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ORDINARY DIFFERENTIAL EQUATIONS

On a Problem with Nonlinear Boundary Conditions for Systems of Functional-Differential Equations

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Abstract— We use the method of a priori estimates to obtain effective sufficient solvability conditions for systems of nonlinear functional-differential equations with nonlinear boundary conditions of periodic type. The results are in a sense optimal.

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1. STATEMENT OF THE MAIN RESULTS

Let $n \geq 2$ be a positive integer, let $\omega > 0$ and $I = [0, \omega]$, and let $F : C(I; R^n) \to L(I; R^n)$ and $\zeta : C(I; R^n) \to R^n$ be continuous operators. Consider the system of functional-differential equations

$$u'(t) = F(u)(t) \tag{1.1}$$

with the nonlinear boundary condition

$$u(0) - u(\omega) = \zeta(u). \tag{1.2}$$

We assume that $F = (f_i)_{i=1}^n$, where the $f_i : C(I; R^n) \to L(I; R)$ are continuous operators.

A *solution* of Eq. (1.1) is defined as an absolutely continuous vector function $v : I \to R^n$ that satisfies Eq. (1.1) almost everywhere on I, and a *solution* of problem (1.1), (1.2) is defined as a solution of Eq. (1.1) satisfying condition (1.2).

In the present paper, we prove existence theorems for problem (1.1), (1.2) on the basis of theorems of Conti–Opial type obtained in [1] and the method suggested in [2, 3] for the analysis of periodic problems for systems of linear functional-differential equations.

In the present paper, we use the following notation: N is the set of positive integers; $R =]-\infty, +\infty[$; $R_+ = [0, +\infty[$; R^n is the space of column n-vectors $x = (x_i)_{i=1}^n$ with components $x_i \in R$ $(i = 1, \ldots, n)$ equipped with the norm

$$\|x\| = \sum_{i=1}^n |x_i|; \qquad R_+^n = \{(x_i)_{i=1}^n \in R^n : \ x_i \in R_+, \quad i = 1, \ldots, n\};$$

if $x, y \in R^n$, then $x \leq y \Leftrightarrow y - x \in R_+^n$; if $x = (x_i)_{i=1}^n \in R^n$, then $|x| = (|x_i|)_{i=1}^n$; $C(I; R^n)$ is the space of continuous vector functions $x : I \to R^n$ with the norm $\|x\|_C = \max_{t \in I}\{\|x(t)\|\}$; if $x = (x_i)_{i=1}^n \in C(I; R^n)$, then $|x|_C = (\|x_i\|_C)_{i=1}^n$; $L(I; R^n)$ is the space of integrable vector functions $x : I \to R^n$ with the norm $\|x\|_L = \int_0^\omega \|x(s)\| \, ds$.

Definition 1.1. We say that a linear operator

$$\ell : C(I; R^m) \to L(I; R^n), \qquad m, n \in N,$$

is *nonnegative* (respectively, *nonpositive*) if

$\ell(x)(t) \ge 0$ (respectively, $\ell(x)(t) \le 0$), $t \in I$, for every function $x \in C(I; R_+^m)$. A linear operator is said to be *monotone* if it is either nonnegative or nonpositive.

Definition 1.2. We say that a function $\delta: I \times R_+ \to R_+^n$ *belongs to the set* $\mathcal{M}_I$ if the function $\delta(t,\cdot): R_+ \to R_+^n$ is nondecreasing with respect to the second argument almost everywhere on I and $\lim_{\varrho\to+\infty} \varrho^{-1}\int_0^\omega \|\delta(s,\varrho)\|_C\, ds = 0$.

Definition 1.3. We say that an operator $\zeta: C(I; R^n) \to R^n$ *belongs to the set* $\mathcal{N}_I$ if $\lim_{\varrho\to+\infty} \varrho^{-1}\zeta^*(\varrho) = 0$, where $\zeta^*(\varrho) = \max\{\|\zeta(u)\|: \|u\|_C \le \varrho\}$.

On I, consider the system of linear differential inequalities

$$|v'(t) - g_0(v)(t)| \le h(|v|)(t) \tag{1.3}$$

with the periodic boundary condition

$$v(0) = v(\omega). \tag{1.4}$$

Definition 1.4. We say that $(p, q, h) \in Q_\omega$ if the following conditions are satisfied.

(i) $p, q, h: C(I; R^n) \to L(I; R^n)$ are nonnegative linear operators.

(ii) For any linear operator $g_0 = (g_{0,i})_{i=1}^n: C(I; R^n) \to L(I; R^n)$ with monotone components satisfying the condition

$$p(|y|)(t) \le |g_0(|y|)(t)| \le q(|y|)(t) \quad \text{for} \quad t \in I, \quad y \in C(I; R^n), \tag{1.5}$$

problem (1.3), (1.4) has only the trivial solution.

Theorem 1.1. *Suppose that the inequalities*

$$|F(x)(t) - g(x,x)(t)| \le h(|x|)(t) + \delta(t, \|x\|_C), \tag{1.6}$$

$$p(|y|)(t) \le |g(x,|y|)(t)| \le q(|y|)(t) \tag{1.7}$$

hold almost everywhere on I *for arbitrary* $x, y \in C(I; R^n)$, *where*

$$(p, q, h) \in Q_\omega, \tag{1.8}$$

$\delta \in \mathcal{M}_I$, $\zeta \in \mathcal{N}_I$, $g \equiv (g_i)_{i=1}^n: C(I; R^n) \times C(I; R^n) \to L(I; R^n)$ *is a continuous operator, and the* $g_i(x,\cdot): C(I; R^n) \to L(I; R)$ $(i = 1, \dots, n)$ *are linear monotone operators for an arbitrary fixed function* $x \in C(I; R^n)$. *Then problem* (1.1), (1.2) *is solvable.*

Now consider the case in which conditions (1.6) and (1.7) acquire the form

$$|f_i(x)(t) - g_i(x, x_{i+1})(t)| \le \sum_{j=2}^{i+1} h_{i,j}(|x_j|)(t) + \delta_i(t, \|x\|_C) \qquad (i = 1, \dots, n), \tag{1.9}$$

$$p_i(|y|)(t) \le |g_i(x,|y|)(t)| \le q_i(|y|)(t), \tag{1.10}$$

where $x = (x_i)_{i=1}^n \in C(I; R^n)$, $y \in C(I; R)$, $\delta = (\delta_i)_{i=1}^n \in \mathcal{M}_I$, $h_{n,n+1} \overset{\text{def}}{\equiv} h_{n,1}$, $x_{n+1} \overset{\text{def}}{\equiv} x_1$, and the $h_{i,j}, p_i, q_i: C(I; R) \to L(I; R)$ $(i, j = 1, \dots, n)$ are nonnegative linear operators. We define a matrix $A_1 = (a_{i,j}^{(1)})_{i,j=1}^n$ by the relations

$$\begin{gathered} a_{1,1}^{(1)} = -1, \qquad a_{n,1}^{(1)} = \|h_{n,1}\| + \|q_n\|/4, \\ a_{i+1,i+1}^{(1)} = \|h_{i+1,i+1}\| - 1, \quad a_{i,i+1}^{(1)} = \|h_{i,i+1}\| + \|q_i\|/4 \quad \text{if} \quad 1 \le i \le n-1, \\ a_{i,1}^{(1)} = 0 \quad \text{if} \quad 2 \le i \le n-1, \quad a_{i,j}^{(1)} = 0 \quad \text{if} \quad i+2 \le j \le n, \\ a_{i,j}^{(1)} = \|h_{i,j}\| \quad \text{if} \quad 3 \le j+1 \le i \le n \end{gathered} \tag{1.11}$$