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A PERIODIC BOUNDARY VALUE PROBLEM FOR FUNCTIONAL DIFFERENTIAL EQUATIONS OF HIGHER ORDER

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Abstract. On the interval $[0,\omega]$, consider the periodic boundary value problem

$$u^{(n)}(t) = \sum_{i=0}^{n-1} \ell_i(u^{(i)})(t) + q(t),$$
$$u^{(j)}(0) = u^{(j)}(\omega) + c_j \qquad (j = 0, \dots, n-1),$$

where $n \geq 2$, $\ell_i : C\big([0,\omega]; R\big) \to L\big([0,\omega]; R\big)$ $(i = 0, \dots, n-1)$ are linear bounded operators, $q \in L\big([0,\omega]; R\big)$, $c_j \in R$ $(j = 0, \dots, n-1)$. The effective sufficient conditions guaranteeing the unique solvability of the considered problem are established.

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Statement of the Problem

Consider the problem on the existence and uniqueness of a solution to the equation

$$u^{(n)}(t) = \sum_{i=0}^{n-1} \ell_i(u^{(i)})(t) + q(t) \qquad \text{for } 0 \leq t \leq \omega \tag{0.1}$$

satisfying the periodic boundary conditions

$$u^{(j)}(0) = u^{(j)}(\omega) + c_j \qquad (j = 0, \dots, n-1), \tag{0.2}$$

where $n \geq 2$, $\ell_i : C\big([0,\omega]; R\big) \to L\big([0,\omega]; R\big)$ are linear bounded operators, $q \in L\big([0,\omega]; R\big)$, and $c_j \in R$ $(i, j = 0, \dots, n-1)$.

By a solution to problem (0.1), (0.2) we understand a function $u \in \widetilde{C}^{n-1}\big([0,\omega]; R\big)$, which satisfies equality (0.1) almost everywhere on $[0,\omega]$ and the boundary condition (0.2).

It is well-known that if the linear operators $\ell_i : C\big([0,\omega]; R\big) \to L\big([0,\omega]; R\big)$ $(i = 0, \dots, n-1)$ are strongly bounded, i.e., if there exist summable functions $\eta_i : [0,\omega] \to [0,+\infty[$ such that

$$|\ell_i(x)(t)| \leq \eta_i(t)\|x\|_C \qquad \text{for } 0 \leq t \leq \omega, \quad x \in C\big([0,\omega]; R\big),$$

then the following theorem on the Fredholm property is valid (see, e.g., [1,10,18])

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Theorem 0.1. *Problem* (0.1), (0.2) *is uniquely solvable iff the corresponding homogeneous problem*

$$v^{(n)}(t)=\sum_{i=0}^{n-1}\ell_i(v^{(i)})(t), \tag{0.3}$$

$$v^{(j)}(0)=v^{(j)}(\omega) \qquad (j=0,\dots,n-1), \tag{0.4}$$

has only the trivial solution.

The above-mentioned Fredholm property for functional differential equations with general bounded linear operators (i.e., not necessarily strongly bounded) had not been investigated before 2000 despite of the fact that in 1972 H. H. Schaefer [17, Theorem 4] proved that there do exist linear bounded operators $\ell : C\big([0,\omega];R\big)\to L\big([0,\omega];R\big)$ which are not strongly bounded. The first important steps in this direction were made by Bravyi in [2], and later in [5], where, among others, the Fredholm property was proved for the first order boundary value problems for functional differential equations with general bounded linear operators. These results were generalized for the n-th order functional differential systems in [7]. Therefore, Theorem 0.1 is also valid if ℓ_i $(i=0,\dots,n-1)$ are bounded (not necessarily strongly bounded) linear operators.

The problem on the existence of a periodic solution to ordinary and functional differential equations was studied very intensively in the past. The first important step was made for linear ordinary differential equations of the type

$$u^{(n)}(t)=p(t)u(t)+q(t) \tag{0.5}$$

by Lasota and Opial in [11]. They showed that problem (0.5), (0.2) is uniquely solvable for $n\geq 4$ if a function $p\in L\big([0,\omega];R\big)$ has the constant sign, $p\not\equiv 0$, and the inequality

$$\int_0^\omega |p(s)|ds<\left(\frac{2}{\omega}\right)^{n-1}\frac{2\cdot 4\cdots(n-2)}{1\cdot 3\cdots(n-3)} \tag{0.6}$$

is fulfilled. This result is far from being optimal, and in [12], condition (0.6) was improved to

$$\int_0^\omega |p(s)|ds<\frac{2}{\omega}\left(\frac{2\pi}{\omega}\right)^{n-2}. \tag{0.7}$$

The next step was made by Kiguradze and Kusano in [8], where the results of [11,12] were essentially improved. In particular, they proved following propositions.

Proposition 0.1. *Let either* $n=2m$, $(-1)^{m-1}p(t)\geq 0$ *for* $t\in[0,\omega]$, $p(t)\not\equiv 0$ *or* $n=2m-1$, $\sigma p(t)\geq 0$ *for* $t\in[0,\omega]$, $p(t)\not\equiv 0$, *where* $\sigma\in\{-1,1\}$. *Then problem* (0.5), (0.2) *has a unique solution.*

Proposition 0.2. *Let* $n=2m$, $(-1)^m p(t)\geq 0$ *for* $t\in[0,\omega]$, $p(t)\not\equiv 0$ *and inequality* (0.7) *be fulfilled. Then problem* (0.5), (0.2) *has a unique solution.*