THE DIRICHLET BVP FOR THE SECOND ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATION AT RESONANCE

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Abstract. Efficient sufficient conditions are established for the solvability of the Dirichlet problem

$$u''(t) = p(t)u(t) + f(t, u(t)) + h(t) \quad \text{for} \quad a \le t \le b,$$

$$u(a) = 0, \quad u(b) = 0,$$

where $h, p \in L([a, b]; R)$ and $f \in K([a, b] \times R; R)$, in the case where the linear problem

$$u''(t) = p(t)u(t), \quad u(a) = 0, \quad u(b) = 0$$

has nontrivial solutions.

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1. Introduction

Consider on the set $I = [a, b]$ the second order nonlinear ordinary differential equation

$$u''(t) = p(t)u(t) + f(t, u(t)) + h(t) \quad \text{for} \quad t \in I \tag{1.1}$$

with the boundary conditions

$$u(a) = 0, \quad u(b) = 0, \tag{1.2}$$

where $h, p \in L(I; R)$ and $f \in K(I \times R; R)$.

By a solution of the problem (1.1), (1.2) we understand a function $u \in \widetilde{C}'(I, R)$, which satisfies the equation (1.1) almost everywhere on I and satisfies the conditions (1.2).

Along with (1.1), (1.2) we consider the homogeneous problem

$$w''(t) = p(t)w(t) \quad \text{for} \quad t \in I, \tag{1.3}$$

$$w(a) = 0, \quad w(b) = 0. \tag{1.4}$$

At present, the foundations of the general theory of two-point boundary value problems are already laid and problems of this type are studied by many authors and investigated in detail (see, for instance, [1], [4], [5], [8], [12],[13], [14]-[16], [17] and references therein). On the other hand, in all of these works, only the case when the homogeneous problem (1.3), (1.4) has only a trivial solution is studied. The case when the problem (1.3), (1.4) has also the nontrivial solution is still little investigated and in the majority of articles, the authors study the case with p constant in the equation (1.1), i.e., when the problem (1.1), (1.2) and the equation (1.3) are of type

$$u''(t) = -\lambda^2 u(t) + f(t, u(t)) + h(t) \quad \text{for} \quad t \in [0, \pi], \tag{1.5}$$

$$u(0) = 0, \quad u(\pi) = 0, \tag{1.6}$$

and

$$w''(t) = -\lambda^2 w(t) \quad \text{for} \quad t \in [0, \pi] \tag{1.7}$$

respectively, with $\lambda = 1$. (see, for instance, [2], [3], [4], [6]-[11], [14]-[16], and references therein).

In the present paper, we study solvability of the problem (1.1), (1.2) in the case when the function $p \in L(I; R)$ is not necessarily constant, under the assumption that the homogeneous problem (1.3), (1.4) has the nontrivial solution with an arbitrary number of zeroes. For the equation (1.7), this is the case when λ is not necessarily the first eigenvalue of the problem (1.7), (1.4), with $a = 0$, $b = \pi$.

The obtained results are new and generalize some well-known results (see,[2], [3], [4], [6], [10]).

The following notation is used throughout the paper: N is the set of all natural numbers. R is the set of all real numbers, $R_+ = [0, +\infty[$. $C(I; R)$ is the Banach space of continuous functions $u : I \to R$ with the norm $\|u\|_C = \max\{|u(t)| : t \in I\}$. $\widetilde{C}'(I; R)$ is the set of functions $u : I \to R$ which are absolutely continuous together with their first derivatives. $L(I; R)$ is the Banach space of Lebesgue integrable functions $p : I \to R$ with the norm $\|p\|_L = \int_a^b |p(s)|ds$.

$K(I \times R; R)$ is the set of functions $f : I \times R \to R$ satisfying the Carathéodory conditions, i.e., $f(\cdot, x) : I \to R$ is a measurable function for all $x \in R$, $f(t, \cdot) : R \to R$ is a continuous function for almost all $t \in I$, and for every $r > 0$ there exists $q_r \in L(I; R_+)$ such that $|f(t,x)| \le q_r(t)$ for almost all $t \in I$, $|x| \le r$.

Having $w : I \to R$, we put: $N_w \overset{def}{=} \{t \in]a,b[: w(t) = 0\}$,

$$\Omega_w^+ \overset{def}{=} \{t \in I : w(t) > 0\},$$
$$\Omega_w^- \overset{def}{=} \{t \in I : w(t) < 0\},$$

and $[w(t)]_+ = (|w(t)| + w(t))/2$, $[w(t)]_- = (|w(t)| - w(t))/2$ for $t \in I$.

Definition 1.1. Let A be a finite (eventually empty) subset of I. We say that $f \in E(A)$, if $f \in K(I \times R; R)$ and, for any measurable set $G \subseteq I$ and an arbitrary constant $r > 0$, we can choose $\varepsilon > 0$ such that if

$$\int_G |f(s,x)| ds \ne 0 \quad \text{for} \quad x \ge r \ (x \le -r)$$

then

$$\int_{G \setminus U_\varepsilon} |f(s,x)| ds - \int_{U_\varepsilon} |f(s,x)| ds \ge 0 \quad \text{for} \quad x \ge r \ (x \le -r),$$

where $U_\varepsilon = I \cap \left(\bigcup_{k=1}^{n}]t_k - \varepsilon/2n,\ t_k + \varepsilon/2n[\right)$ if $A = \{t_1, t_2, ..., t_n\}$, and $U_\varepsilon = \emptyset$ if $A = \emptyset$.

Remark 1.1. If $f \in K(I \times R; R)$ then $f \in E(\emptyset)$.

Remark 1.2. It is clear that if $f_1 \in L(I; R)$ and $f(t,x) \overset{def}{\equiv} f_1(t)$ then $f \in E(A)$ for every finite set $A \subset I$.

Remark 1.3. It is clear that if $f(t,x) \overset{def}{\equiv} f_0(t) g_0(x)$, where $f_0 \in L(I; R)$ and $g_0 \in C(I; R)$, then $f \in E(A)$ for every finite set $A \subset I$.

The example below shows that there exists a function $f \in K(I \times R; R)$ such that $f \notin E(\{t_1, ..., t_k\})$ for some points $t_1, ..., t_k \in I$.

Example 1.1. Let $f(t,x) = |t|^{-1/2} g(t,x)$ for $t \in [-1, 0[\cup]0, 1]$, $x \in R$, and $f(0, .) \equiv 0$, where $g(-t, x) = g(t,x)$ for $t \in]-1, 1]$, $x \in R$, and

$$g(t,x) = \begin{cases} x & \text{for} \quad x \le 1/t,\ t > 0 \\ 1/t & \text{for} \quad x > 1/t,\ t > 0 \end{cases}.$$

Then $f \in K([0,1] \times R; R)$ and it is clear that $f \notin E(\{0\})$ because, for every $\varepsilon > 0$, if $x \ge 1/\varepsilon$ then $\int_\varepsilon^1 f(s,x) ds - \int_0^\varepsilon f(s,x) ds = 4(\varepsilon^{-1/2} - x^{1/2}) - 2 < 0$.

2. Main results

Theorem 2.1. *Let w be a nonzero solution of the problem* (1.3), (1.4),

$$N_w = \emptyset, \tag{2.1}$$

there exist a constant $r > 0$, functions $f^-, f^+ \in L(I; R_+)$ and $g, h_0 \in L(I;]0, +\infty[\,)$ such that

$$f(t,x)\mathrm{sgn}x \le g(t)|x| + h_0(t) \quad \text{for} \quad |x| \ge r \tag{2.2}$$

and

$$\begin{gathered} f(t,x) \le -f^-(t) \quad \text{for} \quad x \le -r, \\ f^+(t) \le f(t,x) \quad \text{for} \quad x \ge r \end{gathered} \tag{2.3}$$

on I. Let, moreover, there exist $\varepsilon > 0$ such that

$$-\int_a^b f^-(s)|w(s)|ds + \varepsilon||\gamma_r||_L \le -\int_a^b h(s)|w(s)|ds \le$$

$$\le \int_a^b f^+(s)|w(s)|ds - \varepsilon||\gamma_r||_L, \tag{2.4$_1$}$$

where

$$\gamma_r(t) = \sup\{|f(t,x)| : |x| \le r\}. \tag{2.5}$$

Then the problem (1.1), (1.2) *has at least one solution.*

Example 2.2. It follows from Theorem 2.1 that the equation

$$u''(t) = -\lambda^2 u(t) + \sigma|u(t)|^\alpha \mathrm{sgn}u(t) + h(t) \quad \text{for} \quad 0 \le t \le \pi \tag{2.6}$$

where $\sigma = 1$, $\lambda = 1$, and $\alpha \in]0,1]$, with the conditions (1.6) has at least one solution for every $h \in L([0,\pi], R)$.

Theorem 2.2. *Let w be a nonzero solution of the problem* (1.3), (1.4), *condition* (2.1) *hold, there exist a constant $r > 0$, functions $f^-, f^+ \in L(I; R_+)$ and $q \in K(I \times R; R_+)$ such that q is non-decreasing in the second argument,*

$$|f(t,x)| \le q(t,x) \quad \text{for} \quad |x| \ge r, \tag{2.7}$$

$$\begin{gathered} f^-(t) \le f(t,x) \quad \text{for} \quad x \le -r, \\ f(t,x) \le -f^+(t) \quad \text{for} \quad x \ge r \end{gathered} \tag{2.8}$$

on I, and

$$\lim_{|x| \to +\infty} \frac{1}{x} \int_a^b q(s,x)ds = 0. \tag{2.9}$$

Let, moreover, there exist $\varepsilon > 0$ *such that*

$$-\int_a^b f^-(s)|w(s)|ds + \varepsilon||\gamma_r||_L \le \int_a^b h(s)|w(s)|ds \le$$

$$\le \int_a^b f^+(s)|w(s)|ds - \varepsilon||\gamma_r||_L, \tag{2.4$_2$}$$

where γ_r *is defined by* (2.5). *Then the problem* (1.1), (1.2) *has at least one solution.*

Example 2.3. From Theorem 2.2 it follows that the problem (2.6), (1.6) with $\sigma = -1$, $\lambda = 1$, and $\alpha \in]0, 1[$ has at least one solution for every $h \in L([0, \pi]; R)$.

Remark 2.4. If $f \not\equiv 0$ the condition (2.4$_i$) of Theorem 2.i ($i = 1, 2$) can be replaced by

$$\begin{aligned} -\int_a^b f^-(s)|w(s)|ds < (-1)^i \int_a^b h(s)|w(s)|ds < \\ < \int_a^b f^+(s)|w(s)|ds, \end{aligned} \tag{2.10$_i$}$$

because, from (2.10$_i$) there follows the existence of a constant $\varepsilon > 0$ such that the condition (2.4$_i$) is satisfied.

Theorem 2.3. *Let* $i \in \{0, 1\}$, w *be a nonzero solution of the problem* (1.3), (1.4), $f \in E(N_w)$, *there exist a constant* $r > 0$ *such that the function* $(-1)^i f$ *is non-decreasing in the second argument for* $|x| \ge r$,

$$(-1)^i f(t, x)\text{sgn}x \ge 0 \quad \textit{for} \quad t \in I, \ |x| \ge r, \tag{2.11}$$

$$\int_{\Omega_w^+} |f(s, r)|ds + \int_{\Omega_w^-} |f(s, -r)|ds \ne 0, \tag{2.12}$$

and

$$\lim_{|x| \to +\infty} \frac{1}{|x|} \int_a^b |f(s, x)|ds = 0. \tag{2.13}$$

Then there exists $\delta > 0$ *such that the problem* (1.1), (1.2) *has at least one solution for every* h *satisfying the condition*

$$\left| \int_a^b h(s)w(s)ds \right| < \delta. \tag{2.14}$$

Corollary 2.1. *Let the assumptions of Theorem* 2.3 *be satisfied and*

$$\int_a^b h(s)w(s)ds = 0. \tag{2.15}$$

Then the problem (1.1), (1.2) *has at least one solution.*

Example 2.4. From Theorem 2.3 it follows that the problem (2.6), (1.6) with $\sigma \in \{-1, 1\}$, $\lambda \in N$, and $\alpha \in]0, 1[$ has at least one solution if $h \in L([0, \pi], R)$ is such that $\int_0^\pi h(s) \sin \lambda s ds = 0$.

Theorem 2.4. *Let* $i \in \{0, 1\}$, w *be a nonzero solution of the problem* (1.3),(1.4), $f(t, x) \overset{def}{\equiv} f_0(t)g_0(x)$ *with* $f_0 \in L(I; R_+)$, $g_0 \in C(R; R)$, *there exist a constant* $r > 0$ *such that* $(-1)^i g_0$ *is non-decreasing for* $|x| \geq r$ *and*

$$(-1)^i g_0(x)\text{sgn}x \geq 0 \qquad \textit{for} \quad |x| \geq r. \tag{2.16}$$

Let, moreover,

$$|g_0(r)| \int_{\Omega_w^+} f_0(s)ds + |g_0(-r)| \int_{\Omega_w^-} f_0(s)ds \neq 0 \tag{2.17}$$

and

$$\lim_{|x| \to +\infty} |g_0(x)| = +\infty, \qquad \lim_{|x| \to +\infty} \frac{g_0(x)}{x} = 0. \tag{2.18}$$

Then, for every $h \in L(I; R)$, *the problem* (1.1), (1.2) *has at least one solution.*

Example 2.5. From Theorem 2.4 it follows that the equation

$$u''(t) = p_0(t)u(t) + p_1(t)|u(t)|^\alpha \text{sgn}u(t) + h(t) \quad \text{for} \quad t \in I, \tag{2.19}$$

where $\alpha \in]0, 1[$ and $p_0, p_1, h \in L(I; R)$, with the conditions (1.2) has at least one solution provided that $p_1(t) > 0$ for $t \in I$.

Theorem 2.5. *Let* $i \in \{0, 1\}$ *and* w *be a nonzero solution of the problem* (1.3), (1.4). *Let, moreover, there exist constants* $r > 0$, $\varepsilon > 0$, *and functions* $\alpha, f^+, f^- \in L(I; R_+)$ *such that the conditions*

$$\begin{gathered} (-1)^i f(t, x) \leq -f^-(t) \quad \textit{for} \quad x \leq -r, \\ f^+(t) \leq (-1)^i f(t, x) \quad \textit{for} \quad x \geq r, \end{gathered} \tag{2.20$_i$}$$

$$\sup\{|f(t, x)| : x \in R\} \leq \alpha(t) \tag{2.21}$$

hold on I, and let

$$
\begin{aligned}
(2.22_i) \qquad & -\int_a^b (f^+(s)[w(s)]_- + f^-(s)[w(s)]_+)ds + \varepsilon||\alpha||_L \le \\
& \le (-1)^{i+1}\int_a^b h(s)w(s)ds \le \\
& \le \int_a^b (f^-(s)[w(s)]_- + f^+(s)[w(s)]_+)ds - \varepsilon||\alpha||_L.
\end{aligned}
$$

Then the problem (1.1), (1.2) *has at least one solution.*

Remark 2.5. If $f \not\equiv 0$ then the condition (2.22_i) $(i = 1, 2)$ of Theorem 2.5 can be replaced by

$$
\begin{aligned}
(2.23_i) \qquad & -\int_a^b (f^+(s)[w(s)]_- + f^-(s)[w(s)]_+)ds < \\
& < (-1)^{i+1}\int_a^b h(s)w(s)ds < \\
& < \int_a^b (f^-(s)[w(s)]_- + f^+(s)[w(s)]_+)ds.
\end{aligned}
$$

because from (2.23_i) there follows the existence of a constant $\varepsilon > 0$ such that the condition (2.22_i) is satisfied.

Remark 2.6. If $\widetilde{f}(t) = \min\{f^+(t), f^-(t)\}$ then the condition (2.22_i) of Theorem 2.5 can be replaced by

$$
\left|\int_a^b h(s)w(s)ds\right| \le \int_a^b \widetilde{f}(s)|w(s)|ds - \varepsilon||\alpha||_L.
$$

Example 2.6. From Theorem 2.5 it follows that the equation

$$
(2.24) \qquad u''(t) = -\lambda^2 u(t) + \frac{|u(t)|^\alpha}{1 + |u(t)|^\alpha}\mathrm{sgn}u(t) + h(t) \quad \text{for} \quad 0 \le t \le \pi,
$$

where $\lambda \in N$ and $\alpha \in]0, +\infty[$, with the conditions (1.6) has at least one solution if $h \in L([0, \pi], R)$ is such that $|h(t)| < 1$ for $0 \le t \le \pi$.

3. Problem (1.5), (1.6).

Throughout this section we will assume that $a = 0$, $b = \pi$, and $I = [0, \pi]$. Since the functions $\beta \sin \lambda t$ $(\beta \in R)$ are nontrivial solutions of the problem (1.7), (1.4), from Theorems 2.1–2.5 it immediately follows:

Corollary 3.2. *Let $\lambda = 1$ and all the assumptions of Theorem* 2.1 (*resp. Theorem* 2.2) *except* (2.1) *be fulfilled with $w(t) = \sin t$. Then the problem* (1.5), (1.6) *has at least one solution.*

Now, note that

$$N_{\sin\lambda t}=\begin{cases}\emptyset & \text{for}\quad \lambda=1\\ \{\pi n/\lambda : n=1,...,\lambda-1\} & \text{for}\quad \lambda\geq 2\end{cases}.$$

Corollary 3.3. *Let $i\in\{0,1\}$, $\lambda\in N$, $f\in E(N_{\sin\lambda t})$, there exist a constant $r>0$ such that the function $(-1)^i f$ is non-decreasing in the second argument for $|x|\geq r$, and let the conditions* (2.11)–(2.13) *be fulfilled with $w(t)=\sin\lambda t$. Then there exists $\delta>0$ such that the problem* (1.5), (1.6) *has at least one solution for every $h\in L(I;R)$ satisfying the condition $|\int_0^\pi h(s)\sin\lambda s ds|<\delta$.*

Corollary 3.4. *Let $i\in\{0,1\}$, $\lambda\in N$, and let all the assumptions of Theorem* 2.4 *be fulfilled with $w(t)=\sin\lambda t$. Then, for any $h\in L(I;R)$, the problem* (1.5), (1.6) *has at least one solution.*

Corollary 3.5. *Let $i\in\{0,1\}$, $\lambda\in N$ and let there exist a constant $r>0$ such that* (2.20_i)–(2.22_i) *be fulfilled with $w(t)=\sin\lambda t$. Then the problem* (1.5), (1.6) *has at least one solution.*

Remark 3.7. If $f\not\equiv 0$ then in Corollary 3.2 (resp. Corollary 3.5), the condition (2.4_i) (resp. (2.22_i)) can be replaced by the condition (2.10_i) (resp. (2.23_i)) with $w(t)=\sin t$ (resp. $w(t)=\sin\lambda t$).

4. Auxiliary propositions

Let $u_n\in\widetilde{C}'(I;R)$, $\|u_n\|_C\neq 0$ $(n\in N)$, w be an arbitrary solution of the problem (1.3), (1.4), and $r>0$. Then, for every $n\in N$, we define:

$$A_{n,1}\overset{def}{=}\{t\in I:|u_n(t)|\leq r\},\qquad A_{n,2}\overset{def}{=}\{t\in I:|u_n(t)|>r\},$$

$$B_{n,i}\overset{def}{=}\{t\in A_{n,2}:\operatorname{sgn}u_n(t)=(-1)^{i-1}\operatorname{sgn}w(t)\}\quad(i=1,2),$$

$$C_{n,1}\overset{def}{=}\{t\in A_{n,2}:|w(t)|\geq 1/n\},\qquad C_{n,2}\overset{def}{=}\{t\in A_{n,2}:|w(t)|<1/n\},$$

$$D_n\overset{def}{=}\{t\in I:|w(t)|>r||u_n||_C^{-1}+1/2n\},$$

$$A^{\pm}_{n,2}\overset{def}{=}\{t\in A_{n,2}:\pm u_n(t)>r\},\quad B^{\pm}_{n,i}\overset{def}{=}A^{\pm}_{n,2}\cap B_{n,i},$$

$$C^{\pm}_{n,i}\overset{def}{=}A^{\pm}_{n,2}\cap C_{n,i}\quad(i=1,2),\ D^{\pm}_n\overset{def}{=}\{t\in I:\pm w(t)>r||u_n||_C^{-1}+1/2n\},$$

From these definitions it is clear that, for any $n\in N$, we have

$$A_{n,1}\cap A_{n,2}=\emptyset, A^+_{n,2}\cap A^-_{n,2}=\emptyset,\quad B_{n,1}\cap B_{n,2}=\emptyset,\ C_{n,1}\cap C_{n,2}=\emptyset,$$

$$D^+_n\cap D^-_n=\emptyset,\ B^+_{n,2}\cap B^-_{n,2}=\emptyset,\ C^+_{n,i}\cap C^-_{n,i}=\emptyset\ (i=1,2),\tag{4.1}$$

and

$$A_{n,1}\cup A_{n,2}=I,\ A^+_{n,2}\cup A^-_{n,2}=A_{n,2},\ B_{n,1}\cup B_{n,2}=A_{n,2}\setminus N_w,$$

$$\begin{gathered}C_{n,1}\cup C_{n,2}=A_{n,2},\ B^+_{n,2}\cup B^-_{n,2}=B_{n,2},\ C^{\pm}_{n,1}\cup C^{\pm}_{n,2}=A^{\pm}_{n,2},\\ C^+_{n,i}\cup C^-_{n,i}=C_{n,i}\ (i=1,2),\ D^+_n\cup D^-_n=D_n.\end{gathered}\tag{4.2}$$

Lemma 4.1. *Let* $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$, $r > 0$, w *be an arbitrary nonzero solution of the problem* (1.3), (1.4), *and*

$$||u_n||_C \geq 2rn \quad \textit{for} \quad n \in N, \tag{4.3}$$

$$||v_n - w||_C \leq 1/2n \quad \textit{for} \quad n \in N, \tag{4.4}$$

where $v_n(t) = u_n(t)||u_n||_C^{-1}$. *Then, for any* $n_0 \in N$, *we have*

$$D^+_{n_0} \subset A^+_{n,2}, \quad D^-_{n_0} \subset A^-_{n,2} \quad \textit{for} \quad n \geq n_0, \tag{4.5}$$

$$C^+_{n_0,1} \subset D^+_n \quad C^-_{n_0,1} \subset D^-_n \quad \textit{for} \quad n \geq n_0. \tag{4.6}$$

Moreover

$$\lim_{n\to+\infty} \mathrm{mes} A_{n,1} = 0, \qquad \lim_{n\to+\infty} \mathrm{mes} A_{n,2} = \mathrm{mes} I, \tag{4.7}$$

$$C_{n,1} \subset B_{n,1}, \qquad B_{n,2} \subset C_{n,2}, \tag{4.8}$$

$$B^+_{n,2} \subset C^+_{n,2}, \qquad B^-_{n,2} \subset C^-_{n,2}, \tag{4.9}$$

$$C^+_{n,1} \subset B^+_{n,1}, \qquad C^-_{n,1} \subset B^-_{n,1}, \tag{4.10}$$

$$\begin{gathered} \lim_{n\to+\infty} \mathrm{mes} C_{n,1} = \lim_{n\to+\infty} \mathrm{mes} B_{n,1} = \mathrm{mes} I, \\ \lim_{n\to+\infty} \mathrm{mes} C_{n,2} = \lim_{n\to+\infty} \mathrm{mes} B_{n,2} = 0, \end{gathered} \tag{4.11}$$

$$r < |u_n(t)| \leq ||u_n||_C/2n \qquad \textit{for} \quad t \in B_{n,2}, \tag{4.12}$$

$$|u_n(t)| \geq ||u_n||_C/2n > r \qquad \textit{for} \quad t \in C_{n,1}, \tag{4.13}$$

$$C^\pm_{n,2} = \{t \in A_{n,2} : 0 \leq \pm w(t) < 1/n\}, \tag{4.14$_1$}$$

$$C^\pm_{n,1} \subset \Omega^\pm_w, \quad \lim_{n\to+\infty} \mathrm{mes} C^\pm_{n,1} = \mathrm{mes}\Omega^\pm_w. \tag{4.15}$$

Proof. From the unique solvability of the Cauchy problem for the equation (1.3) it follows that the set N_w is finite. Consequently, we can assume that $N_w = \{t_1, ..., t_k\}$. Let also $t_0 = a$, $t_{k+1} = b$ and $T_n \overset{def}{=} I \cap \left(\bigcup_{i=0}^{k+1} [t_i - 1/n,\ t_i + 1/n] \right)$. We first show that, for every $n_0 \in N$, there exists $n_1 > n_0$ such that

$$A_{n,1} \subseteq T_{n_0} \quad \text{for} \quad n \geq n_1. \tag{4.16}$$

Suppose on the contrary that, for some $n_0 \in N$, there exists the sequence $t'_{n_j} \in A_{n_j,1}$ $(j \in N)$ with $n_j < n_{j+1}$, such that $t'_{n_j} \notin T_{n_0}$ for $j \in N$. Without loss of generality we can assume that $\lim\limits_{j\to+\infty} t'_{n_j} = t'_0$. Then from the conditions (4.3), (4.4), the definition of the set $A_{n,1}$ and the equality $w(t'_0) = (w(t'_0) - w(t'_{n_j})) + (w(t'_{n_j}) - v_{n_j}(t'_{n_j})) + v_{n_j}(t'_{n_j})$, we get $|w(t'_0)| = 0$, i.e., $t'_0 \in \{t_0, t_1, ..., t_{k+1}\}$. But this contradicts the condition $t'_{n_j} \notin T_{n_0}$ and thus (4.16) is true. Since $\lim\limits_{n\to+\infty} \text{mes}T_n = 0$, it follows from (4.2) and (4.16) that (4.7) is valid.

Let $t_0 \in D^+_{n_0}$. Then from (4.4) it follows that

$$\frac{u_n(t_0)}{||u_n||_C} \geq w(t_0) - |v_n(t_0) - w(t_0)| > \frac{r}{||u_{n_0}||_C} + \frac{1}{2n_0} - \frac{1}{2n} \geq \frac{r}{||u_{n_0}||_C}$$

for $n \geq n_0$, and thus $t_0 \in A^+_{n,2}$ for $n \geq n_0$, i.e., $D^+_{n_0} \subset A^+_{n,2}$ for $n \geq n_0$. The second relation of (4.5) can be proved analogously. Now suppose that $t_0 \in C_{n,1}$ and $t_0 \notin B_{n,1}$. Then, in view of (4.1) and (4.2), it is clear that $t_0 \in B_{n,2}$, and thus

$$|v_n(t_0) - w(t_0)| = |v_n(t_0)| + |w(t_0)| > 1/n\,, \tag{4.17}$$

which contradicts (4.4). Consequently, $C_{n,1} \subset B_{n,1}$ for $n \in N$. This, together with the relations $C_{n,2} = A_{n,2} \setminus C_{n,1}$, $B_{n,2} \subseteq A_{n,2} \setminus B_{n,1}$, implies $B_{n,2} \subset C_{n,2}$, i.e., (4.8) holds. The conditions (4.9) and (4.10) follow immediately from (4.8). In view of the fact that $\lim\limits_{n\to+\infty} \text{mes}C_{n,i} = (2-i)\text{mes}I$, from (4.8) we get (4.11). Now, let $t_0 \in B_{n,2}$ and suppose that $|v_n(t_0)| > 1/2n$. Then from (4.4) we obtain the contradiction $1/2n \geq |v_n(t_0) - w(t_0)| = |v_n(t_0)| + |w(t_0)| > 1/2n$. Thus $\dfrac{|u_n(t_0)|}{||u_n||_C} = |v_n(t_0)| \leq \dfrac{1}{2n}$ and using the definitions of the sets $B_{n,2}$ and $A_{n,2}$ we obtain (4.12). Also, from the inequality $\dfrac{|u_n(t)|}{||u_n||_C} = |v_n(t)| \geq |w(t)| - |v_n(t) - w(t)|$ by (4.3), (4.4) and the definition of the sets $C_{n,1}$ and $A_{n,2}$ we obtain (4.13).

Let there exist $t_0 \in C^+_{n,2}$ such that $t_0 \notin \{t \in A_{n,2} : 0 \leq w(t) \leq 1/n\}$. Then from the definition of the sets $C_{n,2}$ and the inclusion $C^+_{n,2} \subset C_{n,2}$ we get $-1/n < w(t) < 0$ and $t_0 \in A^+_{n,2}$. In this case the inequality (4.17) is fulfilled, which contradicts (4.4). Therefore $C^+_{n,2} \subset \{t \in A_{n,2} : 0 \leq w(t) \leq 1/n\}$. Let now $t_0 \in \{t \in A_{n,2} : 0 \leq w(t) \leq 1/n\}$ and $t_0 \notin C^+_{n,2}$. Then from the definition of the set $C_{n,2}$ and (4.2) it is clear that $t_0 \in C^-_{n,2}$, i.e., $t_0 \in A^-_{n,2}$, and that the inequality (4.17) holds, which contradicts (4.4). Therefore $\{t \in A_{n,2} : 0 \leq w(t) \leq 1/n\} \subset C^+_{n,2}$. From the last two inclusions it follows that (4.14_1) holds for $C^+_{n,2}$. From (4.2) and (4.14_1) for $C^+_{n,1}$ it is clear that (4.14_1) is true for $C^-_{n,1}$ too. Analogously one can prove that

$$C^{\pm}_{n,1} = \{t \in A_{n,2} : \pm w(t) \geq 1/n\} \quad \text{for} \quad n \in N. \tag{4.14$_2$}$$

From (4.14_2), the definition of the sets $D^{\pm}_n$ and (4.3) we obtain (4.6). From the definition of the set $\Omega^{\pm}_w$ and (4.14_2) we have $C^{\pm}_{n,1} \subset \Omega^{\pm}_w$. Hence

$$\text{mes}C^{\pm}_{n,1} \leq \text{mes}\Omega^{\pm}_w.$$

On the other hand $C^{\pm}_{n,1} = \{t \in I : \pm w(t) \geq 1/n\} \setminus (I \setminus A_{n,2})$ and thus

$$\mathrm{mes} C^{\pm}_{n,1} \geq \mathrm{mes}\Omega^{\pm}_{w} - \mathrm{mes}(I \setminus A_{n,2}).$$

In view of (4.7) from last two inequalities we conclude that (4.15) holds. ■

Lemma 4.2. *Let $i \in \{1, 2\}$, $r > 0$, $k \in N$, w_0 be a nonzero solution of the problem* (1.3), (1.4), *$N_{w_0} = \{t_1, ..., t_k\}$, the function $f_1 \in E(N_{w_0})$ be non-decreasing in the second argument for $|x| \geq r$, and*

$$f_1(t, x)\mathrm{sgn} x \geq 0 \qquad \text{for} \quad t \in I, \ |x| \geq r. \tag{4.18}$$

Then:

a) If $G \subset I$ and

$$\int_G |f_1(s, (-1)^i r) w_0(s)| ds \neq 0, \tag{4.19}$$

then there exist $\delta_0 > 0$ and $\varepsilon_1 > 0$ such that

$$\mathbb{I}(G, U_\varepsilon, x) \stackrel{def}{\equiv} \int\limits_{G \setminus U_\varepsilon} |f_1(s, x) w_0(s)| ds - \int\limits_{U_\varepsilon} |f_1(s, x) w_0(s)| ds \geq \delta_0 \tag{4.20}$$

for $(-1)^i x \geq r$ and $0 < \varepsilon \leq \varepsilon_1$, where $U_\varepsilon = I \cap \Big(\cup_{j=1}^{k} [t_j - \varepsilon/2k, \ t_j + \varepsilon/2k] \Big)$.

b) If $u_n \in \widetilde{C}'(I; R)$ $(n \in N)$, $r > 0$, w is an arbitrary nonzero solution of the problem (1.3), (1.4), *and the condition* (4.3) *holds, then there exist $\varepsilon_2 \in]0, \varepsilon_1]$ and $n_0 \in N$ such that*

$$\mathbb{I}(D^+_n, U^+_\varepsilon, x) \geq -\frac{\delta_0}{2} \quad \text{for} \quad x \geq r, \tag{4.21$_1$}$$

$$\mathbb{I}(D^-_n, U^-_\varepsilon, x) \geq -\frac{\delta_0}{2} \quad \text{for} \quad x \leq -r \tag{4.21$_2$}$$

for $n \geq n_0$ and $0 < \varepsilon \leq \varepsilon_2$, where $U^{\pm}_\varepsilon = \{t \in U_\varepsilon : \pm w(t) \geq 0\}$.

Proof. First note that, for any nonzero solution w of the problem (1.3), (1.4), there exists $\beta \neq 0$ such that $w(t) = \beta w_0(t)$ and thus $N_w = N_{w_0}$.

a) For any $\alpha \in R_+$, we put $G_1 = ([a, a + \alpha] \cup [b - \alpha, b]) \cap G$. In view of the condition (4.19), we can choose $\alpha \in]0, (b - a)/2[$ such that if $G_2 = G \setminus G_1$, $t_a = \inf\{G_2\}$ and $t_b = \sup\{G_2\}$, then

$$a < t_a, \quad t_b < b, \tag{4.22}$$

and $\int_{G_1} |f_1(s, (-1)^i r) w_0(s)| ds \neq 0$, $\int_{G_2} |f_1(s, (-1)^i r)| ds \neq 0$. From these inequalities, by virtue of conditions (4.18) and $f_1 \in E(N_{w_0})$, where f_1 is non-decreasing in the second argument, there follows the existence of $\delta_0 > 0$ and $\varepsilon^* > 0$ such that

$$\int_{G_2 \setminus U_{\varepsilon^*}} |f_1(s, x)| ds - \int_{U_{\varepsilon^*}} |f_1(s, x)| ds \geq 0 \quad \text{for} \quad (-1)^i x \geq r, \tag{4.23}$$

(4.24) $$\int_{G_1\setminus U_{\varepsilon^*}} |f_1(s,x)w_0(s)|ds \geq \delta_0 \quad \text{for} \quad (-1)^i x \geq r.$$

Now we put $I^* = [t_a^*,\ t_b^*]$, where $t_a^* = \dfrac{a+\min(t_a,t_1)}{2}$ and $t_b^* = \dfrac{\max(t_k,t_b)+b}{2}$. In view of (4.22), we obtain

(4.25) $$G_2 \subset I^*, \quad N_{w_0} \subset I^*, \quad w_0(t_a^*) \neq 0, \ w_0(t_b^*) \neq 0.$$

Then it is clear that there exists $\gamma_1 > 0$ such that, for any $\gamma \in]0,\gamma_1[$, the equation $|w_0(t)| = \gamma$ has only $t_{\gamma,i}, t^*_{\gamma,i} \in I^*$ $(i = 1,...,k)$ solutions such that

(4.26) $$t_{\gamma,i} < t_i < t^*_{\gamma,i} \quad (i = 1,...,k),$$

(4.27) $$|w_0(t)| \leq \gamma \quad \text{for} \quad t \in H_\gamma, \qquad |w_0(t)| > \gamma \quad \text{for} \quad t \in I^* \setminus H_\gamma,$$

where $H_\gamma = \bigcup\limits_{i=1}^{k}[t_{\gamma,i}, t^*_{\gamma,i}]$, and

(4.28) $$\lim_{\gamma\to+0} t_{\gamma,i} = \lim_{\gamma\to+0} t^*_{\gamma,i} = t_i \quad (i = 1,...,k).$$

The relations (4.26) and (4.28) imply that there exist $\gamma \in]0,\gamma_1]$ and $\varepsilon_1 \in]0,\varepsilon^*]$ such that

(4.29) $$U_{\varepsilon_1} \subset H_\gamma \subset U_{\varepsilon^*}.$$

Moreover, in view of the inclusion $G_1 \subset G$, it is clear that

$$G \setminus U_{\varepsilon_1} = \big[\big(G \setminus G_1\big) \setminus U_{\varepsilon_1}\big] \cup \big(G_1 \setminus U_{\varepsilon_1}\big), \ \big[\big(G \setminus G_1\big) \setminus U_{\varepsilon_1}\big] \cap \big(G_1 \setminus U_{\varepsilon_1}\big) = \emptyset,$$

and thus

$$\mathbb{I}(G, U_{\varepsilon_1}, x) = \int_{G_1\setminus U_{\varepsilon_1}} |f_1(s,x)w_0(s)|ds + \mathbb{I}(G_2, U_{\varepsilon_1}, x) \quad \text{for} \quad (-1)^i x \geq r.$$

By virtue of (4.23), (4.25), (4.27), and (4.29), we get

$$\mathbb{I}(G_2, U_{\varepsilon_1}, x) \geq \gamma\Big(\int_{G_2\setminus H_\gamma} |f_1(s,x)|ds - \int_{H_\gamma} |f_1(s,x)|ds\Big) \geq$$

$$\geq \gamma\Big(\int_{G_2\setminus U_{\varepsilon^*}} |f_1(s,x)|ds - \int_{U_{\varepsilon^*}} |f_1(s,x)|ds\Big) \geq 0$$

for $(-1)^i x \geq r$. In view of the last two relations, (4.24), (4.29), and the fact that $U_\varepsilon \subset U_{\varepsilon_1}$ for $\varepsilon \leq \varepsilon_1$, we conclude that the inequality (4.20) holds.

b) First consider the case when

(4.30) $$\int_{D_n^+} |f_1(s,x)w_0(s)|ds = 0 \ \text{ for } \ x \geq r, \ n \in N.$$

From (4.3) and the definitions of the sets $D_n^{\pm}$ and $U_\varepsilon^{\pm}$ we get

$$\lim_{n\to+\infty} \operatorname{mes}(U_\varepsilon^{\pm} \setminus D_n^{\pm}) = 0. \tag{4.31}$$

Then, in view of (4.30) and the fact that for any $\varepsilon > 0$ and $n \in N$

$$U_\varepsilon^{\pm} = (U_\varepsilon^{\pm} \cap D_n^{\pm}) \cup (U_\varepsilon^{\pm} \setminus D_n^{\pm}), \quad (U_\varepsilon^{\pm} \cap D_n^{\pm}) \cap (U_\varepsilon^{\pm} \setminus D_n^{\pm}) = \emptyset, \tag{4.32}$$

we have $\int_{U_\varepsilon^+} |f_1(s,x)w_0(s)|ds = \int_{U_\varepsilon^+\setminus D_n^+} |f_1(s,x)w_0(s)|ds$ for $x \geq r$, $n \in N$, and $\varepsilon > 0$. Thus by virtue of (4.31), we get $\int_{U_\varepsilon^+} |f_1(s,x)w_0(s)|ds = 0$. From the last equality and (4.30) we conclude that

$$I(D_n^+, U_\varepsilon^+, x) = 0 \quad \text{for} \quad x \geq r,\ n \in N,\ \varepsilon > 0. \tag{4.33}$$

Therefore, in this case the condition (4.21_1) is true.

Now consider the case when for some $r_1 \geq r$ there exists $n_0 \in N$ such that

$$\int_{D_n^+} |f_1(s,x)w_0(s)|ds \neq 0 \ \text{ for } \ x \geq r_1,\ n \geq n_0. \tag{4.34}$$

It is clear that there exist $\eta > 0$ and $\varepsilon_2 \in]0, \varepsilon_1]$ such that

$$\int_{U_\varepsilon^+} |f_1(s,x)w_0(s)|ds \leq \frac{\delta_0}{2} \ \text{ for } \ r \leq x \leq r_1 + \eta,\ \varepsilon \leq \varepsilon_2,$$

and thus

$$I(D_n^+, U_\varepsilon^+, x) \geq -\frac{\delta_0}{2} \quad \text{for} \quad r \leq x \leq r_1 + \eta,\ n \geq n_0,\ \varepsilon \leq \varepsilon_2. \tag{4.35}$$

On the other hand, from (4.34) it is clear that $\int_{D_{n_0}^+} |f_1(s, r_1 + \eta)w_0(s)|ds \neq 0$. Therefore, from the item *a*) of our lemma with $G = D_n^+$, and the inclusions $D_{n_0}^+ \subset D_n^+$, $U_\varepsilon^+ \subset U_\varepsilon$ for $n \geq n_0$, $\varepsilon > 0$, we get $I(D_n^+, U_\varepsilon^+, x) \geq \delta_0$ for $x \geq r_1 + \eta$, $n \geq n_0$, $0 < \varepsilon \leq \varepsilon_2$. From this inequality and (4.35) we obtain (4.21_1) in second case too.

Analogously one can prove (4.21_2). ■

Lemma 4.3. *Let all the conditions of Lemma* 4.1 *be fulfilled and there exist* $r > 0$ *such that the condition* (4.18) *holds, where* $f_1 \in K(I \times R; R)$. *Then*

$$\liminf_{n\to+\infty} \int_s^t f_1(\xi, u_n(\xi)) \operatorname{sgn} u_n(\xi) d\xi \geq 0 \quad \textit{for} \quad a \leq s < t \leq b. \tag{4.36}$$

Proof. Let

$$\gamma_r^*(t) \stackrel{def}{=} \sup\{|f_1(t,x)| : |x| \leq r\} \quad \text{for} \quad t \in I. \tag{4.37}$$

Then, according to (4.1), (4.2), and (4.18), we obtain the estimate

$$\int_s^t f_1(\xi, u_n(\xi)) \operatorname{sgn} u_n(\xi) d\xi \geq$$

$$\geq -\int_{[s,t]\cap A_{n,1}} \gamma_r^*(\xi) d\xi + \int_{[s,t]\cap A_{n,2}} |f_1(\xi, u_n(\xi))| d\xi$$

for $a \leq s < t \leq b$, $n \in N$. This estimate and (4.7) imply (4.36). ■

Lemma 4.4. *Let $r > 0$, the functions $f_1 \in K(I \times R; R)$, $h_1 \in L(I; R)$, $f^+, f^- \in L(I; R_+)$ be such that*

$$f_1(t,x) \le -f^-(t) \quad \text{for} \quad x \le -r, \qquad f^+(t) \le f_1(t,x) \quad \text{for} \quad x \ge r \tag{4.38}$$

on I, and there exist a nonzero solution w_0 of the problem (1.3), (1.4) *and $\varepsilon > 0$ such that*

$$N_{w_0} = \emptyset \tag{4.39}$$

and

$$-\int_a^b f^-(s)|w_0(s)|ds + \varepsilon||\gamma_r^*||_L \le -\int_a^b h_1(s)|w_0(s)|ds \le$$

$$\le \int_a^b f^+(s)|w_0(s)|ds - \varepsilon||\gamma_r^*||_L, \tag{4.40}$$

where γ_r^* is defined by (4.37)*. Then, for every nonzero solution w of the problem* (1.3), (1.4), *and functions $u_n \in \widetilde{C}'(I; R)$ $(n \in N)$ such that the conditions* (4.3),

$$|v_n^{(i)}(t) - w^{(i)}(t)| \le 1/2n \quad \text{for} \quad t \in I,\ n \in N,\ (i = 0, 1) \tag{4.41}$$

where $v_n(t) = u_n(t)||u_n||_C^{-1}$ for $t \in I$ and

$$u_n(a) = 0, \quad u_n(b) = 0 \tag{4.42}$$

are fulfilled, there exists $n_1 \in N$ such that

$$\mathbb{M}_n(w) \overset{def}{\equiv} \int_a^b (h_1(s) + f_1(s, u_n(s)))w(s)ds \ge 0 \quad \text{for} \quad n \ge n_1. \tag{4.43}$$

Proof. First note that, for any nonzero solution w of the problem (1.3), (1.4), there exists $\beta \ne 0$ such that $w(t) = \beta w_0(t)$. Also, it is not difficult to verify that all the assumptions of Lemma 4.1 are satisfied for the function $w(t) = \beta w_0(t)$. From the unique solvability of the Cauchy problem for the equation (1.3) and the conditions (1.4) we conclude that $w'(a) \ne 0$ and $w'(b) \ne 0$. Therefore, in view of (4.41) and (4.42), there exists $n_2 \in N$ such that

$$u_n(t)\operatorname{sgn}\beta w_0(t) > 0 \qquad \text{for} \quad n \ge n_2,\ a < t < b. \tag{4.44}$$

Moreover, by (4.1) and (4.2) we get the estimate

$$\frac{\mathbb{M}_n(w)}{|\beta|} \ge -\int_{A_{n,1}} \gamma_r^*(s)|w_0(s)|ds + \sigma \int_a^b h_1(s)w_0(s)ds + \sigma \int_{A_{n,2}} f_1(s, u_n(s))w_0(s)ds, \tag{4.45}$$

where γ_r^* is given by (4.37) and $\sigma = \operatorname{sgn}\beta$. Now note that $f^- \equiv 0$, $f^+ \equiv 0$ if $f_1(t,x) \equiv 0$. Then by virtue of (4.7), we see that there exist $\varepsilon > 0$ and $n_1 \in N$ $(n_1 \geq n_2)$ such that $\int_a^b f^{\pm}(s)|w_0(s)|ds - \frac{\varepsilon}{2}||\gamma_r^*||_L \leq \int_{A_{n,2}} f^{\pm}(s)|w_0(s)|ds$ and $\frac{\varepsilon}{2}||\gamma_r^*||_L \geq \int_{A_{n,1}} \gamma_r^*(s)|w_0(s)|ds$ for $n \geq n_1$. By these inequalities, (4.3), (4.38) and (4.44), from (4.45) we obtain

$$\frac{\mathbb{M}_n(w)}{|\beta|} \geq -\varepsilon||\gamma_r^*||_L + \int_a^b h_1(s)|w_0(s)|ds + \int_a^b f^+(s)|w_0(s)|ds$$

if $n \geq n_1$, $\sigma w_0(t) \geq 0$, and

$$\frac{\mathbb{M}_n(w)}{|\beta|} \geq -\varepsilon||\gamma_r^*||_L - \int_a^b h_1(s)|w_0(s)|ds + \int_a^b f^-(s)|w_0(s)|ds$$

if $n \geq n_1$, $\sigma w_0(t) \leq 0$. From the last two estimates in view of (4.40) it follows that (4.43) is valid. ■

Lemma 4.5. *Let w_0 be a nonzero solution of the problem* (1.3), (1.4), $r > 0$, *the function $f_1 \in E(N_{w_0})$ be non-decreasing in the second argument for $|x| \geq r$, condition* (4.18) *hold, and*

$$\int_{\Omega_{w_0}^+} |f_1(s,r)|ds + \int_{\Omega_{w_0}^-} |f_1(s,-r)|ds \neq 0. \tag{4.46}$$

Then there exist $\delta > 0$ and $n_1 \in N$ such that if

$$\left| \int_a^b h_1(s)w_0(s)ds \right| < \delta \tag{4.47}$$

then, for every nonzero solution w of the problem (1.3), (1.4) *and the functions $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$ fulfilling the conditions* (4.3), (4.41), (4.42), *the inequality* (4.43) *holds.*

Proof. It is not difficult to verify that all the assumption of Lemma 4.1 are satisfied. Then, by the definition of the sets $B_{n,1}$, $B_{n,2}$, the conditions (4.1), (4.2), and (4.18), we obtain the estimate

$$\int_a^b f_1(s,u_n(s))w(s)ds \geq -\int_{A_{n,1}} \gamma_r^*(s)|w(s)|ds + \widehat{\mathbb{M}}_n(w), \tag{4.48}$$

where

$$\widehat{\mathbb{M}}_n(w) \overset{def}{\equiv} -\int_{B_{n,2}} |f_1(s,u_n(s))w(s)|ds + \int_{B_{n,1}} |f_1(s,u_n(s))w(s)|ds.$$

On the other hand, from the unique solvability of the Cauchy problem for the equation (1.3) it is clear that

$$w'(a) \neq 0, \quad w'(b) \neq 0, \quad w'(t_i) \neq 0 \qquad \text{for } \ i = 1, ..., k \tag{4.49}$$

if $N_{w_0} = \{t_1, ..., t_k\}$. Now note that, for any nonzero solution w of the problem (1.3), (1.4), there exists $\beta \neq 0$ such that $w(t) = \beta w_0(t)$. Consequently,

$$\Omega_w^{\pm} = \Omega_{w_0}^{\pm} \quad \text{if} \quad \beta > 0 \quad \text{and} \quad \Omega_w^{\mp} = \Omega_{w_0}^{\pm} \quad \text{if} \quad \beta < 0. \tag{4.50}$$

Then in view of (4.15) and (4.46), there exists $n_2 \geq n_0$ such that

$$\int_{C^+_{n_2,1}} |f_1(s, r)w_0(s)|ds \neq 0 \ \text{ and/or } \int_{C^-_{n_2,1}} |f_1(s, -r)w_0(s)|ds \neq 0. \tag{4.51}$$

From (4.51), in view of (4.6), it follows that

$$\int_{D_n^+} |f_1(s, r)w_0(s)|ds \neq 0 \quad \text{for} \quad n \geq n_2 \tag{4.52$_1$}$$

and/or

$$\int_{D_n^-} |f_1(s, -r)w_0(s)|ds \neq 0 \quad \text{for} \quad n \geq n_2. \tag{4.52$_2$}$$

Consequently, all the assumptions of Lemma 4.2 are satisfied with $G = D_n^+$ and/or $G = D_n^-$. Therefore, there exist $\varepsilon_0 \in]0, \varepsilon_2[$, $n_3 \geq n_2$, and $\delta_0 > 0$ such that

$$\begin{aligned} &\mathbb{I}(D_n^+, U_{\varepsilon_0}^+, x) \geq \delta_0 \ \text{ for } \ x \geq r, \quad n \geq n_3, \\ &\mathbb{I}(D_n^-, U_{\varepsilon_0}^-, x) \geq -\delta_0/2 \ \text{ for } \ x \leq -r, \ \ n \geq n_3 \end{aligned} \tag{4.53}$$

if (4.52$_1$) holds, and

$$\begin{aligned} &\mathbb{I}(D_n^-, U_{\varepsilon_0}^-, x) \geq \delta_0 \quad \text{for} \quad x \leq -r, \ n \geq n_3, \\ &\mathbb{I}(D_n^+, U_{\varepsilon_0}^+, x) \geq -\delta_0/2 \quad \text{for} \quad x \geq r, \ \ n \geq n_3 \end{aligned} \tag{4.54}$$

if (4.52$_2$) holds.

On the other hand, the definition of the set U_ε and (4.14$_1$), imply that there exists $n_4 > n_3$, such that

$$C_{n,2}^+ \subset U_{\varepsilon_0}^+, \quad C_{n,2}^- \subset U_{\varepsilon_0}^- \quad \text{for} \quad n \geq n_4. \tag{4.55}$$

By these inclusions, (4.2), and (4.5) we obtain

$$C_{n,1}^+ = A_{n,2}^+ \setminus C_{n,2}^+ \supset D_{n_4}^+ \setminus U_{\varepsilon_0}^+, \ C_{n,1}^- = A_{n,2}^- \setminus C_{n,2}^- \supset D_{n_4}^- \setminus U_{\varepsilon_0}^+ \tag{4.56}$$

for $n \geq n_4$. First suppose that $N_{w_0} \neq \emptyset$ and there exists $n \geq n_4$ such that

$$B_{n,2} \neq \emptyset. \tag{4.57}$$

Then, by taking into account that f_1 is non-decreasing in the second argument for $|x| \geq r$, (4.3), (4.12), (4.18) and the definitions of the sets $B^+_{n,2}, B^-_{n,2}$, we get

$$\begin{aligned} &|f_1(t,u_n(t))| = f_1(t,u_n(t)) \leq \\ &\leq f_1\left(t, \frac{\|u_n\|_C}{2n}\right) = \left|f_1\left(t, \frac{\|u_n\|_C}{2n}\right)\right| \quad \text{for } t \in B^+_{n,2}, \\ &|f_1(t,u_n(t))| = -f_1(t,-u_n(t)) \leq \\ &\leq -f_1(t, -\frac{\|u_n\|_C}{2n}) = \left|f_1\left(t, -\frac{\|u_n\|_C}{2n}\right)\right| \quad \text{for } t \in B^-_{n,2}. \end{aligned} \tag{4.58}$$

Analogously, from (4.3), (4.13), (4.18), and the definitions of the sets $C^+_{n,1}, C^-_{n,1}$, we obtain the estimates

$$\begin{aligned} &|f_1(t,u_n(t))| \geq \left|f_1\left(t, \frac{\|u_n\|_C}{2n}\right)\right| \quad \text{for} \quad t \in C^+_{n,1}, \\ &|f_1(t,u_n(t))| \geq \left|f_1\left(t, -\frac{\|u_n\|_C}{2n}\right)\right| \quad \text{for} \quad t \in C^-_{n,1}. \end{aligned} \tag{4.59}$$

Then from (4.1), (4.2), (4.9), (4.58) and respectively from (4.1), (4.2), (4.8), and (4.59) we have

$$\int_{B_{n,2}} |f_1(s,u_n(s))w(s)|ds \leq$$

$$\leq \int_{B^+_{n,2}} \left|f_1\left(s, \frac{\|u_n\|_C}{2n}\right) w(s)\right| ds + \int_{B^-_{n,2}} \left|f_1\left(s, -\frac{\|u_n\|_C}{2n}\right) w(s)\right| ds \leq \tag{4.60}$$

$$\leq \int_{C^+_{n,2}} \left|f_1\left(s, \frac{\|u_n\|_C}{2n}\right) w(s)\right| ds + \int_{C^-_{n,2}} \left|f_1\left(s, -\frac{\|u_n\|_C}{2n}\right) w(s)\right| ds$$

and respectively

$$\int_{B_{n,1}} |f_1(s,u_n(s))w(s)|ds \geq \int_{C_{n,1}} |f_1(s,u_n(s))w(s)|ds \geq$$

$$\geq \int_{C^+_{n,1}} \left|f_1\left(s, \frac{\|u_n\|_C}{2n}\right) w(s)\right| ds + \int_{C^-_{n,1}} \left|f_1\left(s, -\frac{\|u_n\|_C}{2n}\right) w(s)\right| ds. \tag{4.61}$$

If the condition (4.57) holds, from (4.60) and (4.61) we obtain

$$\frac{\widehat{\mathbb{M}}_n(w)}{|\beta|} \geq \left(\int_{C^+_{n,1}} \left|f_1\left(s, \frac{\|u_n\|_C}{2n}\right) w_0(s)\right| ds - \int_{C^+_{n,2}} \left|f_1\left(s, \frac{\|u_n\|_C}{2n}\right) w_0(s)\right| ds\right)$$

$$+\left(\int_{C^-_{n,1}} \left|f_1\left(s, -\frac{\|u_n\|_C}{2n}\right) w_0(s)\right| ds - \int_{C^-_{n,2}} \left|f_1\left(s, -\frac{\|u_n\|_C}{2n}\right) w_0(s)\right| ds\right),$$

Whence, by (4.55) and (4.56) we get

$$\frac{\widehat{\mathbb{M}}_n(w)}{|\beta|} \geq \mathbb{I}\left(D^+_{n_4}, U^+_{\varepsilon_0}, \frac{\|u_n\|_C}{2n}\right) + \mathbb{I}\left(D^-_{n_4}, U^-_{\varepsilon_0}, -\frac{\|u_n\|_C}{2n}\right) \tag{4.62}$$

for $n \geq n_4$. From (4.62) by (4.53) and (4.54) we obtain

$$\widehat{\mathbb{M}}_n(w) \geq \frac{\delta_0|\beta|}{2} \quad \text{for} \quad n \geq n_4. \tag{4.63}$$

On the other hand, in view of (4.10), (4.18), the definition of the sets $A_{n,2}, B_{n,1}$, and the fact that f_1 is non-decreasing in the second argument, we obtain the estimate

$$\int_{B_{n,1}} |f_1(s, u_n(s))w(s)|ds \geq$$

$$\geq \int_{B^+_{n,1}} |f_1(s,r)w(s)|ds + \int_{B^-_{n,1}} |f_1(s,-r)w(s)|ds \geq \tag{4.64}$$

$$\geq \int_{C^+_{n,1}} |f_1(s,r)w(s)|ds + \int_{C^-_{n,1}} |f_1(s,-r)w(s)|ds.$$

Now suppose that there exists $n \geq n_4$ such that

$$B_{n,2} = \emptyset. \tag{4.65}$$

Then from (4.51) and (4.64), (4.65) there follows the existence of $\delta^* > 0$ such that $\widehat{\mathbb{M}}_n(w) \geq |\beta|\delta^*$. From this inequality and (4.63) it follows that, in both cases when (4.57) or (4.65) are fulfilled, the inequality

$$\widehat{\mathbb{M}}_n(w) \geq |\beta|\delta \quad \text{for} \quad n \geq n_4 \tag{4.66}$$

holds with $\delta = \min\{\delta_0/2, \delta^*\}$. From (4.48) by (4.7) and (4.66), we see that for any $\varepsilon \in]0, \delta[$ there exists $n_1 > n_4$ such that

$$\int_a^b f_1(s, u_n(s))w(s)ds \geq |\beta|(\delta - \varepsilon) \quad \text{for} \quad n \geq n_1,$$

and thus

$$\frac{\mathbb{M}_n(w)}{|\beta|} \geq \delta - \varepsilon - \left|\int_a^b h_1(s)w_0(s)ds\right| \quad \text{for} \quad n \geq n_1. \tag{4.67}$$

If $N_{w_0} = \emptyset$ then $|w(t)| > 0$ for $a < t < b$ and in view of (4.3), (4.41), (4.42) and (4.49), the condition (4.65) holds, i.e., the inequality (4.67) also holds.

Consequently, since $\varepsilon > 0$ is arbitrary, the inequality (4.43) from (4.67) and (4.47) follows. ■

Lemma 4.6. *Let w_0 be a nonzero solution of the problem* (1.3), (1.4), $r > 0$, *and the conditions* (4.18), (4.47) *hold with* $f_1(t,x) \overset{def}{\equiv} f_0(t)g_1(x)$, *where* $f_0 \in L(I;R_+)$, $\int_a^b |f_0(s)|ds \neq 0$ *and a non-decreasing function* $g_1 \in C(R;R)$ *be such that*

$$\lim_{|x|\to+\infty} |g_1(x)| = +\infty. \tag{4.68}$$

Then, for every nonzero solution w of the problem (1.3), (1.4) *and functions* $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$ *fulfilling the conditions* (4.3), (4.41), (4.42), *the inequality* (4.43) *holds.*

Proof. From the assumptions of our lemma it is clear that the relations (4.48)–(4.56), (4.58)-(4.61) and (4.64) with $f_1(t,x) = f_0(t)g_1(x)$ and $w(t) = \beta w_0(t)$ $(\beta \neq 0)$ are fulfilled.

Assuming $\int_{C^+_{n_2,1}} |f_1(s,r)w_0(s)|ds \neq 0$, the condition (4.52_1) is satisfied i.e., (4.53) holds.

Now notice that from (4.15) and the equality $C^+_{n,1} = \Omega^+_w \setminus (\Omega^+_w \setminus C^+_{n,1})$ it follows that there exist $\varepsilon > 0$ and $n_0 \in N$ such that

$$\int_{C^+_{n,1}} |f_0(s)w_0(s)|ds \geq \int_{\Omega^+_w} |f_0(s)w_0(s)|ds - \varepsilon > 0 \tag{4.69}$$

for $n \geq n_0$.

First consider the case when there exists $n \geq n_4$ such that the condition (4.65) holds. Without loss of generality we can assume that $n_4 > n_0$. Then by (4.50), (4.64), (4.65) and (4.69), we obtain

$$\widehat{\mathbb{M}}_n(w) \geq |\beta||g_1(r)| \left(\int_{\Theta_\beta} |f_0(s)w_0(s)|ds - \varepsilon \right) > 0, \tag{4.70}$$

where $\Theta_\beta = \Omega^+_{w_0}$ if $\beta > 0$ and $\Theta_\beta = \Omega^-_{w_0}$ if $\beta < 0$.

Consider now the case when there exists $n \geq n_4$ such that (4.57) holds. From (4.3) and the definition of the set D^+_n it follows that $D^+_n \subset D^+_{n+1}$, and since g_1 is non-decreasing, from (4.53) we obtain $\mathbb{I}(D^+_n, U^+_{\varepsilon_0}, x) \geq |g_1(r)|\mu = \mathbb{I}(D^+_{n_4}, U^+_{\varepsilon_0}, r) \geq \delta_0$ for $x \geq r$, with $\mu = \int_{D^+_{n_4}\setminus U^+_{\varepsilon_0}} |f_0(s)w_0(s)|ds - \int_{U^+_{\varepsilon_0}} |f_0(s)w_0(s)|ds$. By the last inequality, (4.3), (4.53), and (4.62) we get $\mu > 0$ and

$$\widehat{\mathbb{M}}_n(w) \geq |\beta|(|g_1(r)|\mu - \delta_0/2). \tag{4.71}$$

Applying (4.70), (4.71) in (4.48) and taking (4.7) into account, we conclude that there exist $\varepsilon_1 > 0$ and $n_1 \geq n_4$ such that

$$|\beta| \left(|g_1(r)|\mu_1 - \frac{\delta_0}{2} - \varepsilon_1 \right) \leq \int_a^b f_1(s,u_n(s))w(s)ds \quad \text{for} \quad n \geq n_1$$

with $\mu_1 = \min(\mu, \int_{\Omega^+_{w_0}} |f_0(s)w_0(s)|ds - \varepsilon)$. From (4.68) and the last inequality it is clear that, for any function h_1, we can choose $r > 0$ such that the inequality (4.43) will be true. Analogously one can prove (4.43) in the case when $\int_{C^-_{n_2,1}} |f_1(s,r)w_0(s)|ds \neq 0$. ∎

Lemma 4.7. *Let $r > 0$, there exist functions $\alpha, f^-, f^+ \in L(I, R_+)$ such that the condition* (4.38) *is satisfied,*

$$\sup\{|f_1(t,x)| : x \in R\} = \alpha(t) \quad \text{for} \quad t \in I, \tag{4.72}$$

and there exist a nonzero solution w_0 of the problem (1.3), (1.4) *and $\varepsilon > 0$ such that*

$$\begin{aligned} &-\int_a^b (f^+(s)[w_0(s)]_- + f^-(s)[w_0(s)]_+)ds + \varepsilon||\alpha||_L \leq \\ &\leq -\int_a^b h_1(s)w_0(s)ds \leq \\ &\leq \int_a^b (f^-(s)[w_0(s)]_- + f^+(s)[w_0(s)]_+)ds - \varepsilon||\alpha||_L. \end{aligned} \tag{4.73}$$

Then, for every nonzero solution w of the problem (1.3), (1.4) *and functions $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$ fulfilling the conditions* (4.3), (4.41), *and* (4.42), *there exists $n_1 \in N$ such that the inequality* (4.43) *holds.*

Proof. First note that, for any nonzero solution w of the problem (1.3), (1.4), there exists $\beta \neq 0$ such that $w(t) = \beta w_0(t)$. Moreover, it is not difficult to verify that all the assumptions of Lemma4.1 are satisfied for the function $w(t) = \beta w_0(t)$. From (4.1), (4.2), and (4.72) we get

$$\begin{aligned} \mathbb{M}_n(w) \geq &-\int_{A_{n,1}\cup B_{n,2}} \alpha(s)|w(s)|ds + \int_{B_{n,1}} f_1(s,u_n)w(s)ds + \\ &+\int_a^b h_1(s)w(s)ds. \end{aligned} \tag{4.74}$$

On the other hand, by the definition of the set $B_{n,1}$ we obtain

$$\text{sgn}u_n(t) = \text{sgn}w(t) \quad \text{for} \quad t \in B^+_{n,1} \cup B^-_{n,1}. \tag{4.75}$$

Hence, by (4.1), (4.2), (4.10), (4.38), and (4.75), from (4.74) we obtain the estimate

$$\begin{aligned} \mathbb{M}_n(w) - \int_a^b h_1(s)w(s)ds \geq &-\int_{A_{n,1}\cup B_{n,2}} \alpha(s)|w(s)|ds + \\ &+\int_{B^+_{n,1}} f^+(s)|w(s)|ds + \int_{B^-_{n,1}} f^-(s)|w(s)|ds \geq \\ \geq -\int_{A_{n,1}\cup B_{n,2}} \alpha(s)|w(s)|ds &+ \int_{C^+_{n,1}} f^+(s)|w(s)|ds + \int_{C^-_{n,1}} f^-(s)|w(s)|ds. \end{aligned} \tag{4.76}$$

Now, note that $f^- \equiv 0$ and $f^+ \equiv 0$ if $f_1(t,x) \equiv 0$. Therefore by (4.7), (4.11), (4.15), and the inclusions $C^+_{n,1} \subset \Omega^+_w$, $C^-_{n,1} \subset \Omega^-_w$, we see that there exist $\varepsilon > 0$

and $n_1 \in N$ such that

$$\begin{aligned} &\frac{1}{3}\varepsilon||\alpha||_L \geq \int_{A_{n,1}\cup B_{n,2}} \alpha(s)|w_0(s)|ds \\ &\int_{\Omega_w^\pm} f^\pm(s)|w_0(s)|ds - \frac{1}{3}\varepsilon||\alpha||_L \leq \int_{C_{n,1}^\pm} f^\pm(s)|w_0(s)|ds \end{aligned} \tag{4.77}$$

for $n \geq n_1$. By virtue of (4.76) and (4.77), we obtain

$$\frac{\mathbb{M}_n(w)}{|\beta|} \geq -\varepsilon||\alpha||_L + \int_{\Omega_w^+} f^+(s)|w_0(s)|ds +$$

$$+ \int_{\Omega_w^-} f^-(s)|w_0(s)|ds + \sigma \int_a^b h_1(s)w_0(s)ds$$

for $n \geq n_1$, where $\sigma = \operatorname{sgn}\beta$. Now, by taking into account that

$$\int_{\Omega_w^\pm} l(s)|w_0(s)|ds = \int_{\Omega_{w_0}^\pm} l(s)|w_0(s)|ds = \int_a^b l(s)[w_0(s)]_\pm ds$$

if $\beta > 0$ and

$$\int_{\Omega_w^\pm} l(s)|w_0(s)|ds = \int_{\Omega_{w_0}^\mp} l(s)|w_0(s)|ds = \int_a^b l(s)[w_0(s)]_\mp ds$$

if $\beta < 0$ for an arbitrary $l \in L(I, R)$, from the last inequalities we get

$$\frac{\mathbb{M}_n(w)}{|\beta|} \geq -\varepsilon||\alpha||_L + \int_a^b (f^+(s)[w_0(s)]_+ + f^-(s)[w_0(s)]_-)ds +$$

$$+ \int_a^b h_1(s)w_0(s)ds \quad \text{for} \quad n \geq n_1$$

if $\sigma = 1$, and

$$\frac{\mathbb{M}_n(w)}{|\beta|} \geq -\varepsilon||\alpha||_L + \int_a^b (f^+(s)[w_0(s)]_- + f^-(s)[w_0(s)]_+)ds -$$

$$- \int_a^b h_1(s)w_0(s)ds \quad \text{for} \quad n \geq n_1$$

if $\sigma = -1$. From the last inequalities and (4.73) we immediately obtain (4.43). ■

Now we consider the definitions of the sets $V_{10}((a, b))$ introduced and described in [12] (see [Definition 1.3, p. 2350])

Definition 4.2. We say that the function $p \in L([a, b])$ belongs to the set $V_{10}((a, b))$ if for any function p^* satisfying the inequality $p^*(t) \geq p(t)$ for $t \in I$ the unique solution of the initial value problem

$$u''(t) = p^*(t)u(t) \quad \text{for} \quad t \in I, \quad u(a) = 0, \quad u'(a) = 1, \tag{4.78}$$

has no zeros in the set $]a, b]$.

Lemma 4.8. *Let $i \in \{1,2\}$, $p \in L(I;R)$, $p_n(t) = p(t) + (-1)^i/n$, and $w_n \in \widetilde{C}'(I;R)$ $(n \in N)$ be a solution of the problem*

$$w_n''(t) = p_n(t)w_n(t) \quad \text{for} \quad t \in I, \quad w_n(a) = 0, \quad w_n(b) = 0. \tag{4.79$_n$}$$

Then:

a) There exists $n_0 \in N$ such that the problem (4.79$_n$) has only the zero solution for $n \geq n_0$.

b) If $i = 2$ and $N_w = \emptyset$, where w is a solution of the problem (1.3), (1.4), then the inclusion $p_n \in V_{10}((a,b))$ for every $n \in N$ holds.

Proof. *a*) Let $N^*_{w_n}$ be the number of zeros of the function w_n on I. Assume on the contrary that there exists a sequence $\{w_n\}_{n\geq n_0}^{+\infty}$ of nonzero solutions of the problem (4.79$_n$).

If $i = 1$ then from the facts that $p_n(t) < p_{n+1}(t)$ and $w_n \not\equiv 0$, by Sturm's comparison theorem, we obtain $N^*_{w_n} - N^*_{w_{n+1}} \geq 1$ $(n \in N)$. Now notice that, in view of (4.79$_n$), the inequality $N^*_{w_n} \geq 2$ holds. Hence there exist $k_0 \geq 2$ and $n_0 \geq 2$ such that $N^*_{w_{n_0}} = k_0$. Therefore, we obtain the contradiction $k_0 = N^*_{w_{n_0}} > N^*_{w_{n_0}} - N^*_{w_{n_0+k_0}} = (N^*_{w_{n_0}} - N^*_{w_{n_0+1}}) + (N^*_{w_{n_0+1}} - N^*_{w_{n_0+2}}) + \ldots + (N^*_{w_{n_0+k_0-1}} - N^*_{w_{n_0+k_0}}) \geq k_0$.

If $i = 2$, from the fact that $p_{n-1}(t) > p_n(t) > p(t)$ and $w_n \not\equiv 0$, by Sturm's comparison theorem, we obtain $N^*_{w_n} - N^*_{w_{n-1}} \geq 1$ and $N^*_w \geq N^*_{w_n} - 1$ $(n \in N)$ if w is a nonzero solution of the equation (1.3). Now notice that, in view of (4.79$_n$), the inequality $N^*_{w_n} \geq 2$ holds for every $n \in N$. Therefore, if we denote $N^*_w = k_0$, we obtain the contradiction $k_0 = N^*_w \geq N^*_{w_{n+k_0}} - 1 > N^*_{w_{n+k_0}} - N^*_{w_n} \geq k_0$.

The contradiction obtained proves the item *a*) of our lemma.

b) Assume on the contrary that there exists $n \in N$ such that $p_n \notin V_{10}([a,b])$. If $p^*(t) \geq p_n(t)$ and u is a solution of the problem (4.78), then there exists $t_0 \in]a,b]$ such that $u(t_0) = 0$. Since $p(t) < p^*(t)$, by Sturm's comparison theorem, we obtain that w, the solution of the problem (1.3), (1.4), has a zero in the interval $]a, t_0[$, which contradicts our assumption $N_w = \emptyset$. The contradiction obtained proves the item *b*) of our lemma. ■

5. Proof of the main results

Proof of Theorem 2.1. Let $p_n(t) = p(t) + 1/n$ and, for any $n \in N$, consider the problem

$$u_n''(t) = p_n(t)u_n(t) + f(t, u_n(t)) + h(t) \quad \text{for} \quad t \in I, \tag{5.1}$$

$$u_n(a) = 0, \quad u_n(b) = 0. \tag{5.2}$$

In view of the condition (2.1) and Lemma 4.8, the inclusion $p_n \in V_{10}((a,b))$ holds for every $n \in N$. On the other hand, from the conditions (2.2) and (2.3) we find

$$0 \leq f(t,x)\operatorname{sgn}x \leq g(t)|x| + h_0(t) \quad \text{for } t \in I, \ |x| \geq r. \tag{5.3}$$

Then the inclusion $p_n \in V_{10}((a,b))$, as is well-known (see [12, Theorem 2.2, p.2367]), guarantees that the problem (5.1), (5.2) has at least one solution, suppose u_n. In view of the condition (2.2), without loss of generality we can assume that there exists $\varepsilon^* > 0$ such that $h_0(t) \geq \varepsilon^*$ on I. Then $g(t)|x| + h_0(t) \geq \varepsilon^*$ for $x \in R$, $t \in I$. Consequently, it is not difficult to verify that u_n is also a solution of the equation

$$u_n''(t) = (p_n(t) + p_0(t, u_n(t))\mathrm{sgn} u_n(t))u_n(t) + p_1(t, u_n(t)) \tag{5.4}$$

with $p_0(t,x) = \dfrac{f(t,x)g(t)}{g(t)|x| + h_0(t)}$, $p_1(t,x) = h(t) + \dfrac{f(t,x)h_0(t)}{g(t)|x| + h_0(t)}$.

Now assume that

$$\lim_{n\to+\infty} ||u_n||_C = +\infty \tag{5.5}$$

and $v_n(t) = u_n(t)||u_n||_C^{-1}$. Then

$$v_n''(t) = (p_n(t) + p_0(t, u_n(t))\mathrm{sgn} u_n(t))v_n(t) + \frac{1}{||u_n||_C} p_1(t, u_n(t)), \tag{5.6}$$

$$v_n(a) = 0 \qquad v_n(b) = 0, \tag{5.7}$$

and

$$||v_n||_C = 1 \tag{5.8}$$

for any $n \in N$. In view of the condition (5.3), the functions $p_0, p_1 \in K(I \times R; R)$ are bounded respectively by the functions $g(t)$ and $h(t) + h_0(t)$. Therefore, from (5.6), by virtue of (5.5), (5.7) and (5.8), we see that there exists $r_0 > 0$ such that $||v_n'||_C \leq r_0$. Consequently in view of (5.8), by Arzela-Ascoli lemma, without loss of generality we can assume that there exists $w \in \widetilde{C}'(I,R)$ such that $\lim\limits_{n\to+\infty} v_n^{(i)}(t) = w^{(i)}(t)$ $(i = 0,1)$ uniformly on I. From the last equality and (5.5) there follows the existence of an increasing sequence $\{\alpha_k\}_{k=1}^{+\infty}$ of a natural numbers, such that $||u_{\alpha_k}||_C \geq 2rk$ and $||v_{\alpha_k}^{(i)} - w^{(i)}||_C \leq 1/2k$ for $k \in N$. Without loss of generality we can suppose that $u_n \equiv u_{\alpha_n}$ and $v_n \equiv v_{\alpha_n}$. In this case we see that u_n and v_n are the solutions of the problems (5.1), (5.2) and (5.6), (5.7) respectively with $p_n(t) = p(t) + 1/\alpha_n$ for $t \in I$, $n \in N$, and that the inequalities

$$||u_n||_C \geq 2rn, \quad ||v_n^{(i)} - w^{(i)}||_C \leq 1/2n \quad \text{for} \quad n \in N \tag{5.9}$$

are fulfilled. Analogously, since the functions $p_0, p_1 \in K(I \times R; R)$ are bounded, in view of (5.5), we can assume without loss of generality that there exists a function $\widetilde{p} \in L(I;R)$ such that

$$\lim_{n\to+\infty} \frac{1}{||u_n||_C^j} \int_a^t p_j(s, u_n(s))\mathrm{sgn} u_n(s)ds = (1-j)\int_a^t \widetilde{p}(s)ds \tag{5.10$_j$}$$

uniformly on I for $j=0,1$. By virtue of (5.8)–(5.10$_j$) $(j=0,1)$, from (5.6) we obtain

$$w''(t)=(p(t)+\widetilde{p}(t))w(t), \tag{5.11}$$

$$w(a)=0, \quad w(b)=0, \tag{5.12}$$

and

$$||w||_C=1. \tag{5.13}$$

From the conditions (2.3) and (5.9) it is clear that all the assumptions of Lemma 4.3 with $f_1(t,x)=f(t,x)$ are satisfied, and thus we obtain from (5.10$_j$) $(j=0)$ the relation $\int_s^t \widetilde{p}(\xi)d\xi \ge 0$ for $a\le s<t\le b$, i.e.,

$$\widetilde{p}(t)\ge 0 \quad \text{for} \quad t\in I. \tag{5.14}$$

Now assume that $\widetilde{p}\not\equiv 0$ and w_0 is a solution of the problem (1.3), (1.4). Then using Sturm's comparison theorem for the equations (1.3) and (5.11), from (5.14) we see that there exists a point $t_0\in]a,b[$ such that $w_0(t_0)=0$, which contradicts (2.1). This contradiction proves that $\widetilde{p}\equiv 0$. Consequently, w is a solution of the problem (1.3), (1.4). Multiplying the equations (5.1) and (1.3) respectively by w and $-u_n$, and therefore integrating their sum from a to b, in view of the conditions (5.2) and (1.4), we obtain

$$-\frac{1}{\alpha_n}\int_a^b w(s)u_n(s)ds=\int_a^b (h(s)+f(s,u_n(s)))w(s)ds \tag{5.15}$$

for $n\ge n_0$. Therefore by virtue of (5.9) we get

$$\int_a^b (h(s)+f(s,u_n(s)))w(s)ds<0 \quad \text{for} \quad n\ge n_0. \tag{5.16}$$

On the other hand, in view the conditions (2.1)-(2.4$_1$), (5.2), and (5.9) it is clear that all the assumption of Lemma 4.4 with $f_1(t,x)=f(t,x)$, $h_1(t)=h(t)$ are fulfilled. Therefore, the inequality (4.43) is true, which contradicts (5.16). This contradiction proves that (5.5) does not hold and thus there exists $r_1>0$ such that $||u_n||_C\le r_1$ for $n\in N$. Consequently, from (5.1) and (5.2) it is clear that there exists $r_1'>0$ such that $||u_n'||_C\le r_1'$ and $|u_n''(t)|\le\sigma(t)$ for $t\in I$, $n\in N$, where $\sigma(t)=(1+|p(t)|)r_1+|h(t)|+\gamma_{r_1}(t)$. Hence, by Arzela-Ascoli lemma, without loss of generality we can assume that there exists a function $u_0\in\widetilde{C}'(I;R)$ such that $\lim\limits_{n\to+\infty} u_n^{(i)}(t)=u_0^{(i)}(t)$ $(i=0,1)$ uniformly on I. Therefore, it follows from (5.1) and (5.2) that u_0 is a solution of the problem (1.1), (1.2). ∎

Proof of Theorem 2.2. Let $p_n(t) = p(t) - 1/n$ and, for any $n \in N$, consider the problems (5.1), (5.2) and (4.79_n). In view of Lemma 4.8, the problem (4.79_n) has only the zero solution for every $n \geq n_0$. Therefore, as is well-known (see [9, Theorem 1.1, p.345]), from the conditions (2.7), (2.9) it follows that the problem (5.1), (5.2) has at least one solution, suppose u_n.

Now assume that (5.5) holds and put $v_n(t) = u_n(t)||u_n||_C^{-1}$. Then the conditions (5.7) and (5.8) are fulfilled, and

$$v_n''(t) = p_n(t)v_n(t) + \frac{1}{||u_n||_C}(f(t, u_n(t))) + h(t)). \tag{5.17}$$

In view the conditions (2.7) and (2.9), from (5.17) there follows the existence of $r_0 > 0$ such that $||v_n'||_C \leq r_0$. Consequently, in view (5.8) by Arzela-Ascoli lemma, without loss of generality we can assume that there exists a function $w \in \widetilde{C}'(I, R)$ such that $\lim\limits_{n\to+\infty} v_n^{(i)}(t) = w^{(i)}(t) \quad (i = 0, 1)$ uniformly on I. Analogously as in the proof of Theorem 2.1, we can find a sequence $\{\alpha_k\}_{n=1}^{+\infty}$ of natural numbers such that, if we suppose $u_n = u_{\alpha_n}$ then the conditions (5.9) will by true when the functions u_n and v_n are the solutions of the problems (5.1), (5.2) and (5.17), (5.7) respectively with $p_n(t) = p(t) - 1/\alpha_n$ for $t \in I,\ n \in N$. From (5.17), by virtue of (5.7), (5.9) and (2.9), we obtain that w is a solution of the problem (1.3), (1.4). In a similar manner as the condition (5.15) in the proof of Theorem 2.1, we show that

$$\frac{1}{\alpha_n}\int_a^b w(s)u_n(s)ds = \int_a^b (h(s) + f(s, u_n(s)))w(s)ds \tag{5.18}$$

for $n \geq n_0$. Now note that, in view of the conditions (2.1), (2.8), (2.4_2), (5.2), and (5.9), all the assumptions of Lemma 4.4 with $f_1(t, x) = -f(t, x),\ h_1(t) = -h(t)$ are satisfied. Hence, analogously as in the proof of Theorem 2.1, from (5.18) we show that the problem (1.1), (1.2) has at least one solution. ■

Proof of Theorem 2.3. Let $p_n(t) = p(t) + (-1)^i/n$ and for any $n \in N$, consider the problems (5.1), (5.2) and (4.79_n). In view of the condition (2.13) and the fact that $(-1)^i f(t, x)$ is non-decreasing in the second argument for $|x| \geq r$, we obtain

$$\lim_{n\to+\infty} \frac{1}{||z_n||_C}\int_a^b |f(s, z_n(s))|ds = 0 \tag{5.19}$$

for an arbitrary sequence $z_n \in C(I; R)$ with $\lim\limits_{n\to+\infty} ||z_n||_C = +\infty$. Moreover, in view of Lemma 4.8, the problem (4.79_n) has only the zero solution for every $n \geq n_0$. Therefore, as it is well-known (see [9, Theorem 1.1, p. 345]), from the inequality (5.19) it follows that the problem (5.1), (5.2) has at least one solution, suppose u_n.

Now assume that (5.5) is fulfilled and put $v_n(t) = u_n(t)||u_n||_C^{-1}$. Then (5.7), (5.8) and (5.17) are also fulfilled. Hence, by the conditions (5.8) and (5.19), from (5.17) we get the existence of $r_0 > 0$ such that $||v_n'||_C \leq r_0$. Consequently, in view of (5.8) by the Arzela-Ascoli lemma, without loss of generality we can assume that there exists a function $w \in \widetilde{C}'(I,R)$ such that $\lim\limits_{n\to+\infty} v_n^{(i)}(t) = w^{(i)}(t) \quad (i = 0,1)$ uniformly on I. Analogously as in the proof of Theorem 2.1, we can find a sequence $\{\alpha_k\}_{n=1}^{+\infty}$ of natural numbers such that, assuming $u_n = u_{\alpha_n}$, the conditions (5.9) is true and the functions u_n and v_n are the solutions of the problems (5.1), (5.2) and (5.17), (5.7) respectively with $p_n(t) = p(t) + (-1)^i/\alpha_n$ for $t \in I,\ n \in N$. From (5.17), by virtue of (5.7), (5.9) and (2.13), we obtain that w is a solution of the problem (1.3), (1.4). In a similar manner as the condition (5.15) in the proof of Theorem 2.1, we show

$$-\frac{1}{\alpha_n}\int_a^b w(s)u_n(s)ds = (-1)^i \int_a^b (h(s) + f(s,u_n(s)))w(s)ds \tag{5.20}$$

for $n \in N \geq n_0$. Now note that, in view the conditions (2.11), (2.12), (2.14), (5.2), and (5.9), all the assumptions of Lemma 4.5 with $f_1(t,x) = (-1)^i f(t,x),\ h_1(t) = (-1)^i h(t)$ are satisfied. Hence, analogously as in the proof of Theorem 2.1, from (5.20) by Lemma 4.5 we obtain that the problem (1.1), (1.2) has at least one solution. ∎

Proof of Corollary 2.1. From the condition (2.15) we immediately obtain (2.14). Therefore all the conditions of Theorem 2.3 are fulfilled. ∎

Proof of Theorem 2.4. The proof is the same as the proof of Theorem 2.3. The only difference is that we use Lemma 4.6 instead of Lemma 4.5. ∎

Proof of Theorem 2.5. From (2.21) it is clear that, for an arbitrary sequence $z_n \in C(I;R)$ such that $\lim\limits_{n\to+\infty} ||z_n||_C = +\infty$, the equality (5.19) is holds. From (5.19) and Lemma 4.7, analogously as in the proof of Theorem 2.3, we show that the problem (1.1), (1.2) has at least one solution. ∎

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