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THE DIRICHLET BOUNDARY VALUE PROBLEMS FOR STRONGLY SINGULAR HIGHER-ORDER NONLINEAR FUNCTIONAL-DIFFERENTIAL EQUATIONS

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Abstract. The a priori boundedness principle is proved for the Dirichlet boundary value problems for strongly singular higher-order nonlinear functional-differential equations. Several sufficient conditions of solvability of the Dirichlet problem under consideration are derived from the a priori boundedness principle. The proof of the a priori boundedness principle is based on the Agarwal-Kiguradze type theorems, which guarantee the existence of the Fredholm property for strongly singular higher-order linear differential equations with argument deviations under the two-point conjugate and right-focal boundary conditions.

Keywords: higher order functional-differential equation, Dirichlet boundary value problem, strong singularity, Fredholm property, a priori boundedness principle

MSC 2010: 34K06, 34K10

1. Statement of the main results

1.1. Statement of the problem and a survey of the literature. Consider the functional differential equation

$$u^{(n)}(t) = F(u)(t) \tag{1.1}$$

with the two-point boundary conditions

$$u^{(i-1)}(a) = 0 \ \ (i = 1, \ldots, m), \quad u^{(i-1)}(b) = 0 \ \ (i = 1, \ldots, n-m). \tag{1.2}$$

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Here $n \geqslant 2$, m is the integer part of $n/2$, $-\infty < a < b < +\infty$, and the operator F is acting from the set of $(m-1)$-th time continuously differentiable on $]a,b[$ functions to the set $L_{\mathrm{loc}}(]a,b[)$. By $u^{(j-1)}(a)$ $(u^{(j-1)}(b))$ we denote the right (the left) limit of the function $u^{(j-1)}$ at the point $a(b)$.

The problem is singular in the sense that for an arbitrary x the right-hand side of equation (1.41) may have nonintegrable singularities at the points a and b.

Throughout the paper we use the following notation:

- ▷ $\mathbb{R}^+ = [0,+\infty[$;
- ▷ $[x]_+$ the positive part of a number x, that is $[x]_+ = \frac{1}{2}(x+|x|)$;
- ▷ $L_{\mathrm{loc}}(]a,b[)(L_{\mathrm{loc}}(]a,b]))$ is the space of functions $y\colon\ ]a,b[\to \mathbb{R}$, which are integrable on $[a+\varepsilon, b-\varepsilon]$ for arbitrarily small $\varepsilon > 0$;
- ▷ $L_{\alpha,\beta}(]a,b[)(L^2_{\alpha,\beta}(]a,b[))$ is the space of integrable (square integrable) with the weight $(t-a)^\alpha(b-t)^\beta$ functions $y\colon\ ]a,b[\to \mathbb{R}$, with the norm

$$\|y\|_{L_{\alpha,\beta}} = \int_a^b (s-a)^\alpha (b-s)^\beta |y(s)|\,\mathrm{d}s$$
$$\left(\|y\|_{L^2_{\alpha,\beta}} = \left(\int_a^b (s-a)^\alpha (b-s)^\beta y^2(s)\,\mathrm{d}s\right)^{1/2}\right);$$

- ▷ $L([a,b]) = L_{0,0}(]a,b[), L^2([a,b]) = L^2_{0,0}(]a,b[)$;
- ▷ $M(]a,b[)$ is the set of measurable functions $\tau\colon\ ]a,b[\to]a,b[$;
- ▷ $\widetilde{L}^2_{\alpha,\beta}(]a,b[)(\widetilde{L}^2_{\alpha}(]a,b])$ is the Banach space of $y \in L_{\mathrm{loc}}(]a,b[)(L_{\mathrm{loc}}(]a,b]))$ functions, with the norm

$$\|y\|_{\widetilde{L}^2_{\alpha,\beta}} \equiv \max\left\{\left[\int_a^t (s-a)^\alpha \left(\int_s^t y(\xi)\,\mathrm{d}\xi\right)^2 \mathrm{d}s\right]^{1/2} : a \leqslant t \leqslant \frac{a+b}{2}\right\}$$
$$+ \max\left\{\left[\int_t^b (b-s)^\beta \left(\int_t^s y(\xi)\,\mathrm{d}\xi\right)^2 \mathrm{d}s\right]^{1/2} : \frac{a+b}{2} \leqslant t \leqslant b\right\} < +\infty.$$

- ▷ $L_n(]a,b[)$ is the Banach space of $y \in L_{\mathrm{loc}}(]a,b[)$ functions, with the norm

$$\|y\|_{\widetilde{L}^2_{\alpha,\beta}} = \sup\left\{[(s-a)(b-t)]^{m-1/2} \int_s^t (\xi-a)^{n-2m}|y(\xi)|\,\mathrm{d}\xi\colon\ a < s \leqslant t < b\right\} < +\infty.$$

- ▷ $C^{n-1}_{\mathrm{loc}}(]a,b[), (\widetilde{C}^{n-1}_{\mathrm{loc}}(]a,b[))$ is the space of functions $y\colon\ ]a,b[\to \mathbb{R}$ which are continuous (absolutely continuous) together with $y', y'', \ldots, y^{(n-1)}$ on $[a+\varepsilon, b-\varepsilon]$ for arbitrarily small $\varepsilon > 0$.
- ▷ $\widetilde{C}^{n-1,m}(]a,b[)$ is the space of functions $y \in \widetilde{C}^{n-1}_{\mathrm{loc}}(]a,b[)$, such that

$$\int_a^b |x^{(m)}(s)|^2\,\mathrm{d}s < +\infty. \tag{1.3}$$

▷ $C_1^{m-1}(]a,b[)$ is the Banach space of functions $y \in C_{\mathrm{loc}}^{m-1}(]a,b[)$, such that

$$\limsup_{t\to a}\frac{|x^{(i-1)}(t)|}{(t-a)^{m-i+1/2}} < +\infty \quad (i=1,\dots,m),$$
$$\limsup_{t\to b}\frac{|x^{(i-1)}(t)|}{(b-t)^{m-i+1/2}} < +\infty \quad (i=1,\dots,n-m), \tag{1.4}$$

with the norm:

$$\|x\|_{C_1^{m-1}} = \sum_{i=1}^{m}\sup\Big\{\frac{|x^{(i-1)}(t)|}{\alpha_i(t)}\colon\ a<t<b\Big\},$$

where $\alpha_i(t) = (t-a)^{m-i+1/2}(b-t)^{m-i+1/2}$.

▷ $\widetilde{C}_1^{m-1}(]a,b[)$ is the Banach space of functions $y \in \widetilde{C}_{\mathrm{loc}}^{m-1}(]a,b[)$, such that conditions (1.3) and (1.4) hold, with the norm:

$$\|x\|_{\widetilde{C}_1^{m-1}} = \sum_{i=1}^{m}\sup\Big\{\frac{|x^{(i-1)}(t)|}{\alpha_i(t)}\colon\ a<t<b\Big\} + \Big(\int_a^b |x^{(m)}(s)|^2\,\mathrm{d}s\Big)^{1/2}.$$

▷ $D_n(]a,b[\times\mathbb{R}^+)$ is the set of such functions $\delta\colon\ ]a,b[\times\mathbb{R}^+ \to L_n(]a,b[)$ that $\delta(t,\cdot)\colon\ \mathbb{R}^+\to\mathbb{R}^+$ is nondecreasing for every $t\in\,]a,b[$, and $\delta(\cdot,\varrho)\in L_n(]a,b[)$ for any $\varrho\in\mathbb{R}^+$.

▷ $D_{2n-2m-2,2m-2}(]a,b[\times\mathbb{R}^+)$ is the set of such functions $\delta\colon\ ]a,b[\times\mathbb{R}^+ \to \widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[)$ that $\delta(t,\cdot)\colon\ \mathbb{R}^+\to\mathbb{R}^+$ is nondecreasing for every $t\in\,]a,b[$, and $\delta(\cdot,\varrho)\in\widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[)$ for any $\varrho\in\mathbb{R}^+$.

▷ A solution of problem (1.1), (1.2) is sought in the space $\widetilde{C}^{n-1,m}(]a,b[)$.

The singular ordinary differential and functional-differential equations have been studied with sufficient completeness under different boundary conditions, see for example [1], [3], [4], [5], [6], [7], [8], [9], [11], [12], [13], [14], [16], [21], [22], [23], [24], [25] and the references cited therein. But the equation (1.1), even under the boundary condition (1.2), have not been studied in the case when the operator F has the form

$$F(x)(t) = \sum_{j=1}^{m} p_j(t)x^{(j-1)}(\tau_j(t)) + f(x)(t), \tag{1.5}$$

where the singularity of the functions $p_j\colon\ L_{\mathrm{loc}}([a,b])$ is such that the inequalities

$$\int_a^b (s-a)^{n-1}(b-s)^{2m-1}[(-1)^{n-m}p_1(s)]_+\,\mathrm{d}s < +\infty,$$
$$\int_a^b (s-a)^{n-j}(b-s)^{2m-j}|p_j(s)|\,\mathrm{d}s < +\infty \quad (j=2,\dots,m), \tag{1.6}$$

are not fulfilled (in this case we say that the linear part of the operator F is strongly singular), the operator f is continuously acting from $C_1^{m-1}(]a,b[)$ to $L_{\widetilde{L}^2_{2n-2m-2,2m-2}}(]a,b[)$, and the inclusion

$$\sup\{f(x)(t)\colon\ \|x\|_{C_1^{m-1}} \leqslant \varrho\} \in \widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[) \tag{1.7}$$

holds. The first step in studying the differential equations with strong singularities was made by R. P. Agarwal and I. Kiguradze in the article [2], where the linear ordinary differential equations under conditions (1.2), in the case when the functions p_j have strong singularities at the points a and b, are studied. Also the ordinary differential equations with strong singularities under two-point boundary conditions are studied in the articles of I. Kiguradze [10], [19], and N. Partsvania [20]. In the papers [18], [15] these results are generalized to linear differential equations with deviating arguments, i.e., the Agarwal-Kiguradze type theorems, which guarantee Fredholm's property for linear differential equations with deviating arguments are proved.

In this paper, on the bases of articles [2] and [17] we prove the a priori boundedness principle for the problem (1.1), (1.2) in the case when the operator has the form (1.5).

Now we introduce some results from the articles [18], [15], which we need for this work. Consider the equation

$$u^{(n)}(t) = \sum_{j=1}^{m} p_j(t)u^{(j-1)}(\tau_j(t)) + q(t) \quad \text{for } a < t < b. \tag{1.8}$$

For problem (1.8), (1.2) we assume, that when $n = 2m$, then the conditions

$$p_j \in L_{\mathrm{loc}}(]a,b[) \quad (j = 1,\dots,m) \tag{1.9}$$

are fulfilled and when $n = 2m+1$, along with (1.9), the condition

$$\limsup_{t\to b} \left|(b-t)^{2m-1}\int_{t_1}^{t} p_1(s)\,\mathrm{d}s\right| < +\infty \quad \Bigl(t_1 = \frac{a+b}{2}\Bigr) \tag{1.10}$$

holds.

By $h_j\colon\ ]a,b[\times]a,b[\ \to \mathbb{R}_+$ and $f_j\colon\ [a,b]\times M(]a,b[) \to C_{\mathrm{loc}}(]a,b[\times]a,b[)$ $(j = 1,\dots,m)$ we denote the functions and operators, respectively, defined by the equalities

$$\begin{aligned} h_1(t,s) &= \left|\int_s^t (\xi-a)^{n-2m}[(-1)^{n-m}p_1(\xi)]_+\,\mathrm{d}\xi\right|, \\ h_j(t,s) &= \left|\int_s^t (\xi-a)^{n-2m}p_j(\xi)\,\mathrm{d}\xi\right| \quad (j = 2,\dots,m), \end{aligned} \tag{1.11}$$

and

$$(1.12)\qquad f_j(c,\tau_j)(t,s) = \left|\int_s^t (\xi-a)^{n-2m}|p_j(\xi)|\left|\int_\xi^{\tau_j(\xi)} (\xi_1-c)^{2(m-j)}\,\mathrm{d}\xi_1\right|^{1/2}\mathrm{d}\xi\right|.$$

Let $k=2k_1+1$ ($k_1\in\mathbb{N}$), then we denote

$$k!! = \begin{cases} 1 & \text{for } k\leqslant 0,\\ 1\cdot 3\cdot 5\cdot\ldots\cdot k & \text{for } k\geqslant 1.\end{cases}$$

Now we can introduce the main theorem of the paper [18].

Theorem 1.1. *Let there exist numbers* $t^*\in\,]a,b[$, $l_{kj}>0$, $\bar l_{kj}\geqslant 0$, *and* $\gamma_{kj}>0$ ($k=0,1$; $j=1,\ldots,m$) *such that along with*

$$(1.13)\quad B_0\equiv\sum_{j=1}^m\Big(\frac{(2m-j)2^{2m-j+1}l_{0j}}{(2m-1)!!(2m-2j+1)!!}+\frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}}\bar l_{0j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{0j}}}\Big) < \frac{1}{2},$$

$$(1.14)\quad B_1\equiv\sum_{j=1}^m\Big(\frac{(2m-j)2^{2m-j+1}l_{1j}}{(2m-1)!!(2m-2j+1)!!}+\frac{2^{2m-j-1}(b-t^*)^{\gamma_{0j}}\bar l_{1j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{1j}}}\Big) < \frac{1}{2},$$

the conditions

$$(1.15)\qquad (t-a)^{2m-j}h_j(t,s)\leqslant l_{0j},\quad (t-a)^{m-\gamma_{0j}-1/2}f_j(a,\tau_j)(t,s)\leqslant \bar l_{0j}$$

for $a<t\leqslant s\leqslant t^*$, *and*

$$(1.16)\qquad (b-t)^{2m-j}h_j(t,s)\leqslant l_{1j},\quad (b-t)^{m-\gamma_{1j}-1/2}f_j(b,\tau_j)(t,s)\leqslant \bar l_{1j}$$

for $t^*\leqslant s\leqslant t<b$ *hold. Then problem* (1.8), (1.2) *is uniquely solvable in the space* $\widetilde C^{n-1,m}(]a,b[)$.

Also, in [15] the following theorem is proved:

Theorem 1.2. *Let all the conditions of Theorem* 1.1 *be satisfied. Then the unique solution* u *of problem* (1.8), (1.2) *for every* $q \in \widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[)$ *admits the estimate*

$$\|u^{(m)}\|_{L^2} \leqslant r\|q\|_{\widetilde{L}^2_{2n-2m-2,2m-2}}, \tag{1.17}$$

with

$$r = \frac{2^m(1+b-a)(2n-2m-1)}{(\nu_n - 2\max\{B_0,B_1\})(2m-1)!!}, \quad \nu_{2m}=1,\ \nu_{2m+1}=\frac{2m+1}{2},$$

and thus the constant $r > 0$ *depends only on the numbers* l_{kj}, $\bar{l}_{kj}$, γ_{kj} $(k=1,2;$ $j=1,\dots,m)$, *and* a, b, t^*, n.

Remark 1.1. Under the conditions of Theorem 1.2, for every

$$q \in \widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[)$$

the unique solution u of problem (1.8), (1.2) admits the estimate

$$\|u^{(m)}\|_{\widetilde{C}_1^{m-1}} \leqslant r_n\|q\|_{\widetilde{L}^2_{2n-2m-2,2m-2}}, \tag{1.18}$$

with

$$r_n = \left(1+\sum_{j=1}^{m}\frac{2^{m-j+1/2}}{(m-j)!(2m-2j+1)^{1/2}(b-a)^{m-j+1/2}}\right) \times \frac{2^m(1+b-a)(2n-2m-1)}{(\nu_n - 2\max\{B_0,B_1\})(2m-1)!!}.$$

1.2. Theorems on solvability of problem (1.1)**,** (1.2)**.**

Define an operator $P\colon C_1^{m-1}(]a,b[) \times C_1^{m-1}(]a,b[) \to L_{\mathrm{loc}}(]a,b[)$ by the equality

$$P(x,y)(t) = \sum_{j=1}^{m} p_j(x)(t)y^{(j-1)}(\tau_j(t)) \quad \text{for } a<t<b \tag{1.19}$$

where $p_j\colon C_1^{m-1}(]a,b[) \to L_{\mathrm{loc}}(]a,b[)$, and $\tau_j \in M(]a,b[)$. Also, for any $\gamma>0$ define a set A_γ by the relation

$$A_\gamma = \{x \in \widetilde{C}_1^{m-1}(]a,b[)\colon \|x\|_{\widetilde{C}_1^{m-1}} \leqslant \gamma\}. \tag{1.20}$$

For formulating the a priori boundedness principle we have to introduce

Definition 1.1. Let γ_0 and γ be positive numbers. We say that the continuous operator $P\colon C_1^{m-1}(]a,b[)\times C_1^{m-1}(]a,b[)\to L_n(]a,b[)$ is γ_0,γ consistent with boundary condition (1.2) if:

(i) For any $x\in A_{\gamma_0}$ and almost all $t\in\,]a,b[$ the inequality

$$\sum_{j=1}^{m}|p_j(x)(t)x^{(j-1)}(\tau_j(t))|\leqslant\delta(t,\|x\|_{\widetilde{C}_1^{m-1}})\|x\|_{\widetilde{C}_1^{m-1}} \tag{1.21}$$

holds, where $\delta\in D_n(]a,b[\times\mathbb{R}^+)$.

(ii) For any $x\in A_{\gamma_0}$ and $q\in\widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[)$ the equation

$$y^{(n)}(t)=\sum_{j=1}^{m}p_j(x)(t)y^{(j-1)}(\tau_j(t))+q(t) \tag{1.22}$$

under boundary conditions (1.2) has a unique solution y in the space $\widetilde{C}^{n-1,m}(]a,b[)$ and

$$\|y\|_{\widetilde{C}_1^{m-1}}\leqslant\gamma\|q\|_{\widetilde{L}^2_{2n-2m-2,2m-2}}. \tag{1.23}$$

Definition 1.2. We say that the operator P is γ consistent with boundary condition (1.2), if the operator P is γ_0,γ consistent with boundary condition (1.2) for any $\gamma_0>0$.

In the sequel it will always be assumed that the operator F_p defined by equality

$$F_p(x)(t)=\left|F(x)(t)-\sum_{j=1}^{m}p_j(x)(t)x^{(j-1)}(\tau_j(t))(t)\right|$$

is continuously acting from $C_1^{m-1}(]a,b[)$ to $L_{\widetilde{L}^2_{2n-2m-2,2m-2}}(]a,b[)$, and

$$\widetilde{F}_p(t,\varrho)\equiv\sup\{F_p(x)(t)\colon\ \|x\|_{C_1^{m-1}}\leqslant\varrho\}\in\widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[) \tag{1.24}$$

for each $\varrho\in[0,+\infty[$.

Then the following theorem is valid

Theorem 1.3. *Let the operator P be γ_0, γ consistent with boundary condition* (1.2), *and let there exist a positive number $\varrho_0 \leqslant \gamma_0$, such that*

$$\|\widetilde{F}_p(\cdot, \min\{2\varrho_0, \gamma_0\})\|_{\widetilde{L}^2_{2n-2m-2,2m-2}} \leqslant \frac{\gamma_0}{\gamma}. \tag{1.25}$$

Let, moreover, for any $\lambda \in]0,1[$ an arbitrary solution $x \in A_{\gamma_0}$ of the equation

$$x^{(n)}(t) = (1-\lambda)P(x,x)(t) + \lambda F(x)(t) \tag{1.26}$$

under the conditions (1.2) *admit the estimate*

$$\|x\|_{\widetilde{C}_1^{m-1}} \leqslant \varrho_0. \tag{1.27}$$

Then problem (1.1), (1.2) *is solvable in the space $\widetilde{C}^{n-1,m}(]a,b[)$.*

Theorem 1.3 with $\varrho_0 = \gamma_0$ immediately yields

Corollary 1.1. *Let the operator P be γ_0, γ consistent with boundary condition* (1.2), *and*

$$\Big|F(x)(t) - \sum_{j=1}^{m} p_j(x)(t)x^{(j-1)}(\tau_j(t))(t)\Big| \leqslant \eta(t, \|x\|_{\widetilde{C}_1^{m-1}}) \tag{1.28}$$

for $x \in A_{\gamma_0}$ and almost all $t \in]a,b[$, and

$$\|\eta(\cdot, \gamma_0)\|_{\widetilde{L}^2_{2n-2m-2,2m-2}} \leqslant \frac{\gamma_0}{\gamma}, \tag{1.29}$$

where $\eta \in D_{2n-2m-2,2m-2}(]a,b[\times \mathbb{R}^+)$. Then problem (1.1), (1.2) *is solvable in the space $\widetilde{C}^{n-1,m}(]a,b[)$.*

Corollary 1.2. *Let the operator P be γ consistent with boundary condition* (1.2), *let inequality* (1.28) *hold for $x \in \widetilde{C}_1^{m-1}(]a,b[)$ and almost all $t \in]a,b[$, where $\eta(\cdot, \varrho) \in \widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[)$ for any $\varrho \in \mathbb{R}^+$, and*

$$\limsup_{\varrho \to +\infty} \frac{1}{\varrho}\|\eta(\cdot, \varrho)\|_{\widetilde{L}^2_{2n-2m-2,2m-2}} < \frac{1}{\gamma}. \tag{1.30}$$

Then problem (1.1), (1.2) *is solvable in the space $\widetilde{C}^{n-1,m}(]a,b[)$.*