

AN OPTIMAL CONDITION FOR THE UNIQUENESS OF A PERIODIC SOLUTION FOR LINEAR FUNCTIONAL DIFFERENTIAL SYSTEMS

S. MUKHIGULASHVILI, I. GRYTSAY

ABSTRACT. Unimprovable effective efficient conditions are established for the unique solvability of the periodic problem

$$\begin{aligned}u'_i(t) &= \sum_{j=2}^{i+1} \ell_{i,j}(u_j)(t) + q_i(t) \quad \text{for } 1 \leq i \leq n-1, \\u'_n(t) &= \sum_{j=1}^n \ell_{n,j}(u_j)(t) + q_n(t), \\u_j(0) &= u_j(\omega) \quad \text{for } 1 \leq j \leq n,\end{aligned}$$

where $\omega > 0$, $\ell_{i,j} : C([0, \omega]) \rightarrow L([0, \omega])$ are linear bounded operators, and $q_i \in L([0, \omega])$.

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1. STATEMENT OF PROBLEM AND FORMULATION OF MAIN RESULTS

Consider on $[0, \omega]$ the system

$$\begin{aligned}u'_i(t) &= \sum_{j=2}^{i+1} \ell_{i,j}(u_j)(t) + q_i(t) \quad \text{for } 1 \leq i \leq n-1, \\u'_n(t) &= \sum_{j=1}^n \ell_{n,j}(u_j)(t) + q_n(t),\end{aligned} \tag{1.1}$$

with the periodic boundary conditions

$$u_j(0) = u_j(\omega) \quad \text{for } 1 \leq j \leq n, \tag{1.2}$$

where $n \geq 2$, $\omega > 0$, $\ell_{i,j} : C([0, \omega]) \rightarrow L([0, \omega])$ are linear bounded operators and $q_i \in L([0, \omega])$.

By a solution of the problem (1.1), (1.2) we understand a vector function $u = (u_i)_{i=1}^n$ with $u_i \in \tilde{C}([0, \omega])$ ($i = \overline{1, n}$) which satisfies system (1.1) almost everywhere on $[0, \omega]$ and satisfies conditions (1.2).

Much work had been carried out on the investigation of the existence and uniqueness of the solution for a periodic boundary value problem for systems of ordinary differential equations and many interesting results have been obtained (see, for instance, [1–3, 7–9, 11, 12, 17] and the references therein). However, an analogous problem for functional differential equations, remains investigated in less detail even for linear equations. In the present paper, we study problem (1.1)–(1.2) under the assumption that $\ell_{n,1}, \ell_{i,i+1}$ ($i = \overline{1, n-1}$) are monotone linear operators. We establish new unimprovable integral conditions sufficient for unique solvability of the problem (1.1)–(1.2) which generalize the well-known results of A. Lasota and Z. Opial (see Remark 1.1) obtained for ordinary differential equations in [13], and on the other hand, extend results obtained for linear functional differential equations in [5, 14–16]. These results are new not only for the systems of functional differential equations (for reference see [2, 4, 6, 10]), but also for the system of ordinary differential equations of the form

$$\begin{aligned} u'_i(t) &= \sum_{j=2}^{i+1} p_{i,j}(t)u_j(t) + q_i(t) \quad \text{for } 1 \leq i \leq n-1, \\ u'_n(t) &= \sum_{j=1}^n p_{n,j}(t)u_j(t) + q_n(t), \end{aligned} \tag{1.3}$$

where $q_i, p_{i,j} \in L([0, \omega])$ (see, for instance, [2, 7–9] and the references therein). The method used for the investigation of the problem considered is based on that developed in our previous papers [14–16] for functional differential equations.

The following notation is used throughout the paper: $N(R)$ is the set of all the natural (real) numbers; R^n is the space of n -dimensional column vectors $x = (x_i)_{i=1}^n$ with elements $x_i \in R$ ($i = \overline{1, n}$); $R_+ = [0, +\infty[$; $C([0, \omega])$ is the Banach space of continuous functions $u : [0, \omega] \rightarrow R$ with the norm $\|u\|_C = \max\{|u(t)| : 0 \leq t \leq \omega\}$; $C([0, \omega]; R^n)$ is the space of continuous functions $u : [0, \omega] \rightarrow R^n$; $\widetilde{C}([0, \omega])$ is the set of absolutely continuous functions $u : [0, \omega] \rightarrow R$; $L([0, \omega])$ is the Banach space of Lebesgue integrable functions $p : [0, \omega] \rightarrow R$ with the norm $\|p\|_L = \int_0^\omega |p(s)|ds$; if $\ell : C([0, \omega]) \rightarrow L([0, \omega])$ is a linear operator, then $\|\ell\| = \sup_{0 < \|x\|_C \leq 1} \|\ell(x)\|_L$.

Definition 1.1. We will say that a linear operator $\ell : C([0, \omega]) \rightarrow L([0, \omega])$ is *nonnegative* (*nonpositive*), if for any nonnegative $x \in C([0, \omega])$ the inequality $\ell(x)(t) \geq 0$ ($\ell(x)(t) \leq 0$) for $0 \leq t \leq \omega$ is satisfied. We will say that an operator ℓ is *monotone* if it is either nonnegative or nonpositive.