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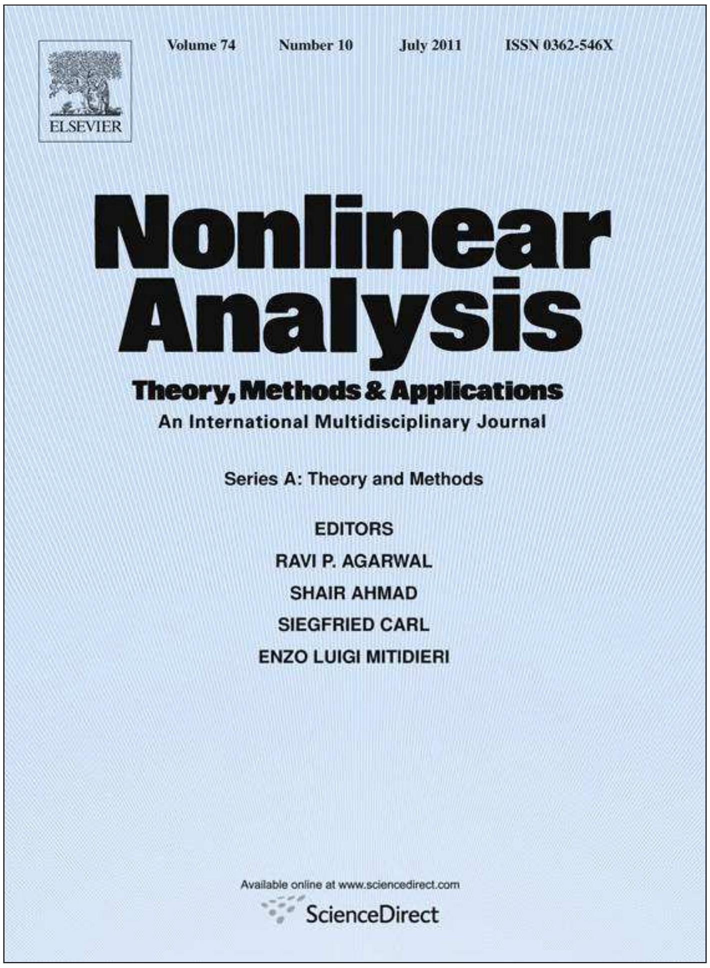

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On a periodic problem for higher-order differential equations with a deviating argument

Sulkhan Mukhigulashvili [a,b], Nino Partsvania [c,d,*], Bedřich Půža [e]

[a] Mathematical Institute, Academy of Sciences of the Czech Republic, Branch in Brno, Brno, Czech Republic
[b] I. Chavchavadze State University, Faculty of Physics and Mathematics, Tbilisi, Georgia
[c] A. Razmadze Mathematical Institute, Georgia
[d] International Black Sea University, Tbilisi, Georgia
[e] Masaryk University, Department of Mathematics and Statistics, Brno, Czech Republic

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ABSTRACT

For higher-order nonautonomous linear and nonlinear differential equations with deviating arguments, new sufficient conditions of existence and uniqueness of a periodic solution are found.

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1. Statement of the main results

1.1. Statement of the problem

In the present paper, for the differential equations with the deviating argument

$$u^{(n)}(t) = p(t)u(\tau(t)) + q(t) \tag{1.1}$$

and

$$u^{(n)}(t) = f(t, u(\tau(t))) + f_0(t), \tag{1.2}$$

a problem on the existence and uniqueness of a periodic solution with a prescribed period $\omega > 0$ is considered. In the case, when $\tau(t) \equiv t$, i.e. when Eqs. (1.1) and (1.2) have the forms

$$u^{(n)}(t) = p(t)u(t) + q(t), \tag{1.3}$$

$$u^{(n)}(t) = f(t, u(t)) + f_0(t), \tag{1.4}$$

* Corresponding address: A. Razmadze Mathematical Institute, 1 M. Aleksidze Str., 0193 Tbilisi, Georgia. Tel.: +995 32 334595; fax: +995 32 3640856.
E-mail addresses: mukhig@ipm.cz (S. Mukhigulashvili), ninopa@rmi.acnet.ge (N. Partsvania), puza@math.muni.cz (B. Půža).

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respectively, the analogous problem was studied sufficiently in detail (see, e.g. [1–24] and the references therein). In particular, by Kiguradze and Kusano [12] and by Kiguradze [13] unimprovable in a certain sense, conditions were found guaranteeing the existence of a unique (of at least one) ω-periodic solution of Eq. (1.3) (Eq. (1.4)). Below, new sufficient conditions are established for the solvability and unique solvability of the periodic problem for Eqs. (1.1) and (1.2), which generalize the above-mentioned results by Kiguradze and Kusano.

Throughout the paper, it is assumed that

$$p \in L_\omega, \qquad q \in L_\omega, \qquad f_0 \in L_\omega, \tag{1.5}$$

and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function from the Carathéodory class such that

$$f(t, 0) \equiv 0, \qquad f(t + \omega, x) \equiv f(t, x). \tag{1.6}$$

As for the function $\tau : \mathbb{R} \to \mathbb{R}$, it is measurable on each finite interval and

$$\frac{\tau(t+\omega) - \tau(t)}{\omega} \quad \text{is an integer for almost all } t \in \mathbb{R}. \tag{1.7}$$

We will use the following notation.

$$[x]_+ = \frac{|x| + x}{2} \quad \text{for } x \in \mathbb{R}.$$

L_ω is the space of ω-periodic and Lebesgue integrable on $[0, \omega]$ functions $y : \mathbb{R} \to \mathbb{R}$ with the norm

$$\|y\|_{L_\omega} = \int_0^\omega |y(t)| \mathrm{d}t.$$

L_ω^∞ is the space of essentially bounded and measurable on $[0, \omega]$ ω-periodic functions $y : \mathbb{R} \to \mathbb{R}$ with the norm

$$\|y\|_{L_\omega^\infty} = \operatorname{ess\,sup}\big\{|y(t)| : \ t \in \mathbb{R}\big\};$$

$$\nu(t) \quad \text{is an integer part of } \frac{\tau(t)}{\omega}, \qquad \tau_0(t) = \tau(t) - \nu(t)\omega; \tag{1.8}$$

$$\delta_0 = \frac{2\pi}{\omega}\left(\int_0^\omega |\tau_0(s) - s| \mathrm{d}s\right)^{1/2}; \tag{1.9}$$

$$\delta(y)(t) = \frac{2\pi}{\omega}\left(\int_0^\omega |\tau_0(s) - s||y(s)| \mathrm{d}s\right)^{1/2} |y(t)|^{1/2} \quad \text{for } y \in L_\omega, \ t \in \mathbb{R}. \tag{1.10}$$

For an arbitrary $y \in L_\omega$, the notation $y(t) \not\equiv 0$ means that y differs from zero on the set of a positive measure.

If m is a natural number, then

$$\gamma_{n,\sigma} = \begin{cases} 1 & \text{for } n = 2m, \sigma = (-1)^m, \\ 0 & \text{for } n = 2m, \sigma = (-1)^{m-1}, \\ 0 & \text{for } n = 2m - 1, \sigma \in \{-1, 1\}. \end{cases} \tag{1.11}$$

Definition 1.1. Let $\sigma \in \{-1, 1\}$. We will say that the function $h : \mathbb{R} \to \mathbb{R}$ belongs to the set $\mathcal{K}_{\omega,\tau}^{n,\sigma}$ if $h \in L_\omega$, $h(t) \geq 0$ for $t \in \mathbb{R}$, $h(t) \not\equiv 0$, and for an arbitrary function $p \in L_\omega$, satisfying the conditions

$$0 \leq \sigma p(t) \leq h(t) \quad \text{for } t \in \mathbb{R}, \ p(t) \not\equiv 0, \tag{1.12}$$

the differential equation

$$u^{(n)}(t) = p(t)u(\tau(t)) \tag{1.13}$$

has no nontrivial ω-periodic solution.

Proposition 1.1. *Let $\sigma \in \{-1, 1\}$, $h \in L_\omega$, $h(t) \geq 0$ for $t \in \mathbb{R}$, $h(t) \not\equiv 0$, and*

$$\gamma_{n,\sigma} h(t) + \delta(h)(t) < \left(\frac{2\pi}{\omega}\right)^n \quad \text{for } t \in \mathbb{R}. \tag{1.14}$$

Then

$$h \in \mathcal{K}_{\omega,\tau}^{n,\sigma}. \tag{1.15}$$

Theorem 1.1. *Let $p(t) \not\equiv 0$ and let there exist a number $\sigma \in \{-1, 1\}$ such that*

$$\sigma p(t) \geq 0 \quad \text{for } t \in \mathbb{R} \tag{1.16}$$