F(u)(t) \tag{1.1}$$

with the two-point boundary conditions

$$u^{(i-1)}(a) = 0 \quad (i = 1, \dots, m), \qquad u^{(j-1)}(b) = 0 \quad (j = m+1, \dots, n). \tag{1.2}$$

Here $n \geq 2$, m is the integer part of $n/2$, $-\infty < a < b < +\infty$, and the operator F acts from the set of $(m-1)$th time continuously differentiable on $]a,b]$ functions to the set $L_{loc}(]a,b])$. By $u^{(i-1)}(a)$ we denote the right limit of the function $u^{(i-1)}$ at the point a.

The problem is singular in the sense that for an arbitrary $u \in C^{m-1}(]a,b])$ the right-hand side of equation (1.1) may have nonintegrable singularities at the point a. Throughout the paper we use the following notations:

$R^+ = [0, +\infty[$; $[x]_+$ the positive part of number x, that is, $[x]_+ = \frac{x+|x|}{2}$;

$L_{loc}(]a,b])$ is the space of functions $y :]a,b] \to R$, which are integrable on $[a+\varepsilon, b]$ for arbitrarily small $\varepsilon > 0$;

$L_\alpha(]a,b])$ ($L^2_\alpha(]a,b])$) is the space of integrable (square integrable) with the weight $(t-a)^\alpha$ functions $y:]a,b]\to R$ with the norm

$$\|y\|_{L_\alpha}=\int_a^b (s-a)^\alpha\big|y(s)\big|\,ds \qquad \biggl(\|y\|_{L^2_\alpha}=\biggl(\int_a^b (s-a)^\alpha y^2(s)\,ds\biggr)^{1/2}\biggr);$$

$L([a,b])=L_0(]a,b])$, $L^2([a,b])=L^2_0(]a,b])$;

$M(]a,b])$ is the set of measurable functions $\tau:]a,b]\to]a,b]$;

$\widetilde{L}^2_\alpha(]a,b])$ is the Banach space of $y\in L_{\mathrm{loc}}(]a,b])$ functions with the norm

$$\|y\|_{\widetilde{L}^2_\alpha}\equiv\max\biggl\{\biggl[\int_a^t (s-a)^\alpha\biggl(\int_s^t y(\xi)\,d\xi\biggr)^2 ds\biggr]^{1/2}: a\le t\le b\biggr\};$$

$L_n(]a,b])$ is the Banach space of $y\in L_{\mathrm{loc}}(]a,b])$ functions with the norm

$$\|y\|_{L_n}=\sup\biggl\{(s-a)^{m-1/2}\int_s^t (\xi-a)^{n-2m}\big|y(\xi)\big|\,d\xi: a<s\le t\le b\biggr\}<+\infty;$$

$\widetilde{C}^{n-1}_{\mathrm{loc}}(]a,b])$ is the space of functions $y:]a,b]\to R$, which are continuous (absolutely continuous) together with $y',y'',\dots,y^{(n-1)}$ on $[a+\varepsilon,b]$ for arbitrarily small $\varepsilon>0$;

$\widetilde{C}^{n-1,m}(]a,b])$ is the space of functions $y\in\widetilde{C}^{n-1}_{\mathrm{loc}}(]a,b])$ such that

$$\int_a^b \big|x^{(m)}(s)\big|^2\,ds<+\infty; \tag{1.3}$$

$C^{m-1}_1(]a,b])$ is the Banach space of functions $y\in C^{m-1}_{\mathrm{loc}}(]a,b])$ such that

$$\limsup_{t\to a}\frac{|x^{(i-1)}(t)|}{(t-a)^{m-i+1/2}}<+\infty \quad (i=1,\dots,m) \tag{1.4}$$

with the norm $\|x\|_{C^{m-1}_1}=\sum_{i=1}^m\sup\{\frac{|x^{(i-1)}(t)|}{(t-a)^{m-i+1/2}}:a<t\le b\}$;

$\widetilde{C}^{m-1}_1(]a,b])$ is the Banach space of functions $y\in\widetilde{C}^{n-1}_{\mathrm{loc}}(]a,b])$ such that conditions (1.3) and (1.4) hold with the norm $\|x\|_{\widetilde{C}^{m-1}_1}=\|x\|_{C^{m-1}_1}+(\int_a^b|x^{(m)}(s)|^2\,ds)^{1/2}$;

$D_n(]a,b]\times R^+)$ is the set of such functions $\delta:]a,b]\times R^+\to L_n(]a,b])$ that $\delta(t,\cdot):R^+\to R^+$ is nondecreasing for every $t\in]a,b]$, and $\delta(\cdot,\rho)\in L_n(]a,b])$ for any $\rho\in R^+$.

A solution of problem (1.1), (1.2) is sought in the space $\widetilde{C}^{n-1,m}(]a,b])$.

The principles of the theory of singular boundary value problems were built by Kiguradze in his study [1]. This theory has been intensively developed and studied with sufficient completeness both for the ordinary differential equations and the functional differential equations (see [2–28]).

But equation (1.1), even under the boundary condition (1.2), is not studied in the case when the operator F has the form $F(x)(t)=\sum_{j=1}^m p_j(t)x^{(j-1)}(\tau_j(t))+q(x)(t)$, where the singularities of the functions $p_j:L_{\mathrm{loc}}(]a,b])$ $(j=2,\dots,m)$ are such that the inequalities

$$\int_a^b (s-a)^{n-1}\big[(-1)^{n-m}p_1(s)\big]_+\,ds<+\infty, \qquad \int_a^b (s-a)^{n-j}\big|p_j(s)\big|\,ds<+\infty \tag{1.5}$$

are not fulfilled (in this case we say that the linear part of the operator F is strongly singular), the operator q continuously acts from $C^{m-1}_1(]a,b])$ to $L_{\widetilde{L}^2_{2n-2m-2}}(]a,b])$, and the inclusion

$$\sup\{q(x)(t) : \|x\|_{C_1^{m-1}} \le \rho\} \in \widetilde{L}^2_{2n-2m-2}(]a,b])$$

holds. The first step in studying the differential equations with strong singularities was made by Agarwal and Kiguradze in the article [29], where the linear ordinary differential equations under conditions (1.2), in the case when the functions p_j have strong singularities at the points a and b, are studied. Also the ordinary differential equations with strong singularities under two-point boundary conditions are studied in the articles [30, 31] by Kiguradze. In the papers [32–34] these results are generalized for a linear differential equation with deviating arguments, *i.e.*, the Agarwal-Kiguradze type theorems are proved, which guarantee the Fredholm property for the linear differential equation with deviating arguments. In this paper, on the basis of articles [33, 34], we prove the *a priori* boundedness principle for problem (1.1), (1.2) from which several sufficient conditions of the solvability of this problem follow.

Now we introduce some results from the articles [33, 34] in this section, which we need for this work. Consider the equation

$$u^{(n)}(t) = \sum_{j=1}^{m} p_j(t)u^{(j-1)}(\tau_j(t)) + q(t) \quad \text{for } a < t < b \tag{1.6}$$

with $q, p_j \in L_{\mathrm{loc}}(]a,b])$.

By $h_j :]a,b] \times]a,b] \to R_+$ and $f_j : [a,b] \times M(]a,b]) \to C_{\mathrm{loc}}(]a,b] \times]a,b])$ $(j = 1,\dots,m)$ we denote the functions and the operator, respectively, defined by the equalities

$$\begin{aligned} h_1(t,s) &= \left| \int_s^t (\xi - a)^{n-2m} \left[(-1)^{n-m} p_1(\xi)\right]_+ d\xi \right|, \\ h_j(t,s) &= \left| \int_s^t (\xi - a)^{n-2m} p_j(\xi)\, d\xi \right|, \end{aligned} \tag{1.7}$$

and

$$f_j(c,\tau_j)(t,s) = \left| \int_s^t (\xi - a)^{n-2m} |p_j(\xi)| \left| \int_\xi^{\tau_j(\xi)} (\xi_1 - c)^{2(m-j)}\, d\xi_1 \right|^{1/2} d\xi \right|. \tag{1.8}$$

Let also $k = 2k_1 + 1$ $(k_1 \in Z)$, then

$$k!! = \begin{cases} 1 & \text{for } k \le 0, \\ 1 \cdot 3 \cdot 5 \cdots k & \text{for } k \ge 1. \end{cases}$$

Now we can introduce the main theorem of the papers [33] and [34].

Theorem 1.1 *Let there exist the numbers* $\ell_j > 0$, $\overline{\ell}_j \ge 0$, *and* $\gamma_j > 0$ $(j = 1,\dots,m)$ *such that along with*

$$B \equiv \sum_{j=1}^{m} \left(\frac{(2m-j)2^{2m-j+1}\ell_j}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(b-a)^{\gamma_j}\overline{\ell}_j}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_j}} \right) < 1, \tag{1.9}$$

the conditions

$$(t-a)^{2m-j}h_j(t,s) \le \ell_j, \qquad (t-a)^{m-\gamma_{0j}-1/2}f_j(a,\tau_j)(t,s) \le \overline{\ell}_j \tag{1.10}$$

hold for $a < t \le s \le b$. *Then problem* (1.6), (1.2) *is uniquely solvable in the space* $\widetilde{C}^{n-1,m}(]a,b])$.

Remark 1.1 From Lemma 2.5 it is clear that any solution of problem (1.6), (1.2) from the space $\widetilde{C}^{n-1,m}(]a,b])$ belongs also to the space $\widetilde{C}_1^{m-1}(]a,b])$.

Theorem 1.2 *Let all the conditions of Theorem* 1.1 *be satisfied. Then the unique solution* u *of problem* (1.6), (1.2) *for every* $q \in \widetilde{L}^2_{2n-2m-2}(]a,b])$ *admits the estimate*

$$\|u^{(m)}\|_{L^2} \le r\|q\|_{\widetilde{L}^2_{2n-2m-2}}, \tag{1.11}$$

with

$$r = \frac{2^{m-1}(2n-2m-1)}{(\nu_n - B)(2m-1)!!}, \qquad \nu_{2m} = 1, \qquad \nu_{2m+1} = \frac{2m+1}{2},$$

and thus constant $r > 0$ *depends only on the numbers* ℓ_j, $\overline{\ell}_j$, γ_j $(j = 1, \dots, m)$, *and* a, b, n.

Remark 1.2 Under the conditions of Theorem 1.2, for every $q \in \widetilde{L}^2_{2n-2m-2}(]a,b])$, the unique solution u of problem (1.6), (1.2) admits the estimate

$$\|u\|_{\widetilde{C}_1^{m-1}} \le r_n\|q\|_{\widetilde{L}^2_{2n-2m-2}}, \tag{1.12}$$

with $r_n = (1 + \sum_{j=1}^m \frac{(2m-2j+1)^{-1/2}}{(m-j)!})\frac{2^{m-1}(2n-2m-1)}{(\nu_n-B)(2m-1)!!}$.

1.2 Theorems on the solvability of problem (1.1), (1.2)

Define the operator $P : C_1^{m-1}(]a,b]) \times C_1^{m-1}(]a,b]) \to L_{\mathrm{loc}}(]a,b])$ by the equality

$$P(x,y)(t) = \sum_{j=1}^{m} p_j(x)(t)y^{(j-1)}\bigl(\tau_j(t)\bigr) \quad \text{for } a < t \le b, \tag{1.13}$$

where $p_j : C_1^{m-1}(]a,b]) \to L_{\mathrm{loc}}(]a,b])$ and $\tau_j \in M(]a,b])$. Also, for any $\gamma > 0$, define the set A_γ by the relation

$$A_\gamma = \bigl\{x \in \widetilde{C}_1^{m-1}\bigl(]a,b]\bigr) : \|x\|_{\widetilde{C}_1^{m-1}} \le \gamma\bigr\}. \tag{1.14}$$

Now, following the article [6] by Kiguradze and Půža, we introduce the following definitions.

Definition 1.1 Let γ_0 and γ be positive numbers. We say that the continuous operator $P : C_1^{m-1}(]a,b]) \times C_1^{m-1}(]a,b]) \to L_n(]a,b])$ is γ_0, γ consistent with boundary condition (1.2) if:

(i) for any $x \in A_{\gamma_0}$ and almost all $t \in]a,b]$, the inequality

$$\sum_{j=1}^{m} \left| p_j(x)(t)x^{(j-1)}\left(\tau_j(t)\right)\right| \le \delta\left(t, \|x\|_{\widetilde{C}_1^{m-1}}\right)\|x\|_{\widetilde{C}_1^{m-1}} \tag{1.15}$$

holds, where $\delta \in D_n(]a,b] \times R^+)$;

(ii) for any $x \in A_{\gamma_0}$ and $q \in \widetilde{L}^2_{2n-2m-2}(]a,b])$, the equation

$$y^{(n)}(t) = \sum_{j=1}^{m} p_j(x)(t)y^{(j-1)}\left(\tau_j(t)\right) + q(t) \tag{1.16}$$

under boundary conditions (1.2) has the unique solution y in the space $\widetilde{C}^{n-1,m}(]a,b])$ and

$$\|y\|_{\widetilde{C}_1^{m-1}} \le \gamma \|q\|_{\widetilde{L}^2_{2n-2m-2}}. \tag{1.17}$$

Definition 1.2 We say that the operator P is γ consistent with boundary condition (1.2) if the operator P is γ_0, γ consistent with boundary condition (1.2) for any $\gamma_0 > 0$.

In the sequel it will always be assumed that the operator F_p is defined by the equality

$$F_p(x)(t) = \left| F(x)(t) - \sum_{j=1}^{m} p_j(x)(t)x^{(j-1)}\left(\tau_j(t)\right)(t)\right|,$$

continuously acting from $C_1^{m-1}(]a,b])$ to $L_{\widetilde{L}^2_{2n-2m-2}}(]a,b])$, and

$$\widetilde{F}_p(t,\rho) \equiv \sup\left\{F_p(x)(t) : \|x\|_{C_1^{m-1}} \le \rho\right\} \in \widetilde{L}^2_{2n-2m-2}\left(]a,b]\right) \tag{1.18}$$

for each $\rho \in [0,+\infty[$. Then the following theorem is valid.

Theorem 1.3 *Let the operator P be γ_0, γ consistent with boundary condition* (1.2), *and let there exist a positive number $\rho_0 \le \gamma_0$ such that*

$$\left\|\widetilde{F}_p\left(\cdot, \min\{2\rho_0, \gamma_0\}\right)\right\|_{\widetilde{L}^2_{2n-2m-2}} \le \frac{\gamma_0}{\gamma}. \tag{1.19}$$

Let, moreover, for any $\lambda \in]0,1[$, an arbitrary solution $x \in A_{\gamma_0}$ of the equation

$$x^{(n)}(t) = (1-\lambda)P(x,x)(t) + \lambda F(x)(t) \tag{1.20}$$

under conditions (1.2) *admit the estimate*

$$\|x\|_{\widetilde{C}_1^{m-1}} \le \rho_0. \tag{1.21}$$

Then problem (1.1), (1.2) *is solvable in the space* $\widetilde{C}^{n-1,m}(]a,b])$.

From Theorem 1.3 with $\rho_0 = \gamma_0$, the corollary immediately follows.

Corollary 1.1 *Let the operator P be* γ_0, γ *consistent with boundary condition* (1.2), *and*

$$\left| F(x)(t) - \sum_{j=1}^{m} p_j(x)(t)x^{(j-1)}\big(\tau_j(t)\big)(t) \right| \le \eta\big(t, \|x\|_{\widetilde{C}_1^{m-1}}\big) \tag{1.22}$$

for $x \in A_{\gamma_0}$ *and almost all* $t \in]a,b]$, *and*

$$\big\|\eta(\cdot,\gamma_0)\big\|_{\widetilde{L}^2_{2n-2m-2}} \le \frac{\gamma_0}{\gamma}, \tag{1.23}$$

where $\eta \in D_{2n-2m-2}(]a,b] \times R^+)$. *Then problem* (1.1), (1.2) *is solvable in the space* $\widetilde{C}^{n-1,m}(]a,b])$.

Corollary 1.2 *Let the operator P be* γ *consistent with boundary condition* (1.2), *let inequality* (1.22) *hold for* $x \in \widetilde{C}_1^{m-1}(]a,b])$ *and almost all* $t \in]a,b]$, *where* $\eta(\cdot,\rho) \in \widetilde{L}^2_{2n-2m-2}(]a,b])$ *for any* $\rho \in R^+$, *and*

$$\limsup_{\rho\to+\infty} \frac{1}{\rho}\big\|\eta(\cdot,\rho)\big\|_{\widetilde{L}^2_{2n-2m-2}} < \frac{1}{\gamma}. \tag{1.24}$$

Then problem (1.1), (1.2) *is solvable in the space* $\widetilde{C}^{n-1,m}(]a,b])$.

Now define the operators $h_j : C_1^{m-1}(]a,b]) \times]a,b] \times]a,b] \to L_{\mathrm{loc}}(]a,b] \times]a,b])$, $f_j : C_1^{m-1}(]a,b]) \times [a,b] \times M(]a,b]) \to C_{\mathrm{loc}}(]a,b] \times]a,b])$ $(j = 1,\dots,m)$ by the equalities

$$\begin{aligned} h_1(x,t,s) &= \left| \int_s^t (\xi-a)^{n-2m}\big[(-1)^{n-m}p_1(x)(\xi)\big]_+ d\xi \right|, \\ h_j(x,t,s) &= \left| \int_s^t (\xi-a)^{n-2m}p_j(x)(\xi)\,d\xi \right| \quad (j=2,\dots,m), \end{aligned} \tag{1.25}$$

$$f_j(x,c,\tau_j)(t,s) = \left| \int_s^t (\xi-a)^{n-2m}\big|p_j(x)(\xi)\big| \left| \int_\xi^{\tau_j(\xi)} (\xi_1-c)^{2(m-j)}\,d\xi_1 \right|^{1/2} d\xi \right| \tag{1.26}$$

and the functions $\alpha_j : [a,b] \to R_+$ by the equality $\alpha_j(t) = (t-a)^{m-j+1/2}$.

Theorem 1.4 *Let the continuous operator* $P : C_1^{m-1}(]a,b]) \times C_1^{m-1}(]a,b]) \to L_n(]a,b])$ *admit condition* (1.15) *where* $\delta \in D_n(]a,b] \times R^+)$, $\tau_j \in M(]a,b])$, *and let the numbers* $\gamma_0 \in]a,b]$, $l_j > 0$, $\bar{l}_j > 0$, $\gamma_j > 0$ $(j = 1,\dots,m)$ *be such that the inequalities*

$$(t-a)^{2m-j}h_j(x,t,s) \le l_j, \qquad \limsup_{t\to a}(t-a)^{m-\frac{1}{2}-\gamma_j}f_j(x,a,\tau_j)(t,s) \le \bar{l}_j \tag{1.27}$$

for $a < t \le s \le b$, $\|x\|_{\widetilde{C}_1^{m-1}} \le \gamma_0$, *and conditions* (1.9) *hold. Let, moreover, the operator F and the function* $\eta \in D_{2n-2m-2}(]a,b] \times R^+)$ *be such that condition* (1.22) *and the inequality*

$$\big\|\eta(\cdot,\gamma_0)\big\|_{\widetilde{L}^2_{2n-2m-2}} < \frac{\gamma_0}{r_n}, \tag{1.28}$$

are fulfilled, where $r_n = (1 + \sum_{j=1}^{m} \frac{(2m-2j+1)^{-1/2}}{(m-j)!}) \frac{2^{m-1}(2n-2m-1)}{(\nu_n-B)(2m-1)!!}$. *Then problem* (1.1), (1.2) *is solvable in the space* $\widetilde{C}^{n-1,m}(]a,b])$.