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ON ONE ESTIMATE FOR SOLUTIONS OF TWO-POINT BOUNDARY VALUE PROBLEMS FOR HIGHER-ORDER STRONGLY SINGULAR LINEAR DIFFERENTIAL EQUATIONS

Abstract. For higher-order strongly singular differential equations with deviating arguments, the estimates for solutions of two-point conjugated and right-focal boundary value problems are established.

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1. Statement of the Main Results

Consider the differential equation with deviating arguments

$$u^{(n)}(t)=\sum_{j=1}^{m}p_j(t)u^{(j-1)}(\tau_j(t))+q(t)\ \text{ for }\ a<t<b \tag{1.1}$$

with the two-point conjugated and right-focal boundary conditions

$$u^{(i-1)}(a)=0\ \ (i=1,\dots,m),\quad u^{(j-1)}(b)=0\ \ (j=1,\dots,n-m), \tag{1.2}$$

and

$$u^{(i-1)}(a)=0\ (i=1,\dots,m),\quad u^{(j-1)}(b)=0\ \ (j=m+1,\dots,n). \tag{1.3}$$

Here $n\geq 2$, m is the integer part of $n/2$, $-\infty<a<b<+\infty$, $p_j,q\in L_{loc}(]a,b[)$ $(j=1,\dots,m)$, and $\tau_j:]a,b[\to]a,b[$ are measurable functions. By $u^{(j-1)}(a)$ $(u^{(j-1)}(b))$ we mean the right (the left) limit of the function $u^{(j-1)}$ at the point a (at the point b).

Following R. P. Agarwal and I. Kiguradze [1], we say that the equation (1.1) is strongly singular if $\int_a^b P(s)ds=+\infty$, where

$$P(t)=(t-a)^{n-1}(b-t)^{n-1}\left[(-1)^{n-m}p_1(t)\right]_++\sum_{i=2}^{m}(t-a)^{n-i}(b-t)^{n-i}|p_i(t)|.$$

If the equation (1.1) is strongly singular, then we say that the problem (1.1), (1.2) (the problem (1.1), (1.3)) is also strongly singular.

In the case, where $\tau_j(t)\equiv t$ $(j=1,\dots,m)$, the strongly singular problems (1.1), (1.2) and (1.1), (1.3) are investigated in detail by I. Kiguradze and R. P. Agarwal [1], [2]. In particular, unimprovable in a certain sense conditions are established by them for the unique solvability of those problems in the spaces $\widetilde{C}^{n-1,m}(]a,b[)$ and $\widetilde{C}^{n-1,m}(]a,b])$. For $\tau_j(t)\not\equiv t$ $(j=1,\dots,m)$, the analogous results are obtained in [5], [6]. In the present paper, on the basis of the results of [6], the estimates for solutions of the strongly singular problems (1.1), (1.2) and (1.1), (1.3) are established.

Throughout the paper we use the following notations.

$R_+=[0,+\infty[$;

$[x]_+$ is the positive part of a number x, i.e., $[x]_+=\frac{x+|x|}{2}$;

$L_{loc}(]a,b[)$ $(L_{loc}(]a,b]))$ is the space of functions $y:]a,b[\to R$, which are integrable on $[a+\varepsilon,b-\varepsilon]$ $([a+\varepsilon,b])$ for an arbitrarily small $\varepsilon>0$;

$L_{\alpha,\beta}(]a,b[)$ $(L^2_{\alpha,\beta}(]a,b[))$ is the space of integrable (square integrable) with the weight $(t-a)^\alpha(b-t)^\beta$ functions $y:]a,b[\to R$, with the norm

$$\|y\|_{L_{\alpha,\beta}}=\int_a^b(s-a)^\alpha(b-s)^\beta|y(s)|\,ds$$

$$\left(\|y\|_{L^2_{\alpha,\beta}}=\left(\int_a^b(s-a)^\alpha(b-s)^\beta y^2(s)\,ds\right)^{1/2}\right);$$

$L([a,b]) = L_{0,0}(]a,b[)$, $L^2([a,b]) = L^2_{0,0}(]a,b[)$;

$M(]a,b[)$ is the set of measurable functions $\tau :]a,b[\to]a,b[$;

$\widetilde{L}^2_{\alpha,\beta}(]a,b[)$ ($\widetilde{L}^2_{\alpha}(]a,b])$) is the Banach space of functions $y \in L_{loc}(]a,b[)$ ($L_{loc}(]a,b])$) such that

$$\mu_1 \equiv \max\bigg\{\bigg[\int_a^t (s-a)^\alpha\Big(\int_s^t y(\xi)\,d\xi\Big)^2 ds\bigg]^{1/2} : \ a \le t \le \frac{a+b}{2}\bigg\} +$$

$$+ \max\bigg\{\bigg[\int_t^b (b-s)^\beta\Big(\int_t^s y(\xi)\,d\xi\Big)^2 ds\bigg]^{1/2} : \ \frac{a+b}{2} \le t \le b\bigg\} < +\infty,$$

$$\mu_2 \equiv \max\bigg\{\bigg[\int_a^t (s-a)^\alpha\Big(\int_s^t y(\xi)\,d\xi\Big)^2 ds\bigg]^{1/2} : \ a \le t \le b\bigg\} < +\infty.$$

Norms in this spaces are defined by the equalities $\|\cdot\|_{\widetilde{L}^2_{\alpha,\beta}} = \mu_1$ ($\|\cdot\|_{\widetilde{L}^2_{\alpha}} = \mu_2$).

$\widetilde{C}^{n-1,m}(]a,b[)$ ($\widetilde{C}^{n-1,m}(]a,b])$) is the space of functions $y \in \widetilde{C}^{n-1}_{loc}(]a,b[)$ ($y \in \widetilde{C}^{n-1}_{loc}(]a,b])$) such that

$$\int_a^b |u^{(m)}(s)|^2\,ds < +\infty. \tag{1.4}$$

When the problem (1.1), (1.2) is discussed, we assume that for $n = 2m$ the conditions

$$p_j \in L_{loc}(]a,b[) \ \ (j = 1,\dots,m) \tag{1.5}$$

are fulfilled, and for $n = 2m+1$, along with (1.5), the condition

$$\limsup_{t\to b}\bigg|(b-t)^{2m-1}\int_{t_1}^t p_1(s)\,ds\bigg| < +\infty \ \ \Big(t_1 = \frac{a+b}{2}\Big) \tag{1.6}$$

is fulfilled. The problem (1.1), (1.3) is discussed under the assumptions

$$p_j \in L_{loc}(]a,b]) \ \ (j = 1,\dots,m). \tag{1.7}$$

A solution of the problem (1.1), (1.2) ((1.1), (1.3)) is sought in the space $\widetilde{C}^{n-1,m}(]a,b[)$ ($\widetilde{C}^{n-1,m}(]a,b])$).

By $h_j :]a,b[\times]a,b[\to R_+$ and $f_j : R \times M(]a,b[) \to C_{loc}(]a,b[\times]a,b[)$ ($j = 1,\dots,m$) we denote, respectively, functions and operators defined by the equalities

$$\begin{aligned} h_1(t,s) &= \bigg|\int_s^t (\xi-a)^{n-2m}\big[(-1)^{n-m}p_1(\xi)\big]_+\,d\xi\bigg|, \\ h_j(t,s) &= \bigg|\int_s^t (\xi-a)^{n-2m}p_j(\xi)\,d\xi\bigg| \ \ (j = 2,\dots,m), \end{aligned} \tag{1.8}$$

and

$$f_j(c,\tau_j)(t,s)=\left|\int\limits_s^t(\xi-a)^{n-2m}|p_j(\xi)|\left|\int\limits_\xi^{\tau_j(\xi)}(\xi_1-c)^{2(m-j)}\,d\xi_1\right|^{1/2}d\xi\right|. \tag{1.9}$$

Suppose also that

$$m!!=\begin{cases}1 & \text{for } m\le 0\\ 1\cdot 3\cdot 5\cdots m & \text{for } m\ge 1\end{cases},$$

if $m=2k+1$.

In [6] (see, Theorems 1.4 and 1.5), the following two theorems are proved.

Theorem 1.1. *Let there exist numbers* $t^*\in\,]a,b[\,$, $\ell_{kj}>0$, $\bar l_{kj}\ge 0$, *and* $\gamma_{kj}>0$ $(k=0,1;\ j=1,\dots,m)$ *such that along with*

$$B_0\equiv\sum_{j=1}^m\bigg(\frac{(2m-j)2^{2m-j+1}l_{0j}}{(2m-1)!!(2m-2j+1)!!}+\\ +\frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}}\bar l_{0j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{0j}}}\bigg)<\frac12, \tag{1.10}$$

$$B_1\equiv\sum_{j=1}^m\bigg(\frac{(2m-j)2^{2m-j+1}l_{1j}}{(2m-1)!!(2m-2j+1)!!}+\\ +\frac{2^{2m-j-1}(b-t^*)^{\gamma_{0j}}\bar l_{1j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{1j}}}\bigg)<\frac12, \tag{1.11}$$

the conditions

$$(t-a)^{2m-j}h_j(t,s)\le l_{0j},\quad (t-a)^{m-\gamma_{0j}-1/2}f_j(a,\tau_j)(t,s)\le\bar l_{0j}\\ \textit{for}\quad a<t\le s\le t^*, \tag{1.12}$$

$$(b-t)^{2m-j}h_j(t,s)\le l_{1j},\quad (b-t)^{m-\gamma_{1j}-1/2}f_j(b,\tau_j)(t,s)\le\bar l_{1j}\\ \textit{for}\quad t^*\le s\le t<b \tag{1.13}$$

hold. Then for every $q\in\widetilde L^2_{2n-2m-2,2m-2}(]a,b[)$ *the problem* (1.1), (1.2) *is uniquely solvable in the space* $\widetilde C^{n-1,m}(]a,b[)$.

Theorem 1.2. *Let there exist numbers* $t^*\in\,]a,b[\,$, $\ell_{0j}>0$, $\bar\ell_{0j}\ge 0$, *and* $\gamma_{0j}>0$ $(j=1,\dots,m)$ *such that the conditions*

$$(t-a)^{2m-j}h_j(t,s)\le l_{0j},\quad (t-a)^{m-\gamma_{0j}-1/2}f_j(a,\tau_j)(t,s)\le\bar l_{0j}\\ \textit{for}\quad a<t\le s\le b, \tag{1.14}$$

and

$$B_3 \equiv \sum_{j=1}^{m} \bigg(\frac{(2m-j)2^{2m-j+1} l_{0j}}{(2m-1)!!(2m-2j+1)!!} + \\ + \frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}} \bar{l}_{0j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{0j}}} \bigg) < 1 \quad (1.15)$$

hold. Then for every $q \in \widetilde{L}^2_{2n-2m-2}(]a,b])$*, the problem* (1.1), (1.3) *is uniquely solvable in the space* $\widetilde{C}^{n-1,m}(]a,b])$.

In the paper, we prove the following two theorems on the estimates of solutions of the problems (1.1), (1.2) and (1.1), (1.3), the existence of which is guaranteed by Theorems 1.1 and 1.2.

Theorem 1.3. *Let all the conditions of Theorem* 1.1 *be satisfied. Then the unique solution* u *of the problem* (1.1), (1.2) *for every* $q \in \widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[)$ *admits the estimate*

$$\|u^{(m)}\|_{L^2} \le r\|q\|_{\widetilde{L}^2_{2n-2m-2,2m-2}}, \quad (1.16)$$

where

$$r = \frac{(1+b-a)(2n-2m-1)2^m}{(\nu_n - 2\max\{B_0,B_1\})(2m-1)!!}, \quad \nu_{2m}=1, \ \ \nu_{2m+1} = \frac{2m+1}{2},$$

and thus the constant $r > 0$ *depends only on the numbers* l_{kj}, $\bar{l}_{kj}$, γ_{kj} ($k=1,2$; $j=1,\dots,m$), *and* a, b, t^*, n.

Theorem 1.4. *Let all the conditions of Theorem* 1.2 *be satisfied. Then the unique solution* u *of the problem* (1.1), (1.3) *for every* $q \in \widetilde{L}^2_{2n-2m-2}(]a,b])$ *admits the estimate*

$$\|u^{(m)}\|_{L^2} \le r\|q\|_{\widetilde{L}^2_{2n-2m-2}}, \quad (1.17)$$

where

$$r = \frac{2^{m-1}(2n-2m-1)}{(\nu_n - B_3)(2m-1)!!}, \quad \nu_{2m}=1, \ \ \nu_{2m+1} = \frac{2m+1}{2},$$

end thus the constant $r > 0$ *depends only on the numbers* l_{0j}, $\bar{l}_{0j}$, γ_{0j} ($j=1,\dots,m$), *and* a, b, n.

2. Auxiliary Propositions

To prove Theorems 1.3 and 1.4, we need Lemmas 2.1–2.6 below.

Lemma 2.1. *Let* $\in \widetilde{C}^{m-1}_{loc}(]t_0,t_1[)$ *and*

$$u^{(j-1)}(t_0) = 0 \ \ (j=1,\dots,m), \quad \int_{t_0}^{t_1} |u^{(m)}(s)|^2\, ds < +\infty. \quad (2.1)$$

Then

$$\int\limits_{t_0}^{t} \frac{(u^{(j-1)}(s))^2}{(s-t_0)^{2m-2j+2}}\,ds \leq$$

$$\leq \Big(\frac{2^{m-j+1}}{(2m-2j+1)!!}\Big)^2 \int\limits_{t_0}^{t} |u^{(m)}(s)|^2\,ds \ \ \textit{for}\ \ t_0 \leq t \leq t_1. \tag{2.2}$$

Lemma 2.2. *Let* $u \in \widetilde{C}_{loc}^{m-1}(]t_0,t_1[)$, *and*

$$u^{(j-1)}(t_1) = 0 \ \ (j=1,\dots,m), \quad \int\limits_{t_0}^{t_1} |u^{(m)}(s)|^2\,ds < +\infty. \tag{2.3}$$

Then

$$\int\limits_{t}^{t_1} \frac{(u^{(j-1)}(s))^2}{(t_1-s)^{2m-2j+2}}\,ds \leq$$

$$\leq \Big(\frac{2^{m-j+1}}{(2m-2j+1)!!}\Big)^2 \int\limits_{t}^{t_1} |u^{(m)}(s)|^2\,ds \ \ \textit{for}\ \ t_0 \leq t \leq t_1. \tag{2.4}$$

Let $t_0, t_1 \in]a,b[$, $u \in \widetilde{C}_{loc}^{m-1}(]t_0,t_1[)$, and $\tau_j \in M(]a,b[)$ $(j=1,\dots,m)$. Then we define the functions $\mu_j : [a,(a+b)/2] \times [(a+b)/2,b] \times [a,b] \to [a,b]$, $\rho_k : [t_0,t_1] \to R_+$ $(k=0,1)$, $\lambda_j : [a,b] \times]a,(a+b)/2] \times [(a+b)/2,b[\times]a,b[\to R_+$ by the equalities

$$\mu_j(t_0,t_1,t) = \begin{cases} \tau_j(t) & \text{for } \ \tau_j(t) \in [t_0,t_1] \\ t_0 & \text{for } \ \tau_j(t) < t_0 \\ t_1 & \text{for } \ \tau_j(t) > t_1 \end{cases},$$

$$\rho_k(t) = \Big|\int\limits_{t}^{t_k} |u^{(m)}(s)|^2\,ds\Big|, \tag{2.5}$$

$$\lambda_j(c,t_0,t_1,t) = \Big|\int\limits_{t}^{\mu_j(t_0,t_1,t)} (s-c)^{2(m-j)}\,ds\Big|^{1/2}.$$

Moreover, we define the functions $\alpha_j : R_+^3 \times [0,1[\to R_+$ and $\beta_j \in R_+ \times [0,1[\to R_+$ $(j=1,\dots,m)$ as follows

$$\begin{gathered}\alpha_j(x,y,z,\gamma) = x + \frac{2^{m-j} y z^{\gamma}}{(2m-2j-1)!!}, \\ \beta_j(y,\gamma) = \frac{2^{2m-j-1}}{(2m-2j-1)!!(2m-3)!!}\,\frac{y^{\gamma}}{\sqrt{2\gamma}}.\end{gathered} \tag{2.6}$$

Lemma 2.3. *Let $a_0 \in]a,b[$, $t_0 \in]a,a_0[$, $t_1 \in]a_0,b[$, and a function $u \in \widetilde{C}_{loc}^{m-1}(]t_0,t_1[)$ be such that the conditions (2.1) hold. Moreover, let constants $l_{0j} > 0$, $\bar{l}_{0j} \geq 0$, $\gamma_{0j} > 0$, and functions $\overline{p}_j \in L_{loc}(]t_0,t_1[)$, $\tau_j \in M(]a,b[)$ be such that the inequalities*

$$(t-t_0)^{2m-1}\int_t^{a_0}\big[\overline{p}_1(s)\big]_+\,ds \leq l_{01}, \tag{2.7}$$

$$(t-t_0)^{2m-j}\bigg|\int_t^{a_0}\overline{p}_j(s)\,ds\bigg| \leq l_{0j} \quad (j=2,\dots,m), \tag{2.8}$$

$$(t-t_0)^{m-\frac{1}{2}-\gamma_{0j}}\bigg|\int_t^{a_0}\overline{p}_j(s)\lambda_j(t_0,t_0,t_1,s)\,ds\bigg| \leq \bar{l}_{0j} \quad (j=1,\dots,m), \tag{2.9}$$

hold for $t_0 < t \leq a_0$. Then

$$\begin{gathered}\int_t^{a_0}\overline{p}_j(s)u(s)u^{(j-1)}(\mu_j(t_0,t_1,s))\,ds \leq \\ \leq \alpha_j(l_{0j},\bar{l}_{0j},a_0-a,\gamma_{0j})\rho_0^{1/2}(\tau^*)\rho_0^{1/2}(t)+\bar{l}_{0j}\beta_j(a_0-a,\gamma_{0j})\rho_0^{1/2}(\tau^*)\rho_0^{1/2}(a_0)+ \\ +l_{0j}\,\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\,\rho_0(a_0) \ \text{ for } \ t_0 < t \leq a_0,\end{gathered} \tag{2.10}$$

where $\tau^* = \sup\big\{\mu_j(t_0,t_1,t):\ t_0 \leq t \leq a_0,\ j=1,\dots,m\big\} \leq t_1$.

Lemma 2.4. *Let $b_0 \in]a,b[$, $t_1 \in]b_0,b[$, $t_0 \in]a,b_0[$, and a function $u \in \widetilde{C}_{loc}^{m-1}(]t_0,t_1[)$ be such that the conditions (2.3) hold. Moreover, let constants $l_{1j} > 0$, $\bar{l}_{1j} \geq 0$, $\gamma_{1j} > 0$, and functions $\overline{p}_j \in L_{loc}(]t_0,t_1[)$, $\tau_j \in M(]a,b[)$ be such that the inequalities*

$$(t_1-t)^{2m-1}\int_{b_0}^{t}\big[\overline{p}_1(s)\big]_+\,ds \leq l_{11}, \tag{2.11}$$

$$(t_1-t)^{2m-j}\bigg|\int_{b_0}^{t}\overline{p}_j(s)\,ds\bigg| \leq l_{1j} \quad (j=2,\dots,m), \tag{2.12}$$

$$(t_1-t)^{m-\frac{1}{2}-\gamma_{1j}}\bigg|\int_{b_0}^{t}\overline{p}_j(s)\lambda_j(t_1,t_0,t_1,s)\,ds\bigg| \leq \bar{l}_{1j} \quad (j=1,\dots,m) \tag{2.13}$$

hold for $b_0 < t \le t_1$. *Then*

$$\int\limits_{b_0}^{t} \overline{p}_j(s)u(s)u^{(j-1)}(\mu_j(t_0,t_1,s))\,ds \le$$

$$\le \alpha_j(l_{1j},\bar{l}_{1j},b-b_0,\gamma_{1j})\rho_1^{1/2}(\tau_*)\rho_1^{1/2}(t)+\bar{l}_{1j}\beta_j(b-b_0,\gamma_{1j})\rho_1^{1/2}(\tau_*)\rho_1^{1/2}(b_0)+$$
$$+\,l_{1j}\,\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\,\rho_1(b_0)\ \ \textit{for}\ \ b_0 \le t < t_1, \quad (2.14)$$

where $\tau_* = \inf\big\{\mu_j(t_0,t_1,t):\ b_0 \le t \le t_1,\ j=1,\dots,m\big\} \ge t_0$.

Lemma 2.5. *If* $u \in C^{n-1}_{loc}(]a,b[)$, *then for any* $s,t \in]a,b[$ *the equality*

$$(-1)^{n-m}\int\limits_{s}^{t}(\xi-a)^{n-2m}u^{(n)}(\xi)u(\xi)\,d\xi =$$
$$= w_n(t)-w_n(s)+\nu_n\int\limits_{s}^{t}|u^{(m)}(\xi)|^2\,d\xi \quad (2.15)$$

is valid, where

$$\nu_{2m}=1, \quad \nu_{2m+1}=\frac{2m+1}{2}, \quad w_{2m}(t)=\sum_{j=1}^{m}(-1)^{m+j-1}u^{(2m-j)}(t)u(t),$$
$$w_{2m+1}(t)=\sum_{j=1}^{m}(-1)^{m+j}\Big[(t-a)u^{(2m+1-j)}(t)-ju^{(2m-j)}(t)\Big]u^{(j-1)}(t)-$$
$$-\frac{t-a}{2}\,|u^{(m)}(t)|^2.$$

Lemma 2.6. *Let*

$$w(t)=\sum_{i=1}^{n-m}\sum_{k=i}^{n-m}c_{ik}(t)u^{(n-k)}(t)u^{(i-1)}(t),$$

where $\widetilde{C}^{n-1,m}(]a,b[)$, *and each* $c_{ik}:[a,b]\to R$ *is an* $(n-k-i+1)$*-times continuously differentiable function. If, moreover,* $u^{(i-1)}(a)=0$ $(i=1,\dots,m)$,

$$\limsup_{t\to a}\frac{|c_{ii}(t)|}{(t-a)^{n-2m}} < +\infty \ \ (i=1,\dots,n-m),$$

then $\liminf\limits_{t\to a}|w(t)| = 0$, *and if* $u^{(i-1)}(b) = 0$ $(i = 1,\dots,n-m)$, *then* $\liminf\limits_{t\to b}|w(t)|=0$.

Lemmas 2.1, 2.2 are proved in [1], Lemmas 2.3, 2.4 are proved in [6]. The proof of Lemma 2.6 can be found in [4]. As for Lemma 2.5, it is a particular case of Lemma 4.1 from [3].

3. Proofs

Proof of Theorem 1.3. Let u be a solution of the problem (1.1), (1.2). Then in view of Theorem 1.1, the inclusion $u \in \widetilde{C}^{n\,m-1}(]a,b[)$ holds, i.e.,

$$\rho = \int_a^b |u^{(m)}(s)|^2\,ds < +\infty. \tag{3.1}$$

Multiplying the equation (1.1) by $(-1)^{n-m}(t-a)^{n-2m}u(t)$ and then integrating from t_0 to t_1, by Lemma 2.5 we obtain

$$w_n(t)-w_n(s)+\nu_n\int_s^t |u^{(m)}(\xi)|^2\,d\xi=(-1)^{n-m}\int_s^t (s-a)^{n-2m}q(s)u(s)\,ds+$$
$$+(-1)^{n-m}\sum_{j=1}^m\int_s^t (\xi-a)^{n-2m}p_j(\xi)u^{(j-1)}(\tau_j(\xi))u(\xi)\,d\xi \tag{3.2}$$

for $a<s\le t<b$. Hence by Lemma 2.6 it is evident that

$$\liminf_{s\to a}|w_n(s)|=0, \quad \liminf_{t\to b}|w_n(t)|=0. \tag{3.3}$$

Moreover, due to the conditions (1.10) and (1.11), a number $\nu\in]0,1[$ can be chosen so that the inequalities

$$\begin{aligned} B_0 &\equiv \sum_{j=1}^m\left(l_{0j}\,\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}+\bar l_{0j}\beta_j(t^*-a,\gamma_{0j})\right)< \\ &\quad <(\nu_n-\nu)/2, \\ B_1 &\equiv \sum_{j=1}^m\left(l_{1j}\,\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}+\bar l_{1j}\beta_j(b-t^*,\gamma_{1j})\right)< \\ &\quad <(\nu_n-\nu)/2, \end{aligned} \tag{3.4}$$

would be satisfied, and then

$$0<\nu<\nu_n-2\max\{B_0,B_1\}. \tag{3.5}$$

It is obvious that the maximum of ν depends only on the numbers l_{kj}, $\bar l_{kj}$, γ_{kj} ($k=1,2$; $j=1,\dots,m$), and a, b, t^*, n. Now, if we put $c=(a+b)/2$, then by virtue of Lemmas 2.1, 2.2, and Young's inequality we get

$$\left|\int_s^t (\psi-a)^{n-2m}q(\psi)u(\psi)\,d\psi\right|\le$$
$$\le\left|\int_s^c (\psi-a)^{n-2m}q(\psi)u(\psi)\,d\psi\right|+\left|\int_c^t (\psi-a)^{n-2m}q(\psi)u(\psi)\,d\psi\right|=$$

$$= \Big| \int\limits_s^c \Big[(n-2m)u(\psi) + (\psi-a)^{n-2m} u'(\psi) \Big] \Big(\int\limits_\psi^c q(\xi)\, d\xi \Big)\, d\psi \Big| +$$

$$+ \Big| \int\limits_c^t \Big[(n-2m)u(\psi) + (\psi-a)^{n-2m} u'(\psi) \Big] \Big(\int\limits_c^\psi q(\xi)\, d\xi \Big)\, d\psi \Big| \le$$

$$\le \bigg[(n-2m) \Big(\int\limits_s^c \frac{u^2(\psi)}{(\psi-a)^{2m}}\, d\psi \Big)^{1/2} + \Big(\int\limits_s^c \frac{u'^2(\psi)}{(\psi-a)^{2m-2}}\, d\psi \Big)^{1/2} \bigg] \times$$

$$\times \bigg(\int\limits_s^c (\psi-a)^{2n-2m-2} \Big(\int\limits_\psi^c q(\xi)\, d\xi \Big)^2 d\psi \bigg)^{1/2} +$$

$$+ (1+b-a) \bigg[(n-2m) \Big(\int\limits_c^t \frac{u^2(\psi)}{(b-\psi)^{2m}}\, d\psi \Big)^{1/2} + \Big(\int\limits_c^t \frac{u'^2(\psi)}{(b-\psi)^{2m-2}}\, d\psi \Big)^{1/2} \bigg] \times$$

$$\times \bigg(\int\limits_c^t (b-\psi)^{2m-2} \Big(\int\limits_c^\psi q(\xi)\, d\xi \Big)^2 d\psi \bigg)^{1/2} \le$$

$$\le \frac{(1+b-a)(2n-2m-1)2^{m-1}}{(2m-1)!!} \, \|q\|_{\widetilde{L}^2_{2n-2m-2,2m-2}} \times$$

$$\times \bigg[\Big(\int\limits_a^c |u^{(m)}(s)|^2\, ds \Big)^{1/2} + \Big(\int\limits_c^b |u^{(m)}(s)|^2\, ds \Big)^{1/2} \bigg] \le \frac{\nu}{2} \int\limits_a^b |u^{(m)}(s)|^2\, ds +$$

$$+ \frac{1}{2\nu} \Big(\frac{(1+b-a)(2n-2m-1)2^m}{(2m-1)!!} \Big)^2 \|q\|^2_{\widetilde{L}^2_{2n-2m-2,2m-2}} \tag{3.6}$$

for $a < s \le t^* \le t < b$. Due to Lemmas 2.3 and 2.4 with $a_0 = t^*$, $t_0 = a$, $b_0 = t^*$, $t_1 = b$, $\overline{p}_j(t) = (t-a)^{n-2m}(-1)^{n-m} p_j(t)$, and the equalities $\rho_0(a) = \rho_1(b) = 0$, $\mu_j(a, b, t) = \tau_j(t)$, we have

$$(-1)^{n-m} \int\limits_s^t (\xi-a)^{n-2m} p_j(\xi) u^{(j-1)}(\tau_j(\xi)) u(\xi)\, d\xi \le$$

$$\le \bar{l}_{0j} \beta_j(t^*-a, \gamma_{0j}) \rho_0^{1/2}(b) \rho_0^{1/2}(t^*) +$$

$$+ l_{0j} \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} \rho_0(t^*) + \bar{l}_{1j} \beta_j(b-t^*, \gamma_{1j}) \rho_1^{1/2}(a) \rho_1^{1/2}(t^*) +$$

$$+ l_{1j} \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} \rho_1(t^*) \tag{3.7}$$

for $a < s \le t^* \le t < b$. Thus according to (3.3)–(3.7), and the inequalities $\rho_0^{1/2}(b)\rho_0^{1/2}(t^*) \le \rho$, $\rho_1^{1/2}(a)\rho_1^{1/2}(t^*) \le \rho$, we have the estimate

$$\nu_n\rho \le (\nu_n - \nu)\rho + \frac{\nu}{2}\,\rho + \\ + \frac{1}{2\nu}\Big(\frac{(1+b-a)(2n-2m-1)2^m}{(2m-1)!!}\Big)^2 \|q\|^2_{\widetilde{L}^2_{2n-2m-2,2m-2}}. \tag{3.8}$$

From (3.5) and (3.8) it immediately follows that

$$\|u^{(m)}\|_{L^2} \le r_\nu \|q\|_{\widetilde{L}^2_{2n-2m-2,2m-2}} \quad \text{for } 0 < \nu < \nu_n - 2\max\{B_0, B_1\}, \tag{3.9}$$

where $r_\nu = [(1+b-a)(2n-2m-1)2^m]/[\nu(2m-1)!!]$. Thus from (3.9) we obtain

$$\|u^{(m)}\|_{L^2} \le r\|q\|_{\widetilde{L}^2_{2n-2m-2,2m-2}}, \tag{3.10}$$

where

$$r = \frac{(1+b-a)(2n-2m-1)2^m}{(\nu_n - 2\max\{B_0,\,B_1\})(2m-1)!!}\,.$$

Hence, by the definition of the numbers ν_n, B_0, B_1, it is clear that r depends only on the numbers l_{kj}, $\bar{l}_{kj}$, γ_{kj} ($k = 1,2$; $j = 1,\dots,m$), and a, b, t^*, n. □

The proof of Theorem 1.4 is analogous to that of Theorem 1.3. The only difference is that instead of Theorem 1.1, Theorem 1.2 is applied, and we put $t = c = b$.

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