THE DIRICHLET BOUNDARY VALUE PROBLEMS FOR
STRONGLY SINGULAR HIGHER-ORDER NONLINEAR
FUNCTIONAL-DIFFERENTIAL EQUATIONS

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Abstract. The a priori boundedness principle is proved for the Dirichlet boundary value problems for strongly singular higher-order nonlinear functional-differential equations. Several sufficient conditions of solvability of the Dirichlet problem under consideration are derived from the a priori boundedness principle. The proof of the a priori boundedness principle is based on the Agarwal-Kiguradze type theorems, which guarantee the existence of the Fredholm property for strongly singular higher-order linear differential equations with argument deviations under the two-point conjugate and right-focal boundary conditions.

Keywords: higher order functional-differential equation, Dirichlet boundary value problem, strong singularity, Fredholm property, a priori boundedness principle

MSC 2010: 34K06, 34K10

1. Statement of the main results

1.1. Statement of the problem and a survey of the literature. Consider the functional differential equation

\[(1.1) \quad u^{(m)}(t) = F(u)(t)\]

with the two-point boundary conditions

\[(1.2) \quad u^{(i-1)}(a) = 0 \quad (i = 1, \ldots, m), \quad u^{(i-1)}(b) = 0 \quad (i = 1, \ldots, n - m).\]

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Here \( n \geq 2 \), \( m \) is the integer part of \( n/2 \), \(-\infty < a < b < +\infty\), and the operator \( F \) is acting from the set of \((m-1)\)-th time continuously differentiable on \([a, b]\) functions to the set \( L_{\text{loc}}([a, b]) \). By \( u^{(j-1)}(a) \) (\( u^{(j-1)}(b) \)) we denote the right (the left) limit of the function \( u^{(j-1)} \) at the point \( a(b) \).

The problem is singular in the sense that for an arbitrary \( x \) the right-hand side of equation (1.41) may have nonintegrable singularities at the points \( a \) and \( b \).

Throughout the paper we use the following notation:

\( \mathbb{R}^+ = [0, +\infty[; \)

\( [x]_+ \) the positive part of a number \( x \), that is \( [x]_+ = \frac{1}{2}(x + |x|) \);

\( L_{\text{loc}}([a, b])\left(L_{\text{loc}}([a, b])\right) \) is the space of functions \( y : [a, b] \rightarrow \mathbb{R} \), which are integrable on \([a + \varepsilon, b - \varepsilon]\) for arbitrarily small \( \varepsilon > 0 \);

\( L_{\alpha, \beta}([a, b])\left(L_{\alpha, \beta}([a, b])\right) \) is the space of integrable (square integrable) with the weight \((s-a)^\alpha (b-s)^\beta\) functions \( y : [a, b] \rightarrow \mathbb{R} \), with the norm

\[
\|y\|_{L_{\alpha, \beta}} = \left( \int_a^b (s-a)^\alpha (b-s)^\beta |y(s)|^2 \, ds \right)^{1/2}.
\]

\( L([a, b]) = L_{0,0}([a, b]), L^2([a, b]) = L_{2,0}([a, b]) \);

\( M([a, b]) \) is the set of measurable functions \( \tau : [a, b[ \rightarrow [a, b[; \)

\( L_{\alpha, \beta}([a, b])\left(L_{\alpha, \beta}([a, b])\right) \) is the Banach space of \( y \in L_{\text{loc}}([a, b])\left(L_{\text{loc}}([a, b])\right) \) functions, with the norm

\[
\|y\|_{L_{\alpha, \beta}} = \max \left\{ \left[ \int_a^t (s-a)^\alpha \left( \int_s^t y(\xi) \, d\xi \right)^2 \, ds \right]^{1/2} : a \leq t \leq \frac{a+b}{2} \right\} + \left[ \int_a^b (b-s)^\beta \left( \int_s^b y(\xi) \, d\xi \right)^2 \, ds \right]^{1/2} : \frac{a+b}{2} \leq t \leq b \right\} < +\infty.
\]

\( L([a, b]) \) is the Banach space of \( y \in L_{\text{loc}}([a, b]) \) functions, with the norm

\[
\|y\|_{L_{\alpha, \beta}} = \sup \left\{ \left[ (s-a)(b-s) \right]^{m-1/2} \int_s^b (\xi-a)^{n-2m} |y(\xi)| \, d\xi : a < s < t < b \right\} < +\infty.
\]

\( C_{\text{loc}}^{n-1}([a, b]), (\vec{C}_{\text{loc}}^{n-1}([a, b]) \) is the space of functions \( y : [a, b] \rightarrow \mathbb{R} \) which are continuous (absolutely continuous) together with \( y', y'', \ldots, y^{(n-1)} \) on \([a+\varepsilon, b-\varepsilon]\) for arbitrarily small \( \varepsilon > 0 \);

\( \vec{C}_{\text{loc}}^{n-1,m}([a, b]) \) is the space of functions \( y \in \vec{C}_{\text{loc}}^{n-1}([a, b]) \), such that

\[
(1.3) \quad \int_a^b |x^{(m)}(s)|^2 \, ds < +\infty.
\]
$C_{m-1}^m([a,b])$ is the Banach space of functions $y \in C_{loc}^{m-1}([a,b])$, such that

$$
\limsup_{t \to a^+} \frac{|x^{(i-1)}(t)|}{(t-a)^{m-i+1/2}} < +\infty \quad (i = 1, \ldots, m),
$$

$$
\limsup_{t \to b^-} \frac{|x^{(i-1)}(t)|}{(b-t)^{m-i+1/2}} < +\infty \quad (i = 1, \ldots, n-m),
$$

with the norm:

$$
\|x\|_{C_{m-1}^m} = \sum_{i=1}^m \sup_{a < t < b} \left\{ \frac{|x^{(i-1)}(t)|}{\alpha_i(t)} : a < t < b \right\},
$$

where $\alpha_i(t) = (t-a)^{m-i+1/2}(b-t)^{m-i+1/2}$.

$D_{m-1}([a,b] \times \mathbb{R}^+)$ is the set of such functions $\delta : [a,b] \times \mathbb{R}^+ \to L_{loc}([a,b])$ that $\delta(t, \cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing for every $t \in [a,b]$, and $\delta(\cdot, \phi) \in L_{loc}([a,b])$ for any $\phi \in \mathbb{R}^+$.

$D_{2m-2m-2}([a,b] \times \mathbb{R}^+) \subset [a,b] \times [a,b]$ is the set of such functions $\delta : [a,b] \times \mathbb{R}^+ \to \mathcal{L}_{2m-2m-2}([a,b], [a,b])$ that $\delta(t, \cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing for every $t \in [a,b]$, and $\delta(\cdot, \phi) \in \mathcal{L}_{2m-2m-2}([a,b], [a,b])$ for any $\phi \in \mathbb{R}^+$.

A solution of problem (1.1), (1.2) is sought in the space $C_{m-1}^{m}([a,b])$.

The singular ordinary differential and functional-differential equations have been studied with sufficient completeness under different boundary conditions, see for example [1], [3], [4], [5], [6], [7], [8], [9], [11], [12], [13], [14], [16], [21], [22], [23], [24], [25] and the references cited therein. But the equation (1.1), even under the boundary condition (1.2), have not been studied in the case when the operator $F$ has the form

$$
F(x)(t) = \sum_{j=1}^m p_j(t)x^{(j-1)}(\tau_j(t)) + f(x)(t),
$$

where the singularity of the functions $p_j : L_{loc}([a,b])$ is such that the inequalities

$$
\int_a^b (s-a)^{n-1}(b-s)^{2m-1}|(-1)^{n-m}p_1(s)|_+ \, ds < +\infty,
$$

$$
\int_a^b (s-a)^{n-j}(b-s)^{2m-j}|p_j(s)| \, ds < +\infty \quad (j = 2, \ldots, m),
$$

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are not fulfilled (in this case we say that the linear part of the operator $F$ is strongly singular), the operator $f$ is continuously acting from $C^{n-1}_a([a, b])$ to $L_{2, -2m - 2, 2m - 2}([a, b])$, and the inclusion

\begin{equation}
\sup\{f(x)(t): \|x\|_{C^{n-1}_a} \leq \varepsilon\} \in L_{2, -2m - 2, 2m - 2}([a, b])
\end{equation}

holds. The first step in studying the differential equations with strong singularities was made by R. P. Agarwal and I. Kiguradze in the article [2], where the linear ordinary differential equations under conditions (1.2), in the case when the functions $p_j$ have strong singularities at the points $a$ and $b$, are studied. Also the ordinary differential equations with strong singularities under two-point boundary conditions are studied in the articles of I. Kiguradze [10], [19], and N. Partsvania [20]. In the papers [18], [15] these results are generalized to linear differential equations with deviating arguments, i.e., the Agarwal-Kiguradze type theorems, which guarantee Fredholm's property for linear differential equations with deviating arguments are proved.

In this paper, on the bases of articles [2] and [17] we prove the a priori boundedness principle for the problem (1.1), (1.2) in the case when the operator has the form (1.5).

Now we introduce some results from the articles [18], [15], which we need for this work. Consider the equation

\begin{equation}
u^{(n)}(t) = \sum_{j=1}^{m} p_j(t) u^{(j-1)}(\tau_j(t)) + q(t) \quad \text{for } a < t < b.
\end{equation}

For problem (1.8), (1.2) we assume, that when $n = 2m$, then the conditions

\begin{equation}p_j \in L_{\text{loc}}([a, b]) \quad (j = 1, \ldots, m)
\end{equation}

are fulfilled and when $n = 2m + 1$, along with (1.9), the condition

\begin{equation}\lim_{t \to a^+} \left| (b - t)^{2m-1} \int_{t_1}^{t} p_1(s) \, ds \right| < +\infty \quad \left( t_1 = \frac{a + b}{2} \right)
\end{equation}

holds.

By $h_j: [a, b] \times [a, b] \to \mathbb{R}_+$ and $f_j: [a, b] \times M([a, b]) \to C_{\text{loc}}([a, b] \times [a, b])$ ($j = 1, \ldots, m$) we denote the functions and operators, respectively, defined by the equalities

\begin{equation}h_1(t, s) = \left| \int_{s}^{t} (\xi - a)^{n-2m}[(\xi - a)^{n-m}p_1(\xi)]_+ \, d\xi \right|,
\end{equation}

\begin{equation}h_j(t, s) = \left| \int_{s}^{t} (\xi - a)^{n-2m}p_j(\xi) \, d\xi \right| \quad (j = 2, \ldots, m),
\end{equation}

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and

(1.12) \[ f_j(c, \tau_j)(t, s) = \left| \int_s^t (\xi - a)^{n-2m} \left| p_j(\xi) \right| \left( \int_\xi^{\tau_j(\xi)} (\xi_1 - c)^{2(m-j)} \, d\xi_1 \right)^{1/2} \, d\xi \right|. \]

Let \( k = 2k_1 + 1 \) (\( k_1 \in \mathbb{N} \)), then we denote

\[
\begin{aligned}
k!! &= \begin{cases} 1 & \text{for } k \leq 0, \\ 1 \cdot 3 \cdot 5 \cdots k & \text{for } k \geq 1. \end{cases}
\end{aligned}
\]

Now we can introduce the main theorem of the paper [18].

**Theorem 1.1.** Let there exist numbers \( t^* \in ]a, b[ \), \( l_{kj} > 0 \), \( \tilde{l}_{kj} \geq 0 \), and \( \gamma_{kj} > 0 \) (\( k = 0, 1; j = 1, \ldots, m \)) such that along with

(1.13) \[ B_0 = \sum_{j=1}^{m} (2m-j)^{2m-j+1}(2m-2j+1)!! \left( \frac{(2j-1)(2m-2j+1)!!}{(2m-2j+1)!!} \right) + \frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}}}{(2m-2j+1)!!(2m-3)!! \sqrt{2\gamma_{0j}}} \leq \frac{1}{2}, \]

(1.14) \[ B_1 = \sum_{j=1}^{m} (2m-j)^{2m-j+1}(2m-2j+1)!! \left( \frac{(2j-1)(2m-2j+1)!!}{(2m-2j+1)!!} \right) + \frac{2^{2m-j-1}(b-t)^{\gamma_{1j}}}{(2m-2j+1)!!(2m-3)!! \sqrt{2\gamma_{1j}}} \leq \frac{1}{2}, \]

the conditions

(1.15) \[ (t-a)^{2m-j}h_j(t, s) \leq l_{0j}, \quad (t-a)^{m-\gamma_{0j}-1/2} f_j(a, \tau_j)(t, s) \leq \tilde{l}_{0j} \]

for \( a < t \leq s \leq t^* \), and

(1.16) \[ (b-t)^{2m-j}h_j(t, s) \leq l_{1j}, \quad (b-t)^{m-\gamma_{1j}-1/2} f_j(b, \tau_j)(t, s) \leq \tilde{l}_{1j} \]

for \( t^* \leq s \leq t < b \) hold. Then problem (1.8), (1.2) is uniquely solvable in the space \( \tilde{C}^{n-1,m}([a, b]) \).

Also, in [15] the following theorem is proved:

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Theorem 1.2. Let all the conditions of Theorem 1.1 be satisfied. Then the unique solution \( u \) of problem (1.8), (1.2) for every \( q \in \mathcal{L}^2_{2n-2m-2,2m-2}([a, b]) \) admits the estimate

\[
(1.17) \quad \|u^{(m)}\|_{\mathcal{L}^2} \leq r \|q\|_{\mathcal{L}^2_{2n-2m-2,2m-2}},
\]

with

\[
 r = \frac{2^m(1+b-a)(2n-2m-1)}{(\nu_n - 2\max\{B_0, B_1\})(2m-1)!!}, \quad \nu_{2m} = 1, \quad \nu_{2m-1} = \frac{2m+1}{2},
\]

and thus the constant \( r > 0 \) depends only on the numbers \( l_{kj}, \tilde{l}_{kj}, \gamma_{kj} \) (\( k = 1, 2; j = 1, \ldots, m \)), and \( a, b, t^*, n \).

Remark 1.1. Under the conditions of Theorem 1.2, for every

\[
q \in \mathcal{L}^2_{2n-2m-2,2m-2}([a, b])
\]

the unique solution \( u \) of problem (1.8), (1.2) admits the estimate

\[
(1.18) \quad \|u^{(m)}\|_{\mathcal{C}^{m-1}_1} \leq r_n \|q\|_{\mathcal{L}^2_{2n-2m-2,2m-2}},
\]

with

\[
r_n = \left(1 + \sum_{j=1}^{m} \frac{2^{m-j+1/2}}{(m-j)! (2m-2j+1)^{1/2} (b-a)^{m-j+1/2}} \right) \times \frac{2^m(1+b-a)(2n-2m-1)}{(\nu_n - 2\max\{B_0, B_1\})(2m-1)!!}.
\]

1.2. Theorems on solvability of problem (1.1), (1.2).

Define an operator \( P \colon C^{m-1}_1([a, b]) \times C^{m-1}_1([a, b]) \to \mathcal{L}_{loc}([a, b]) \) by the equality

\[
(1.19) \quad P(x, y)(t) = \sum_{j=1}^{m} p_j(x)(t)y^{(j-1)}(\tau_j(t)) \quad \text{for} \quad a < t < b
\]

where \( p_j : C^{m-1}_1([a, b]) \to \mathcal{L}_{loc}([a, b]) \), and \( \tau_j \in M([a, b]) \). Also, for any \( \gamma > 0 \) define a set \( A_\gamma \) by the relation

\[
(1.20) \quad A_\gamma = \{ x \in \mathcal{C}^{m-1}_1([a, b]) : \|x\|_{\mathcal{C}^{m-1}_1} \leq \gamma \}.
\]

For formulating the a priori boundedness principle we have to introduce
Definition 1.1. Let $\gamma_0$ and $\gamma$ be positive numbers. We say that the continuous operator $P: C_1^{m-1}([a, b]) \times C_1^{m-1}([a, b]) \to L_n([a, b])$ is $\gamma_0, \gamma$ consistent with boundary condition (1.2) if:

(i) For any $x \in A_{\gamma_0}$ and almost all $t \in [{a, b}]$ the inequality

\begin{equation}
\sum_{j=1}^{m} |p_j(x)(t)x^{(j-1)}(\tau_j(t))| \leq \delta(t, \|x\|_{C_1^{m-1}}) \|x\|_{C_1^{m-1}}
\end{equation}

holds, where $\delta \in D_n([a, b] \times \mathbb{R}^+)$.

(ii) For any $x \in A_{\gamma_0}$ and $q \in \tilde{L}_{2n-2m-2,2m-2}([a, b])$ the equation

\begin{equation}
y^{(n)}(t) = \sum_{j=1}^{m} p_j(x)(t)y^{(j-1)}(\tau_j(t)) + q(t)
\end{equation}

under boundary conditions (1.2) has a unique solution $y$ in the space $C^{m-1,0}([a, b])$ and

\begin{equation}
\|y\|_{C_1^{m-1}} \leq \gamma \|q\|_{\tilde{L}_{2n-2m-2,2m-2}}.
\end{equation}

Definition 1.2. We say that the operator $P$ is $\gamma$ consistent with boundary condition (1.2), if the operator $P$ is $\gamma_0, \gamma$ consistent with boundary condition (1.2) for any $\gamma_0 > 0$.

In the sequel it will always be assumed that the operator $F_p$ defined by equality

\[ F_p(x)(t) = \left| F(x)(t) - \sum_{j=1}^{m} p_j(x)(t)x^{(j-1)}(\tau_j(t))\right| \]

is continuously acting from $C_1^{m-1}([a, b])$ to $L_{2n-2m-2,2m-2}([a, b])$, and

\begin{equation}
\tilde{F}_p(t, \varrho) \equiv \sup\{F_p(x)(t): \|x\|_{C_1^{m-1}} \leq \varrho\} \in \tilde{L}_{2n-2m-2,2m-2}([a, b])
\end{equation}

for each $\varrho \in [0, +\infty]$.

Then the following theorem is valid
Theorem 1.3. Let the operator \( P \) be \( \gamma_0, \gamma \) consistent with boundary condition (1.2), and let there exist a positive number \( \varrho_0 \leq \gamma_0 \), such that

\[
\| \bar{F}_p(\cdot, \min\{2\varrho_0, \gamma_0\}) \|_{\bar{L}^{2n-2m-2,2m-2}_2} \leq \frac{\gamma_0}{\gamma}.
\]

Let, moreover, for any \( \lambda \in [0,1] \) an arbitrary solution \( x \in A_{\varrho_0} \) of the equation

\[
x^{(n)}(t) = (1 - \lambda)P(x,x)(t) + \lambda F(x)(t)
\]

under the conditions (1.2) admit the estimate

\[
\| x \|_{\bar{C}^{n-1}_1} \leq \varrho_0.
\]

Then problem (1.1), (1.2) is solvable in the space \( \bar{C}^{n-1,m}_1([a,b]) \).

Theorem 1.3 with \( \varrho_0 = \gamma_0 \) immediately yields

Corollary 1.1. Let the operator \( P \) be \( \gamma_0, \gamma \) consistent with boundary condition (1.2), and

\[
|F(x)(t) - \sum_{j=1}^{m} p_j(x)(t)x^{(j-1)}(\tau_j(t))(t)| \leq \eta(t, \| x \|_{\bar{C}^{n-1}_1})
\]

for \( x \in A_{\varrho_0} \) and almost all \( t \in [a,b] \), and

\[
\| \eta(\cdot, \gamma_0) \|_{\bar{L}^{2n-2m-2,2m-2}_2} \leq \frac{\gamma_0}{\gamma},
\]

where \( \eta \in D_{2n-2m-2,2m-2}([a,b] \times \mathbb{R}^+) \). Then problem (1.1), (1.2) is solvable in the space \( \bar{C}^{n-1,m}_1([a,b]) \).

Corollary 1.2. Let the operator \( P \) be \( \gamma \) consistent with boundary condition (1.2), let inequality (1.28) hold for \( x \in \bar{C}^{m-1}_1([a,b]) \) and almost all \( t \in [a,b] \), where \( \eta(\cdot, \theta) \in \bar{L}^{2n-2m-2,2m-2}_2([a,b]) \) for any \( \theta \in \mathbb{R}^+ \), and

\[
\limsup_{\varrho \to +\infty} \frac{1}{\varrho}\| \eta(\cdot, \varrho) \|_{\bar{L}^{2n-2m-2,2m-2}_2} < \frac{1}{\gamma}.
\]

Then problem (1.1), (1.2) is solvable in the space \( \bar{C}^{n-1,m}_1([a,b]) \).