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Short Communication

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ON PERIODIC SOLUTIONS OF THE SYSTEM OF TWO LINEAR DIFFERENTIAL EQUATIONS

Abstract. For two-dimensional linear differential systems with periodic coefficients, optimal in a certain sense conditions are established guaranteeing the existence and uniqueness of a periodic solution.

რეზიუმე. პერიოდულკოეფიციენტებიანი ორგანზომილებიანი წრფივი დიფერენციალური სისტემებისათვის დადგენილია გარკვეული აზრით ოპტიმალური პირობები, რაც უზრუნველყოფს პერიოდული ამონახსნის არსებობასა და ერთადერთობას.

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Problems on the existence and uniqueness of a periodic solution of nonautonomous ordinary differential equations and systems have long been attracting the attention of mathematicians and used as the subject of many studies (see, for example, [1]–[29] and the references therein). And all the same these problems still remain topical for the linear differential system

$$u_i' = p_{i1}(t)u_1 + p_{i2}(t)u_2 + q_i(t) \quad (i = 1, 2) \tag{1}$$

with ω-periodic coefficients. In this paper new and, in a certain sense, optimal sufficient conditions for the existence of a unique ω-periodic solution of system (1) are given.

We denote by L_ω the space of functions $p : \mathbb{R} \to \mathbb{R}$ which are periodic with period $\omega > 0$ and Lebesgue integrable on $[0, \omega]$.

Throughout the paper it is assumed that $p_{ik} \in L_\omega$, $q_i \in L_\omega$ $(i, k = 1, 2)$ and the following notation is used:

$$[x]_- = \frac{1}{2}\,(|x| - x) \quad \text{for} \quad x \in \mathbb{R},$$

$$p_i(t) = p_{i\,3-i}(t) \exp\bigg(\int\limits_0^t \big(p_{3-i\,3-i}(s) - p_{ii}(s)\big)\, ds \bigg) \quad (i = 1, 2),$$

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$$\ell = \int_0^\omega |p_1(s)|\,ds \int_0^\omega |p_2(s)|\,ds, \quad \lambda_i = \exp\bigg(-\int_0^\omega p_{ii}(s)\,ds\bigg) \quad (i=1,2),$$

$$\nu_1 = \min\{\lambda_1,\lambda_2\}, \quad \nu_2 = \max\{\lambda_1,\lambda_2\}, \quad \varkappa = \int_0^1 (1-x^4)^{-1/2}\,dx,$$

By k_γ, where $\gamma > 0$, we undestand the functions given by the equalities

$$k_\gamma(x) = (\gamma+3)x^\gamma - x^{2\gamma+2} \ \text{ for } \ 0 \le x \le 1,$$
$$k_\gamma(x) = k_\gamma(2-x) \ \text{ for } \ 1 \le x \le 2, \quad k_\gamma(x+2) = k_\gamma(x) \ \text{ for } \ x \in \mathbb{R}.$$

For any function $p : \mathbb{R} \to \mathbb{R}$ the notation $p(t) \not\equiv 0$ means that p is different from zero on the set of positive measure.

The case, where for some $\sigma \in \{-1,1\}$ the inequalities

$$\sigma p_1(t) \ge 0, \quad \sigma p_2(t) \ge 0 \ \text{ for } \ t \in \mathbb{R}$$

hold, is considered in [16].

Theorems formulated below refer to the case where the functions p_1 and p_2 satisfy, for some $\sigma \in \{-1,1\}$, one of the following four conditions:

$$\sigma p_1(t) \ge 0, \quad \sigma p_2(t) \le 0 \ \text{ for } \ t \in \mathbb{R}; \tag{3_1}$$

$$\sigma p_1(t) \ge 0, \quad \sigma \int_t^{t+\omega} p_2(\tau)\,d\tau < 0 \ \text{ for } \ t \in \mathbb{R}; \tag{3_2}$$

$$\sigma p_1(t) > 0, \quad \sigma p_2(t) \le 0 \ \text{ for } \ t \in \mathbb{R}; \tag{4_1}$$

$$\sigma p_1(t) > 0, \quad \sigma \int_t^{t+\omega} p_2(\tau)\,d\tau \le 0 \ \text{ for } \ t \in \mathbb{R}. \tag{4_2}$$

It is also required in these theorems that

$$(1-\lambda_1)(\lambda_2-1) \notin \,]\ell\nu_1, \ell\nu_2[\,, \tag{5_1}$$

or

$$\int_0^\omega \big(p_{22}(s) - p_{11}(s)\big)\,ds \int_0^\omega p_{11}(s)\,ds \ge 0. \tag{5_2}$$

Theorem 1. *Let, for some $\sigma \in \{-1,1\}$, either conditions (3_1) and (5_1) or conditions (3_2) and (5_2) or conditions (4_2) and (5_2) be fulfilled. Let, furthermore, $p_1(t)[\sigma p_2(t)]_- \not\equiv 0$ and*

$$\int_t^{t+\omega} |p_1(s)|\,ds \int_t^{t+\omega} [\sigma p_2(s)]_-\,ds \le 16 \ \text{ for } \ t \in \mathbb{R}. \tag{6}$$

Then system (1) *has a unique ω-periodic solution.*

Example 1. For arbitrarily given $\varepsilon \in]0,1[$, choose $\varepsilon_0 > 0$, δ and δ_0 such that

$$(1+\varepsilon_0)^2\varepsilon_0^2 < \varepsilon, \quad \exp(\delta\omega) - 1 = \varepsilon_0, \quad \delta_0 = \frac{2}{\omega}\,\varepsilon_0.$$

Let

$$p_{11}(t) = -\delta, \quad p_{22}(t) = \delta \ \text{ for } \ t \in \mathbb{R},$$

$$\Delta_1(t) = \begin{cases} 0 & \text{for } \ 0 \le t \le \dfrac{\omega}{2} \\ \delta_0 & \text{for } \ \dfrac{\omega}{2} < t < \omega \end{cases}, \quad \Delta_2(t) = \begin{cases} \delta_0 & \text{for } \ 0 \le t \le \dfrac{\omega}{2} \\ 0 & \text{for } \ \dfrac{\omega}{2} < t < \omega \end{cases},$$

and p_{12} and p_{21} be ω-periodic functions such that

$$p_{12}(t) = \Delta_1(t)\exp(-2\delta t), \quad p_{21}(t) = -\Delta_2(t)\exp(2\delta t) \ \text{ for } \ 0 \le t < \omega.$$

Then

$$\lambda_1 = \exp(\delta\omega), \quad \lambda_2 = \exp(-\delta\omega),$$
$$p_1(t) = \Delta_1(t), \quad p_2(t) = -\Delta_2(t) \ \text{ for } \ 0 \le t < \omega.$$

By the identities $p_i(t+\omega) \equiv \frac{\lambda_i}{\lambda_{3-i}}\, p_i(t)$ $(i=1,2)$ we have $\ell = \varepsilon_0^2$,

$$\int\limits_t^{t+\omega} |p_1(s)|\,ds \int\limits_t^{t+\omega} |p_2(s)|\,ds =$$

$$= \Bigg(\int\limits_t^{\omega} |p_1(s)|\,ds + \frac{\lambda_1}{\lambda_2}\int\limits_0^{t} |p_1(s)|\,ds\Bigg)\Bigg(\int\limits_t^{\omega} |p_2(s)|\,ds + \frac{\lambda_2}{\lambda_1}\int\limits_0^{t} |p_2(s)|\,ds\Bigg) \le$$

$$\le \frac{\lambda_1}{\lambda_2}\int\limits_0^{\omega} |p_1(s)|\,ds \int\limits_0^{\omega} |p_2(s)|\,ds = \exp(2\delta\omega)\ell = (1+\varepsilon_0)^2\varepsilon_0^2 \ \text{ for } \ 0 \le t \le \omega,$$

$$\int\limits_t^{t+\omega} |p_1(s)|\,ds \int\limits_t^{t+\omega} |p_2(s)|\,ds < \varepsilon \ \text{ for } \ t \in \mathbb{R}, \tag{7}$$

and

$$(1-\lambda_1)(\lambda_2-1) = \exp(-\delta\omega)\big(\exp(2\delta\omega)-1\big)^2 = \ell\nu_1.$$

Hence it is clear that, along with condition (6), conditions (3_1) and (5_1), where $\sigma = 1$, are fulfilled too. However, the condition $p_1(t)[\sigma p_2(t)]_- \not\equiv 0$ is violated. Nevertheless the homogeneous system

$$u_i' = p_{i1}(t)u_1 + p_{i2}(t)u_2 \ \ (i=1,2) \tag{1_0}$$

has a nontrivial ω-periodic solution (u_1,u_2) with the components

$$u_1(t) = \Big[1 + \Delta_1(t)\Big(t - \frac{\omega}{2}\Big)\Big]\exp(-\delta t), \ \ u_2(t) = \Big[1 - \Delta_2(t)\Big(t - \frac{\omega}{2}\Big)\Big]\exp(\delta t)$$
$$\text{for } \ 0 \le t < \omega.$$

The constructed example shows that the condition $p_1(t)[\sigma p_2(t)]_- \not\equiv 0$ in Theorem 1 is essential and cannot be neglected even if condition (7), where ε is an arbitrarily small positive number, is fulfilled instead of (6).

Example 2. For arbitrary $\varepsilon \in]0, 1/2[$, choose $\delta > 0$ such that

$$(1-\varepsilon)^{-1/2} < \exp(\delta\omega) < 2(1-\varepsilon)^{-1/2} - 1$$

and put

$$p_{12}(t) \equiv p_{22}(t) \equiv \delta, \quad p_{11}(t) \equiv p_{21}(t) \equiv -\delta.$$

Then conditions (3_k) and (4_k) $(k = 1, 2)$, where $\sigma = 1$, are fulfilled. Moreover, $\nu_2 = \lambda_1 = \exp(\delta\omega)$, $\nu_1 = \lambda_2 = \exp(-\delta\omega)$, $p_1(t) = \delta\exp(2\delta t)$, $p_2(t) = -\delta\exp(-2\delta t)$. Hence

$$\int\limits_t^{t+\omega} |p_1(s)|\,ds \int\limits_t^{t+\omega} |p_2(s)|\,ds \equiv \ell = \frac{1}{4}\,\exp(-2\delta\omega)\big(\exp(2\delta\omega) - 1\big)^2 <$$

$$< \frac{1}{4}\,\exp(2\delta\omega) < (1-\varepsilon)^{-1} < 2,$$

$$(\lambda_1 - 1)(1 - \lambda_2) = \exp(-\delta\omega)\big(\exp(\delta\omega) - 1\big)^2 =$$

$$= 4\lambda_1\big(\exp(\delta\omega) + 1\big)^{-2}\ell > (1-\varepsilon)\ell\nu_2 > \ell\nu_1.$$

Thus condition (6) is fulfilled, but condition (5_1) is violated and instead of the latter condition we have

$$(1 - \lambda_1)(\lambda_2 - 1) \notin]\ell\nu_1, (1-\varepsilon)\ell\nu_2[\,. \tag{8}$$

On the other hand, the homogeneous system (1_0) has a nontrivial ω-periodic solution (u_1, u_2) with the components $u_i(t) \equiv 1$ $(i = 1, 2)$. The constructed example shows that condition (5_1) in Theorem 1 cannot be replaced by condition (8) no matter how small $\varepsilon > 0$ is.

To construct the next example showing the optimality of condition (6) in Theorem 1, we have to introduce, for any $\gamma > 0$, the function $y_\gamma : \mathbb{R} \to \mathbb{R}$ by means of the following equalities:

$$y_\gamma(x) = x\exp\Big(-\frac{x^{\gamma+2}}{\gamma+2}\Big) \ \text{ for } \ 0 \le x \le 1, \tag{9}$$

$$y_\gamma(x) = y_\gamma(2-x) \ \text{ for } \ 1 \le x \le 2, \quad y_\gamma(x+2) = -y_\gamma(x) \ \text{ for } \ x \in \mathbb{R}. \tag{10}$$

By definition of the function k_γ it is clear that

$$y_\gamma''(x) = -k_\gamma(x)y_\gamma(x), \quad y_\gamma(x+4) = y_\gamma(x) \ \text{ for } \ x \in \mathbb{R}. \tag{11}$$

Example 3. Let $\varepsilon \in]0, 1[$, $\gamma = 24/\varepsilon$, $p \in L_\omega$, $p(t) > 0$ for $t \in \mathbb{R}$ and

$$\delta = 4\Big(\int\limits_0^\omega p(s)\,ds\Big)^{-1}.$$

Put

$$p_{11}(t) \equiv p_{22}(t) \equiv 0, \quad p_{12}(t) = p(t), \quad p_{21}(t) = -\delta^2 p(t) k_\gamma \Bigg(\delta \int\limits_0^t p(s)\, ds \Bigg).$$

Then $p_{ik} \in L_\omega$ $(i,k = 1,2)$ and the functions $p_i(t) \equiv p_{i\,3-i}(t)$ $(i = 1,2)$ satisfy conditions (3_k), (4_k) and (5_k) $(k = 1,2)$, where $\sigma = 1$. Moreover,

$$\int\limits_t^{t+\omega} |p_1(s)|\, ds \int\limits_t^{t+\omega} |p_2(s)|\, ds = \delta^2 \int\limits_0^\omega p(s)\, ds \int\limits_0^\omega p(s) k_\gamma \Bigg(\delta \int\limits_0^s p(\tau)\, d\tau \Bigg) ds =$$

$$= 4 \int\limits_0^4 k_\gamma(x)\, dx = 16 \int\limits_0^1 k_\gamma(x)\, dx = 16 + \frac{16(3\gamma+5)}{(\gamma+1)(2\gamma+3)} <$$

$$< 16 + \frac{24}{\gamma} = 16 + \varepsilon.$$

From (9)–(11) it follows that the vector function (u_1, u_2) with the components

$$u_1(t) = y_\gamma \Bigg(\delta \int\limits_0^t p(\tau)\, d\tau \Bigg), \quad u_2(t) = \delta y'_\gamma \Bigg(\delta \int\limits_0^t p(\tau)\, d\tau \Bigg)$$

is a nontrivial ω-periodic solution of system (1_0). The constructed example shows that in the right-hand part of inequality (6) in Theorem 1 we cannot replace 16 by $16 + \varepsilon$ no matter how small $\varepsilon > 0$ is.

If we replace conditions (5_1) and (6) in Theorem 1 by the more strong conditions

$$(1-\lambda_1)(\lambda_2 - 1) \notin [\ell\nu_1, \ell\nu_2] \tag{$5'_1$}$$

and

$$\int\limits_t^{t+\omega} |p_1(s)|\, ds \int\limits_t^{t+\omega} [\sigma p_2(s)]_-\, ds < 16 \quad \text{for} \quad t \in \mathbb{R}, \tag{$6'$}$$

respectively, then the condition $p_1(t)[\sigma p_2(t)]_- \not\equiv 0$ can be replaced by the condition $p_i(t) \not\equiv 0$ $(i = 1,2)$. More exactly, the following theorem is valid.

Theorem 2. *Let $p_i(t) \not\equiv 0$ $(i = 1,2)$ and there exist $\sigma \in \{-1,1\}$ such that either conditions (3_1) and $(5'_1)$ or conditions (3_2) and (5_2) or conditions (4_2) and (5_2) are fulfilled. Let, furthermore, inequality $(6')$ be fulfilled too. Then system (1) has a unique ω-periodic solution.*

Theorem 3. *Let $p_i(t) \not\equiv 0$ $(i = 1,2)$, and conditions (4_k) and (5_k) be fulfilled for some $\sigma \in \{-1,1\}$ and $k \in \{1,2\}$. Let, furthermore, either*

$$\sigma p_2(t) > -4\pi^2 |p_1(t)| \Bigg(\int\limits_{t_0}^{t_0+\omega} |p_1(s)|\, ds \Bigg)^{-2} \quad \textit{for} \;\; t_0 < t < t_0 + \omega, \;\; t_0 \in \mathbb{R}, \tag{12}$$

or

$$\left(\int_t^{t+\omega} |p_1(s)|\,ds\right)^3 \int_t^{t+\omega} |p_1(s)|^{-1}[\sigma p_2(s)]_-^2\,ds < \frac{1024}{3}\,\varkappa^4 \quad \text{for } t \in \mathbb{R}. \tag{13}$$

Then system (1) *has a unique ω-periodic solution.*

Note that in Theorem 3 condition (12) cannot be replaced by the condition

$$\sigma p_2(t) \geq -4\pi^2 |p_1(t)| \left(\int_{t_0}^{t_0+\omega} |p_1(s)|\,ds\right)^{-2} \quad \text{for } t_0 < t < t_0+\omega, \;\; t_0 \in \mathbb{R}$$

and condition (13) cannot be replaced by the condition

$$\left(\int_t^{t+\omega} |p_1(s)|\,ds\right)^3 \int_t^{t+\omega} |p_1(s)|^{-1}[\sigma p_2(s)]_-^2\,ds \leq \frac{1024}{3}\,\varkappa^4 \quad \text{for } t \in \mathbb{R}.$$

Now let us consider the differential equation of second order

$$u'' = g_1(t)u + g_2(t)u' + h(t), \tag{14}$$

where $g_i \in L_\omega$ $(i = 1, 2)$, $h \in L_\omega$.

Put

$$r(t) = \exp\left(\int_0^t g_2(s)\,ds\right).$$

Theorems 1–3 immediately give rise to the following proposition.

Corollary. *Let $g_1(t) \not\equiv 0$ and*

$$\int_t^{t+\omega} \frac{g_1(s)}{r(s)}\,ds \leq 0 \quad \text{for } t \in \mathbb{R}.$$

Let, furthermore, one of the following three conditions be fulfilled:

$$\int_t^{t+\omega} r(s)\,ds \int_t^{t+\omega} \frac{[g_1(s)]_-}{r(s)}\,ds \leq 16 \quad \text{for } t \in \mathbb{R};$$

$$g_1(t) > -4\pi^2 r^2(t) \left(\int_{t_0}^{t_0+\omega} r(s)\,ds\right)^{-2} \quad \text{for } t_0 < t < t_0+\omega, \;\; t_0 \in \mathbb{R};$$

$$\left(\int_t^{t+\omega} r(s)\,ds\right)^3 \int_t^{t+\omega} r^{-3}(s)[g_1(s)]_-^2\,ds < \frac{1024}{3}\,\varkappa^2 \quad \text{for } t \in \mathbb{R}.$$

Then equation (14) *has a unique ω-periodic solution.*

This corollary is a generalization of the well-known results by Lasota–Opial [19] and J. Mawhin and J. R. Ward [23], [24] concerning the existence of a unique ω-periodic solution of equation (14) for $g_2(t) \equiv 0$.

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