THE NONLOCAL BOUNDARY VALUE PROBLEMS FOR STRONGLY SINGULAR HIGHER-ORDER NONLINEAR FUNCTIONAL-DIFFERENTIAL EQUATIONS

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Abstract. A priori boundedness principle is proven for the nonlocal boundary value problems for strongly singular higher-order nonlinear functional-differential equations. Several sufficient conditions of solvability of the Dirichlet problem under consideration are derived from the a priori boundedness principle. The proof of the a priori boundedness principle is based on the Agarwal–Kiguradze type theorems, which guarantee the existence of the Fredholm property for strongly singular higher-order linear differential equations with argument deviations under the nonlocal boundary conditions.

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1. Statement of the main results

1.1. Statement of the problem and a survey of the literature. Consider the functional differential equation

$$u^{(2m+1)}(t) = F(u)(t) \tag{1.1}$$

with the boundary conditions

$$\int_a^b u(s)d\varphi(s) = 0 \tag{1.2}$$

where $\varphi(b) - \varphi(a) \neq 0,\ u^{(i)}(a) = 0,\ u^{(i)}(b) = 0\ (i = 1, ..., m)$.

Here, $-\infty < a < b < +\infty$, $\varphi : [a,b] \to R$ is a function of bounded variation, and the operator F acting from the set of $(m-1)$-th time continuously differentiable on $]a,b[$ functions, to the set $L_{loc}(]a,b[)$. By $u^{(i)}(a)$ $(u^{(i)}(b))$, we denote the right (the left) limit of the function $u^{(i)}$ at the point $a\,(b)$.

The problem is singular in the sense that for an arbitrary x the right-hand side of equation (1.1) may have nonintegrable singularities at the points a and b.

Throughout the paper we use the following notations:

$R^+ = [0,+\infty[$;

$[x]_+$ the positive part of number x, that is $[x]_+ = \frac{x+|x|}{2}$;

$L_{loc}(]a,b[)$ $(L_{loc}(]a,b]))$ is the space of functions $y :]a,\, b[\to R$, which are integrable on $[a+\varepsilon,\, b-\varepsilon]$ for arbitrary small $\varepsilon > 0$;

$L_{\alpha,\beta}(]a,b[)$ $(L^2_{\alpha,\beta}(]a,b[))$ is the space of integrable (square integrable) with the weight $(t-a)^\alpha(b-t)^\beta$ functions $y :]a,b[\to R$, with the norm

$$||y||_{L_{\alpha,\beta}}=\int_a^b (s-a)^\alpha(b-s)^\beta|y(s)|ds \ \Big(||y||_{L^2_{\alpha,\beta}}=\Big(\int_a^b (s-a)^\alpha(b-s)^\beta y^2(s)ds\Big)^{1/2}\Big);$$

$L([a,b]) = L_{0,0}(]a,b[),\ L^2([a,b]) = L^2_{0,0}(]a,b[)$;

$M(]a,b[)$ is the set of the measurable functions $\tau :]a,b[\to]a,b[$;

$\widetilde{L}^2_{\alpha,\beta}(]a,b[)$ $(\widetilde{L}^2_{\alpha}(]a,b]))$ is the Banach space of $y \in L_{loc}(]a,b[)$ $(L_{loc}(]a,b]))$ functions, with the norm

$$||y||_{\widetilde{L}^2_{\alpha,\beta}} \equiv \max\Big\{\Big[\int_a^t (s-a)^\alpha\Big(\int_s^t y(\xi)d\xi\Big)^2 ds\Big]^{1/2} : a \le t \le \frac{a+b}{2}\Big\}$$
$$+ \max\Big\{\Big[\int_t^b (b-s)^\beta\Big(\int_t^s y(\xi)d\xi\Big)^2 ds\Big]^{1/2} : \frac{a+b}{2} \le t \le b\Big\} < +\infty.$$

$L_m(]a,b[)$ is the Banach space of $y \in L_{loc}(]a,b[)$ functions, with the norm

$$||y||_{L_m} = \sup\Big\{[(s-a)(b-t)]^{m-1/2}\int_s^t |y(\xi)|d\xi : \ \ a < s \le t < b\Big\} < +\infty.$$

$C^n_{loc}(]a,b[)$, $(\widetilde{C}^{m-1}_{loc}(]a,b[))$ is the space of the functions $y :]a,b[\to R$, which are continuous (absolutely continuous) together with $y', y'', ..., y^{(n-1)}$ on $[a+\varepsilon,$ $b-\varepsilon]$ for arbitrarily small $\varepsilon > 0$.

$\widetilde{C}^{n,m}(]a,b[)$ $(m \le n)$ is the space of the functions $y \in \widetilde{C}^n_{loc}(]a,b[)$, such that

$$\int_a^b |x^{(m)}(s)|^2 ds < +\infty. \tag{1.3}$$

$C_2^m(]a,b[)$ is the Banach space of the functions $y \in C_{loc}^m(]a,b[)$, such that

$$\begin{aligned} &\limsup_{t\to a} \frac{|x^{(i)}(t)|}{(t-a)^{m-i+1/2}} < +\infty \ (i=1,...,m), \\ &\limsup_{t\to b} \frac{|x^{(i)}(t)|}{(b-t)^{m-i+1/2}} < +\infty \ (i=1,...,m), \end{aligned} \tag{1.4}$$

with the norm:

$$||x||_{C_2^m} = ||x||_C + \sum_{i=1}^m \sup\Big\{\frac{|x^{(i)}(t)|}{\alpha_i(t)} : a<t<b\Big\},$$

where $\alpha_i(t) = (t-a)^{m-i+1/2}(b-t)^{m-i+1/2}$.

$\widetilde{C}_2^m(]a,b[)$ is the Banach space of the functions $y \in \widetilde{C}_{loc}^m(]a,b[)$, such that conditions $\Big(\int\limits_a^b |x^{(m+1)}(s)|^2 ds\Big)^{1/2} < +\infty$ and (1.4) hold, with the norm:

$$||x||_{\widetilde{C}_2^m} = ||x||_{C_1^m} + \Big(\int\limits_a^b |x^{(m+1)}(s)|^2 ds\Big)^{1/2}.$$

$D_n(]a,b[\times R^+)$ is the set of such functions $\delta :]a,b[\times R^+ \to L_n(]a,b[)$ that $\delta(t,\cdot) : R^+ \to R^+$ is nondecreasing for every $t \in]a,b[$, and $\delta(\cdot,\rho) \in L_n(]a,b[)$ for any $\rho \in R^+$.

$D_{2m-2,2m-2}(]a,b[\times R^+)$ is the set of such functions $\delta :]a,b[\times R^+ \to \widetilde{L}^2_{2m-2,2m-2}(]a,b[)$ that $\delta(t,\cdot) : R^+ \to R^+$ is nondecreasing for every $t \in]a,b[$, and $\delta(\cdot,\rho) \in \widetilde{L}^2_{2m-2,2m-2}(]a,b[)$ for any $\rho \in R^+$.

A solution of problem (1.1), (1.2) is sought in the space $\widetilde{C}^{2m,m+1}(]a,b[)$.

The singular ordinary differential and functional-differential equations, have been studied with sufficient completeness under different boundary conditions, see for example [1], [2], [4] – [14], [17], [25]-[29] and the references cited therein. But the equation (1.1), under the boundary condition (1.2), is not studied even in the case when equation (1.1) has the form

$$x^{(2m+1)}(t) = \sum_{j=0}^m p_j(t)x^{(j)}(\tau_j(t)) + f(x)(t), \tag{1.5}$$

where the singularity of the functions $p_j : L_{loc}([a,\, b])$ be such that the inequalities

$$\begin{aligned} &\int\limits_a^b (s-a)^{2m-1}(b-s)^{2m-1}[(-1)^{n-m}p_1(s)]_+ ds < +\infty, \\ &\int\limits_a^b (s-a)^{2m-j}(b-s)^{2m-j}|p_j(s)| ds < +\infty \quad (j=2,...,m), \end{aligned} \tag{1.6}$$

are not fulfilled (in this case we sad that the linear part of the operator F is a strongly singular), the operator f continuously acting from $C_2^m(]a,b[)$ to $L_{\widetilde{L}^2_{2m-2,2m-2}}(]a,b[)$, and the inclusion

$$\sup\{f(x)(t) : ||x||_{C_2^m} \leq \rho\} \in \widetilde{L}^2_{2m-2,2m-2}(]a,b[). \tag{1.7}$$

holds. The first step in studying of the differential equations with strong singularities was made by R.P. Agarwal and I. Kiguradze in the article [3], where the linear ordinary differential equations under conditions (1.2), in the case when the functions p_j have strong singularities at the points a and b, are studied. Also the ordinary differential equations with strong singularities under two-point boundary conditions are studied in the articles of I. Kiguradze [15], [16], and N. Partsvania [24]. In the papers [20], [21] these results are generalized for linear differential equation with deviating arguments i.e., are proven the Agarwal-Kiguradze type theorems, which guarantee Fredholm's property for linear differential equation with deviating arguments.

In this paper, on the basis of articles [3] and [19], we prove a priori boundedness principle for the problem (1.1), (1.2) in the case where equation (1.1) is in form (1.5).

Now, we introduce some results from articles [20], [21], which we need for this work. Consider the equation

$$u^{(2m+1)}(t) = \sum_{j=1}^{m} p_j(t)u^{(j-1)}(\tau_j(t)) + q(t) \qquad \text{for} \qquad a < t < b. \tag{1.8}$$

By $h_j :]a,b[\times]a,b[\to R_+$ and $f_j : [a,b]\times M(]a,b[) \to C_{loc}(]a,b[\times]a,b[)$ $(j = 1,...,m)$ we denote the functions and operator, respectively defined by the equalities

$$\begin{aligned} h_1(t,s) &= \Big|\int_s^t (\xi-a)^{n-2m}[(-1)^{n-m}p_1(\xi)]_+ d\xi\Big|, \\ h_j(t,s) &= \Big|\int_s^t (\xi-a)^{n-2m}p_j(\xi)d\xi\Big| \quad (j=2,...,m), \end{aligned} \tag{1.9}$$

and

$$f_j(c,\tau_j)(t,s) = \Big|\int_s^t (\xi-a)^{n-2m}|p_j(\xi)|\Big|\int_\xi^{\tau_j(\xi)} (\xi_1-c)^{2(m-j)}d\xi_1\Big|^{1/2} d\xi\Big|, \tag{1.10}$$

and also we put that

$$f_0(t,s)=\Big|\int_s^t |p_0(\xi)|d\xi\Big|.$$

Let $k = 2k_1 + 1$ $(k_1 \in N)$, then

$$k!! = \begin{cases} 1 & \text{for } \ k \le 0, \\ 1 \cdot 3 \cdot 5 \cdots k & \text{for } \ k \ge 1. \end{cases}$$

Now, we can introduce the main theorem of the paper [20].

Theorem 1.1. *Let there exist numbers* $t^* \in]a, b[$, $l_{k0} > 0$, $l_{kj} > 0$, $\bar{l}_{kj} \ge 0$, *and* $\gamma_{kj} > 0$ $(k = 0, 1;\ j = 1, ..., m)$ *such that along with*

$$\begin{aligned} B_0 \equiv & \bar{l}_{00} \Big(\frac{2^{m-1}}{(2m-3)!!} \Big)^2 \frac{(b-a)^{m-1/2}}{(2m-1)^{1/2}} \frac{(t^*-a)^{\gamma_{00}}}{\sqrt{2\gamma_{00}}} \int\limits_a^b \frac{|\varphi(\xi)-\varphi(a)|+|\varphi(\xi)-\varphi(b)|}{|\varphi(b)-\varphi(a)|} d\xi \\ & + \sum_{j=1}^{m} \Big(\frac{(2m-j)2^{2m-j+1}\, l_{0j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}}\, \bar{l}_{0j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{0j}}} \Big) < \frac{1}{2}, \end{aligned} \tag{1.11}$$

$$\begin{aligned} B_1 \equiv & \bar{l}_{10} \Big(\frac{2^{m-1}}{(2m-3)!!} \Big)^2 \frac{(b-a)^{m-1/2}}{(2m-1)^{1/2}} \frac{(b-t^*)^{\gamma_{10}}}{\sqrt{2\gamma_{10}}} \int\limits_a^b \frac{|\varphi(\xi)-\varphi(a)|+|\varphi(\xi)-\varphi(b)|}{|\varphi(b)-\varphi(a)|} d\xi \\ & + \sum_{j=1}^{m} \Big(\frac{(2m-j)2^{2m-j+1}\, l_{1j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(b-t^*)^{\gamma_{0j}}\, \bar{l}_{1j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{1j}}} \Big) < \frac{1}{2}, \end{aligned} \tag{1.12}$$

the conditions

$$\begin{aligned} & (t-a)^{m-\gamma_{00}-1/2} f_0(t,s) \le \bar{l}_{00}, \qquad (t-a)^{2m-j} h_{j_1}(t,s) \le l_{0j_1}, \\ & (t-a)^{m-\gamma_{0j}-1/2} f_j(a,\tau_j)(t,s) \le \bar{l}_{0j} \qquad \textit{for } a<t\le s\le t^*, \end{aligned} \tag{1.13}$$

$$\begin{aligned} & (b-t)^{m-\gamma_{10}-1/2} f_0(t,s) \le \bar{l}_{10}, \qquad (b-t)^{2m-j} h_{j_1}(t,s) \le l_{1j_1}, \\ & (b-t)^{m-\gamma_{1j}-1/2} f_j(b,\tau_j)(t,s) \le \bar{l}_{1j} \qquad \textit{for } t^*\le s\le t<b \end{aligned} \tag{1.14}$$

$j = 1, \cdots, m$ *hold. Then for every* $q \in \widetilde{L}^2_{2m-2,\,2m-2}(]a, b[)$ *problem* (1.8), (1.2) *is uniquely solvable in the space* $\widetilde{C}^{2m,\,m+1}(]a,\, b[)$.

Also, in [21], the following theorem is proven:

Theorem 1.2. *Let all the conditions of Theorem* 1.5 *are satisfied. Then the unique solution* u *of problem* (1.8), (1.2) *for every* $q \in \widetilde{L}^2_{2m-2,2m-2}(]a, b[)$ *admits the estimate*

$$||u^{(m+1)}||_{L^2} \le r||q||_{\widetilde{L}^2_{2m-2,\,2m-2}}, \tag{1.15}$$

with

$$r = \frac{2^m}{(1 - 2\max\{B_0, B_1\})(2m-1)!!},$$

and thus constant $r > 0$ *depends only on the numbers* l_{kj}, $\bar{l}_{kj}, \gamma_{kj}$ $(k = 0, 1;$ $j = 0, ..., m)$, *and* a, b, t^*.