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ON A PERIODIC BOUNDARY VALUE PROBLEM FOR THIRD ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Unimprovable efficient sufficient conditions are established for the unique solvability of the periodic problem

$$u'''(t) = \sum_{j=0}^{2} \ell_j(u^{(j)})(t) + q(t) \quad \text{for} \quad 0 \le t \le \omega,$$

$$u^{(i)}(0) = u^{(i)}(\omega) \ (i = 0, 1, 2),$$

where $\omega > 0$, $\ell_j : C([0,\omega]) \to L([0,\omega])$ $(j = 0, 1, 2)$ are a linear bounded operators and $q \in L([0,\omega])$.

Key Words. Linear functional differential equation, ordinary differential equation, concentrated operator, periodic boundary value problem.

AMS(MOS) subject classification. 34K06, 34K10

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Introduction. Consider the equation

$$u'''(t) = \sum_{j=0}^{2} \ell_j(u^{(j)})(t) + q(t) \quad \text{for} \quad 0 \le t \le \omega, \tag{0.1}$$

with the periodic boundary conditions

$$u^{(i)}(0) = u^{(i)}(\omega) \qquad (i = 0, 1, 2), \tag{0.2}$$

where $\omega > 0$, $\ell_j : C([0,\omega]) \to L([0,\omega])$ $(j = 0, 1, 2)$ are a linear bounded operators and $q \in L([0,\omega])$.

By a solution of the problem (0.1), (0.2) we understand a function $u \in \widetilde{C}^2([0,\omega])$, which satisfies the equation (0.1) almost everywhere on $[0,\omega]$ and satisfies the conditions (0.2).

The periodic boundary value problem for higher order ordinary differential equations has been studied by many authors (see, for instance, [2, 5, 9, 10, 15] and the references therein). But an analogous problem for functional differential equations, even in the case of linear equations remains little investigated. Results obtained in this paper are nonimprovable and on the one hand generalise the well-known results of A. Lasota and Z. Opial (see [10, Theorem 6, p. 88]) for linear ordinary differential equations, and on the other hand describe some properties which belong only to functional differential equations. In the paper [13], it was proved that the problem (0.1), (0.2) has a unique solution if the inequality

$$0 < \int_0^{\omega} |\ell_0(1)(s)| ds \le \frac{d}{\omega^2} \left(1 - \frac{\omega}{4}||\ell_1|| - ||\ell_2||\right) \tag{0.3}$$

with $d = 128$ is fulfilled. Moreover, there was also shown that the condition (0.3) is nonimprovable. This paper attempts to find a specific subset of the set of linear monotone operators, in which the condition (0.3) guarantees the unique solvability of the problem (0.1), (0.2) even for $d \ge 128$ (see Corollary 1.1). It turned out that if A satisfies some conditions dependent only on the constants d and ω, then $K_{[0,\omega]}(A)$ (see Definition 0.2) is such a subset of the set of linear monotone operators.

In 1972 H.H. Schaefer [14] proved that there exists a linear bounded operator $\ell \in C([0,\omega]) \to C([0,\omega])$ which is not strongly bounded, that is, it does not have the following property: there exists a summable function $\eta : [0,\omega] \to [0,+\infty]$ such that

$$|\ell(x)(t)| \le \eta(t)||x||_C \quad \text{for} \quad 0 \le t \le \omega, \quad x \in C([0,\omega]).$$

It is well-known (see, e.g., [7, 8]) that the general boundary value problem for linear functional differential equations with a strongly bounded linear operator has the Fredholm property, i.e., it is uniquely solvable iff the corresponding homogeneous problem has only the trivial solution. The same property (Fredholmity) for functional differential equations with a nonstrongly bounded linear operator was not investigated till 2000. The first step was made in [1] for scalar first order functional differential equations. Those results were generalized for the n-th order functional differential systems in [4].

Thus, in the present paper, we study the problem (0.1), (0.2) under the assumptions that ℓ_0 is a strongly bounded operator and ℓ_j $(j = 1, 2)$ are bounded, not necessarily strongly bounded, operators. We establish new nonimprovable, integral, sufficient conditions of unique solvability of the problem (0.1), (0.2).

The following notation is used throughout:

N is the set of all natural numbers.

R is the set of all real numbers, $R_+ = [0, +\infty[$.

$C([a,b])$ is the Banach space of continuous functions $u : [a,b] \to R$ with the norm $\|u\|_C = \max\{|u(t)| : a \le t \le b\}$.

$\widetilde{C}^2([a,b])$ $(\widetilde{C}^3([a,b]))$ is the set of functions $u : [a,b] \to R$ which are absolutely continuous together with their second (third) derivatives.

$L([a,b])$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to R$ with the norm $\|p\|_L = \int_a^b |p(s)|ds$.

If $\ell : C([a,b]) \to L([a,b])$ is a linear operator, then

$$||\ell|| = \sup_{0<||x||_C<1} ||\ell(x)||_L.$$

If $x \in R$, then $[x]_+ = (|x| + x)/2$, $[x]_- = (|x| - x)/2$.

Definition 1. *We will say that an operator $\ell : C([a,b]) \to L([a,b])$ is nonnegative (nonpositive), if for any nonnegative $x \in C([a,b])$ the inequality $\ell(x)(t) \ge 0$ $(\ell(x)(t) \le 0)$ for $a \le t \le b$ is satisfied. We will say that an operator ℓ is monotone if it is nonnegative or nonpositive.*

Definition 2. *Let $A \subset [a,b]$ be a nonempty set. We will say that a linear operator $\ell : C([a,b]) \to L([a,b])$ belongs to the set $K_{[a,b]}(A)$ if for any $x \in C([a,b])$, satisfying $x(t) = 0$ for $t \in A$, the equality*

$$\ell(x)(t) = 0 \qquad \textit{for} \quad a \le t \le b$$

holds. We will say that $K_{[a,b]}(A)$ is the set of operators concentrated on the set $A \subset [a,b]$.

1. Main Results. Define, for any nonempty set $A \subseteq R$, the continuous (see [12] Lemma 2.1) functions:

$$\rho_A(t) = \inf\{|t-s| : s \in A\}, \quad \sigma_A(t) = \rho_A(t) + \rho_A(t + \frac{\omega}{2}) \quad \text{for} \quad t \in R. \tag{1.1}$$

THEOREM 1. *Let $A \subset [0,\omega]$, $A \neq \emptyset$ and a linear monotone operator $\ell_0 \in K_{[0,\omega]}(A)$ be such that*

$$\int_0^\omega \ell_0(1)(s)ds \neq 0. \tag{1.2}$$

Moreover, let the linear bounded operators ℓ_1, ℓ_2, $\delta \in [0, \omega/2]$ and the set A be such that the conditions

$$0 < 1 - \frac{\omega}{4}||\ell_1|| - ||\ell_2||, \tag{1.3}$$

$$\left(1 - 4\left(\frac{\delta}{\omega}\right)^2\right) \int_0^\omega |\ell_0(1)(s)|ds \leq \frac{128}{\omega^2}\left(1 - \frac{\omega}{4}||\ell_1|| - ||\ell_2||\right), \tag{1.4}$$

$$\delta \leq \min\left\{\sigma_A(t) : 0 \leq t \leq \frac{0}{\omega}\right\} \tag{1.5}$$

are satisfied. Then the problem (0.1), (0.2) *has a unique solution.*

EXAMPLE 1. *The example below shows that condition* (1.4) *in Theorem 1 is optimal and it cannot be replaced by the condition*

$$\left(1 - 4\left(\frac{\delta}{\omega}\right)^2\right) \int_0^\omega |\ell_0(1)(s)|ds \leq \frac{128}{\omega^2}\left(1 - \frac{\omega}{4}||\ell_1|| - ||\ell_2||\right) + \varepsilon, \tag{1.4$_\varepsilon$}$$

no matter how small $\varepsilon \in]0,1]$ would be. Let $\omega = 1$, $A = [0,\omega]$, $\alpha_k = \frac{1}{32} + \frac{1}{16\pi^2k^2} - \frac{1}{128k^2}$, $\beta_k = \frac{1}{8k} - \frac{1}{4}$, $k \in N$, and the function $u_0 \in \widetilde{C}^3([0,1])$ be defined by the equality

$$u_0(t) = \begin{cases} \widetilde{u}(t) & \text{for } 0 \leq t \leq 1/2 \\ -\widetilde{u}(t - 1/2) & \text{for } 1/2 < t \leq 1 \end{cases},$$

where

$$\widetilde{u}(t) = \begin{cases} 1 - \frac{t^2}{2\alpha_{k_0}}, & 0 \le t \le 1/4 - 1/8k_0 \\ 1 + \frac{\beta_{k_0}}{\alpha_{k_0}} t + \frac{\beta_{k_0}^2}{2\alpha_{k_0}} - \frac{1 - \sin \pi k_0 (1-4t)}{16\pi^2 k_0^2 \alpha_{k_0}}, & 1/4 - 1/8k_0 < t \le 1/4 + 1/8k_0 \\ -1 - \frac{t(1-t)}{2\alpha_{k_0}} + \frac{1}{8\alpha_{k_0}}, & 1/4 + 1/8k_0 < t \le 1/2 \end{cases},$$

and $k_0 \in N$ *is such that*

$$\frac{4}{(128+\varepsilon)\alpha_{k_0}} < 1. \tag{1.6}$$

Then it is clear that $u_0^{(j)}(0) = u_0^{(j)}(1)$ $(j = 0, 1, 2)$, *and there exist constants* $\lambda_1 > 0, \lambda_2 > 0$ *such that*

$$\left(\frac{\lambda_1}{4} + \lambda_2 \right) \int_0^1 |u_0'''(s)| ds = 1 - \frac{4}{(128+\varepsilon)\alpha_{k_0}}. \tag{1.7}$$

Now, let a measurable function $\tau : [0,1] \to [0,1]$ *and the linear operators* $\ell_i : C([0,1]) \to L([0,1])$ $(i = 0, 1, 2)$ *be given by the equalities:*

$$\tau(t) = \begin{cases} 0 & \text{for } u_0'''(t) > 0 \\ 1/2 & \text{for } u_0'''(t) \le 0 \end{cases},$$

$$\ell_0(x)(t) = |u_0'''(t)| x(\tau(t)), \;\; \ell_i(x)(t) = \lambda_i |u_0'''(t)| x\left(\frac{i-1}{4} \right) \;\; (i = 1, 2).$$

From (1.6) *and* (1.7) *follows that*

$$1 - \frac{1}{4}\|\ell_1\| - \|\ell_2\| = 1 - \left(\frac{\lambda_1}{4} + \lambda_2 \right) \int_0^1 |u_0'''(s)| ds = \frac{4}{(128+\varepsilon)\alpha_{k_0}},$$

$$\int_0^1 |\ell_0(1)(s)|\, ds \le \int_0^1 |u_0'''(s)|\, ds = \frac{4}{\alpha_{k_0}} < 128 \left(1 - \frac{1}{4}\|\ell_1\| - \|\ell_2\| \right) + \varepsilon.$$

Thus, all the assumptions of Theorem 1 are satisfied except (1.4) *and instead of* (1.4) *the condition* (1.4_ε) *is fulfilled with* $\omega = 1$, $\delta = 0$. *On the* *other hand, from the definition of functions* u_0, τ *and operators* ℓ_i *it follows that*

$$u_0'''(t) = |u_0'''(t)| \operatorname{sign} u_0'''(t) = |u_0'''(t)| u_0(\tau(t)) = \ell_0(u_0)(t),$$
$$\ell_1(u_0')(t) + \ell_2(u_0'')(t) = (\lambda_1 u_0'(0) + \lambda_2 u_0''(1/4)) \, |u_0'''(t)| = 0,$$

that is, u_0 *and* $u_1(t) \equiv 0$ *are the different solutions of the problem* (0.1),(0.2) *with* $\omega = 1$, $q(t) \equiv 0$, *which contradicts the conclusion of Theorem 1.*

COROLLARY 1. *Let the set* $A \subset [0, \omega]$ *and a linear monotone operator* $\ell_0 \in K_{[0,\omega]}(A)$ *be such that the conditions* (1.2),

$$\int_0^\omega |\ell_0(1)(s)| ds \leq \frac{d}{\omega^2} \tag{1.8}$$

are satisfied. Moreover, let the linear bounded operators ℓ_1, ℓ_2, *and the set* A *be such that*

$$0 < \frac{128}{d}\left(1 - \frac{\omega}{4}||\ell_1|| - ||\ell_2||\right) \leq 1, \tag{1.9}$$

and

$$\sigma_A(t) \geq \frac{\omega}{2}\sqrt{1 - \frac{128}{d}\left(1 - \frac{\omega}{4}||\ell_1|| - ||\ell_2||\right)} \quad \text{for} \quad 0 \leq t \leq \frac{\omega}{2}. \tag{1.10}$$

Then the problem (0.1), (0.2) *has a unique solution.*

COROLLARY 2. *Let* $\alpha \in [0, \omega]$, $\beta \in [\alpha, \omega]$, *a linear monotone operator* $\ell_0 \in K_{[0,\omega]}(A)$ *and the linear bounded operators* ℓ_1, ℓ_2 *be such that the conditions* (1.2)– (1.4) *are satisfied, where*

$$A = [\alpha, \beta], \quad \delta = \left[\frac{\omega}{2} - (\beta - \alpha)\right]_+ \tag{1.11$_1$}$$

or

$$A = [0, \alpha] \cup [\beta, \omega], \quad \delta = \left[\frac{\omega}{2} - (\beta - \alpha)\right]_- . \tag{1.11$_2$}$$

Then the problem (0.1),(0.2) *has a unique solution.* Consider the equation with deviating arguments

$$u'''(t) = \sum_{j=0}^{2} p_j(t) u^{(j)}(\tau_j(t)) + q(t) \quad \text{for} \quad 0 \leq t \leq \omega, \tag{1.12}$$

where $p_j \in L([0,\omega])$ and $\tau_j : [0,\omega] \to [0,\omega]$ are the measurable functions.

COROLLARY 3. *Let there exist* $\sigma \in \{-1,1\}$ *such that*

$$\sigma p_0(t) \geq 0 \qquad \text{for} \quad 0 \leq t \leq \omega. \tag{1.13}$$

Moreover, let $\delta \in [0,\omega/2]$, *the functions* p_j, $(j=0,1,2)$ *be such that*

$$0 < 1 - \frac{\omega}{4}||p_1||_L + ||p_2||_L, \quad \int_0^\omega p_0(s)ds \neq 0. \tag{1.14}$$

$$\left(1 - 4\left(\frac{\delta}{\omega}\right)^2\right) \int_0^\omega |p_0(s)|ds \leq \frac{128}{\omega^2}\left(1 - \frac{\omega}{4}||p_1||_L + ||p_2||_L\right), \tag{1.15}$$

and let at least one of the following items be fulfilled:

a) the set $A \subset [0,\omega]$ *is such that the condition* (1.5) *holds and*

$$p_0(t) = 0 \quad \text{if} \quad \tau_0(t) \notin A \tag{1.16}$$

on $[0,\omega]$;

b) the constants $\alpha \in [0,\omega]$, $\beta \in [\alpha,\omega]$ *are such that*

$$\tau_0(t) \in [\alpha,\beta] \qquad \text{for} \quad 0 \leq t \leq \omega, \tag{1.17}$$

and

$$\delta = \left[\frac{\omega}{2} - (\beta - \alpha)\right]_+. \tag{1.18}$$

Then the problem (1.12), (0.2) *has a unique solution.*

Now consider the ordinary differential equation

$$u'''(t) = \sum_{j=0}^{2} p_j(t)u^{(j)}(t) + q(t) \qquad \text{for} \quad 0 \leq t \leq \omega, \tag{1.19}$$

where $p_j, q \in L([0,\omega])$.

COROLLARY 4. *Let there exist* $\sigma \in \{-1,1\}$, *such that the condition* (1.13) *be satisfied. Moreover, let* $\delta \in [0,\omega/2]$, *the functions* p_j $(j=0,1,2)$ *be such that the conditions* (1.14), (1.15) *hold, and let at least one of the*

following items be fulfilled:
a) the set $A \subset [0,\omega]$ *is such that* $\operatorname{mes} A \neq 0$, *the condition* (1.5) *holds and*

$$p_0(t) = 0 \qquad \textit{for} \quad t \notin A; \tag{1.20}$$

b) the constants $\alpha \in [0,\omega]$, $\beta \in [\alpha,\omega]$ *are such that*

$$p_0(t) = 0 \qquad \textit{for} \quad t \in [0,\alpha[\, \cup \,]\beta,\omega], \tag{1.21}$$

and δ *satisfies* (1.18).

Then the problem (1.19), (0.2) *has a unique solution.*

REMARK 1. *As for the case where exist* $\sigma \in \{-1,1\}$, *such that the condition* (1.13) *is satisfied and* $p_1(t) \equiv 0$, $p_2(t) \equiv 0$, *the necessary condition for the unique solvability of* (1.19), (0.2) *is* $p_0(t) \not\equiv 0$ *(see [5, Proposition 1.1, p. 72]).*

2. Auxiliary Propositions. In the paper [12] the following three lemmas are proved:

LEMMA 1. *Let* $A \subseteq [0,\omega]$ *be a nonempty set,* $A_1 = \{t+\omega : t \in A\}, B = A \cup A_1$. *Then*

$$\min\left\{\sigma_A(t) : 0 \le t \le \frac{\omega}{2}\right\} = \min\left\{\sigma_B(t) : 0 \le t \le \frac{3\omega}{2}\right\}.$$

LEMMA 2. *Let* $\sigma \in \{-1,1\}, A \subset [0,\omega]$, $A \not\equiv \emptyset$, $\ell_0 \in K_{[0,\omega]}(A)$, *and let* $\sigma\ell_0$ *be nonnegative. Then, for an arbitrary* $v \in C([0,\omega])$,

$$\min\{v(s) : s \in \bar{A}\}|\ell_0(1)(t)| \le$$

$$\le \sigma\ell_0(v)(t) \le \max\{v(s) : s \in \bar{A}\}|\ell_0(1)(t)| \qquad \textit{for} \quad 0 \le t \le \omega.$$

LEMMA 3. *Let* $a \in [0,\omega[$, $D \subset [a,a+\omega]$, $c \in]a,a+\omega[$, *and* $\delta \in [0,\omega/2]$ *be such that*

$$\sigma_D(t) \ge \delta \qquad \textit{for} \quad a \le t \le a + \frac{\omega}{2},$$

and $A_c = \bar{D} \cap [a,c] \neq \emptyset$, $\quad B_c = \bar{D} \cap [c,a+\omega] \neq \emptyset$. *Then the estimate*

$$\left(\frac{(c-t_1)(t_1-a)(a+\omega-t_2)(t_2-c)}{(c-a)(a+\omega-c)}\right)^{1/2} \le \frac{\omega^2-4\delta^2}{8\omega} \tag{2.1}$$

for all $t_1 \in A_c$, $t_2 \in B_c$ *is satisfied.*

Let $\omega > 0$, $a \in R$, and define the functional $\Delta : C([a, a+\omega]) \to R_+$ by the equality

$$\Delta(x) = \max\{x(t) : a \le t \le a+\omega\} + \max\{-x(t) : a \le t \le a+\omega\}.$$

To prove Theorem 1 we need the following lemma which is a consequence of the more general result obtained in [3] (see [5, Theorem 1.1])

LEMMA 4. *Let* $v \in \widetilde{C}^2([0,\omega])$, *and*

$$v(t) \not\equiv const, \quad v^{(j)}(0) = v^{(j)}(\omega) \ (j = 0, 1, 2). \tag{2.2}$$

Then the estimate

$$\Delta(v') < \frac{\omega}{4}\Delta(v'') \tag{2.3}$$

is satisfied.

Proof. Let

$$w(t) = \begin{cases} v'(t) & \text{for } 0 \le t \le \omega \\ v'(t-\omega) & \text{for } \omega < t \le 2\omega \end{cases}.$$

In view of (2.2), $w \in \widetilde{C}^2([0, 2\omega])$ and

$$\Delta(w^{(j)}) = \Delta(v^{(j+1)}) \quad (j = 0, 1). \tag{2.4}$$

Define $t_{j1} \in [0, \omega[$, $t_{j2} \in]t_{j1}, t_{j1} + \omega[$, $(j = 0, 1)$ by the equalities

$$w^{(j)}(t_{jk}) = (-1)^{k-1} \max\{(-1)^{k-1} w^{(j)}(t) : 0 \le t \le 2\omega\} \quad j = 0, 1; k = 1, 2.$$

It follows from the conditions (2.2) that the functions w and w' changes its sign on $[0, 2\omega]$. Thus

$$w^{(j)}(t_{j1}) > 0, \ w^{(j)}(t_{j2}) < 0 \quad (j = 0, 1), \tag{2.5}$$

$$0 < \Delta(w) = -\int_{t_{11}}^{t_{12}} w'(s)ds, \quad 0 < \Delta(w) = \int_{t_{12}}^{t_{11}+\omega} w'(s)ds. \tag{2.6}$$

In view of the conditions (2.2)

$$w'(t) \not\equiv const \quad \text{for } t_{11} \le t \le t_{12}, \tag{2.7}$$

and/or $w'(t) \not\equiv const$ for $t_{12} \le t \le t_{11} + \omega$. Without a loss of generality we can assume that the condition (2.7) is satisfied. Then from (2.6) by (2.5) and (2.7) we get

$$\Delta(w) < -w'(t_{12})(t_{12} - t_{11}), \quad \Delta(w) \le w'(t_{11})(t_{11} + \omega - t_{12}).$$

By multiplying these estimates applying the numerical inequality

$$4AB \le (A+B)^2, \tag{2.8}$$

with regard to (2.4) we obtain (2.3). □

LEMMA 5. *Let* $v \in \widetilde{C}^2([0,\omega])$,

$$v(t) \not\equiv const, \ v^{(j)}(0) = v^{(j)}(\omega) \ (j = 0,1,2). \tag{2.9}$$

Then for any $t_1 \in [0,\omega[$ *and* $t_2 \in]t_1,\omega[$, *exist* $a \in [0,t_1[$ *and* $c \in]t_1,t_2[$, *such that*

$$|v(t_2) - v(t_1)| < \frac{\omega}{4}\left(\frac{(c-t_1)(t_1-a)(a+\omega-t_2)(t_2-c)}{(c-a)(a+\omega-c)}\right)^{1/2}\Delta(v''). \tag{2.10}$$

Proof. If $v(t_1) = v(t_2)$ then (2.10) is obvious. Assume that $v(t_1) - v(t_2) > 0$ (if $v(t_1) - v(t_2) < 0$, then, in view of the equality $\Delta(v) = \Delta(-v)$, one can consider $-v$ instead of v). Then if $v_0(t) = v(t) - \alpha$ with $\alpha = (v(t_1) + v(t_2))/2$, and v_1 is the ω-periodic extention of v_0 to R , we get

$$v_1(t_1) = \frac{v(t_1) - v(t_2)}{2} > 0, \quad v_1(t_2) = -\frac{v(t_1) - v(t_2)}{2} < 0. \tag{2.11}$$

From (2.9) and (2.11) it follows the existence of $a, c \in R$ such that $a < t_1 < c < t_2 < a + \omega$, $v_1(a) = v_1(c) = v_1(a+\omega) = 0$. Then, by using the Green's function of the problem

$$z''(t) = 0 \quad \text{for} \quad a \le t \le c \ \ (c \le t \le a + \omega),$$

$$z(a) = 0, \ z(c) = 0 \ \ (z(c) = 0, \ z(a+\omega) = 0),$$

in view the condition (2.11), we obtain the representations

$$|v_1(t_1)| = -\frac{c - t_1}{c - a}\int_a^{t_1}(s-a)v_1''(s)ds - \frac{t_1 - a}{c - a}\int_{t_1}^{c}(c-s)v_1''(s)ds,$$

$$|v_1(t_2)| = \frac{a+\omega-t_2}{a+\omega-c}\int\limits_c^{t_2}(s-c)v_1''(s)ds + \frac{t_2-c}{a+\omega-c}\int\limits_{t_2}^{a+\omega}(a+\omega-s)v_1''(s)ds,$$

respectively. In view of the conditions (2.9)

$$v''(t) \not\equiv const \quad \text{for} \quad a \le t \le c, \tag{2.12}$$

and/or $v''(t) \not\equiv const$ for $c \le t \le a+\omega$. Without a loss of generality we can assume that the condition (2.12) is satisfied. Then from the last two equalities on account of (2.11) and (2.12) we get the following estimates

$$0 < v(t_1) - v(t_2) < (c-t_1)(t_1-a)\max\{-v_1''(t) : a \le t \le c\},$$

$$0 < v(t_1) - v(t_2) \le (a+\omega-t_2)(t_2-c)\max\{v_1''(t) : c \le t \le a+\omega\}.$$

By multiplyng these estimates and applying the numerical inequality (2.8), in view the definition of the function v_1, we obtain

$$v(t_1) - v(t_2) < \frac{1}{2}\left((c-t_1)(t_1-a)(a+\omega-t_2)(t_2-c)\right)^{1/2}\Delta(v'').$$

On the other hand, applying the numerical inequality (2.8) we obtain the estimate $((c-a)(a+\omega-c))^{1/2} \le \omega/2$. From last two inequalities immediately follows (2.10). □

3. Proof of the Main Results. *Proof of Theorem 1.* Consider the homogeneous problem

$$v'''(t) = \sum_{j=0}^{2} \ell_j(v^{(j)})(t) \qquad \text{for} \quad 0 \le t \le \omega, \tag{3.1}$$

$$v^{(i)}(0) = v^{(i)}(\omega) \quad (i = 0,1,2). \tag{3.2}$$

In the work (see [4, Theorem 1.1]) it is proved that if ℓ_j $(j = 0,1,2)$ are bounded operators then the problem (0.1), (0.2) has the Fredholm property. Thus, the problem (0.1), (0.2) is uniquely solvable iff the homogeneous problem (3.1), (3.2) has only the trivial solution.

Assume that, on the contrary, the problem (3.1), (3.2) has a nontrivial solution v and let $t_1', t_2' \in [0,\omega[$ be defined by relations

$$v''(t_i') = (-1)^{i-1}\max\{(-1)^{i-1}v''(t) : 0 \le t \le \omega\} \quad \text{for} \quad i = 1,2.$$

Without loss of generality we can assume that $t_1' \le t_2'$. Then, if $I_1 = [t_1', t_2']$ and $I_2 = [0, t_1'] \cup [t_2', \omega]$, in view of (3.2) it is clear

$$\int\limits_{I_1} v'''(s)ds = -\Delta(v''), \quad \int\limits_{I_2} v'''(s)ds = \Delta(v''). \tag{3.3}$$

If $v(t) \equiv const$, then from (3.1) we obtain a contradiction with the condition (1.2), i.e. $v(t) \not\equiv const$. Consequently, in view of the conditions (3.2), the function $v^{(i)}$ $(i = 1, 2)$ changes its sign on $[0, \omega]$ and $||v^{(i)}||_C < \Delta(v^{(i)})$ $(i = 1, 2)$. From these inequalities and Lemma 4 we obtain

$$\int\limits_0^\omega |\ell_i(v^{(i)})(s)|ds \le ||\ell_i||\Delta(v^{(i)}) \le \left(\frac{\omega}{4}\right)^{2-i} ||\ell_i||\Delta(v'') \quad (i = 1, 2). \tag{3.4}$$

Then the integration of (3.1) on I_1 and I_2 respectively, in view of (1.3), (3.3) and (3.4) yields

$$0 < \beta\Delta(v'') \le -\int\limits_{I_1} \ell_0(v)(s)ds, \quad 0 < \beta\Delta(v'') \le \int\limits_{I_2} \ell_0(v)(s)ds, \tag{3.5}$$

where $\beta = 1 - ||\ell_1||\omega/2 - ||\ell_2||$. Now, let ℓ_0 be a nonpositive operator. Then from (3.5) and Lemma 2 it follows that

$$0 < (-1)^i \int\limits_{I_i} \ell_0(v)(s)ds \le (-1)^{i-1} v(t_i) \int\limits_{I_i} |\ell_0(1)(s)|ds \quad (i = 1, 2), \tag{3.6}$$

where $t_1,\ t_2 \in A$ are such that

$$v(t_1) = \max\{v(t) : t \in A\}, \quad v(t_2) = \min\{v(t) : t \in A\}. \tag{3.7}$$

From (3.6) and (3.7) it is claer that function v change its sign on A and then

$$v(t_1) > 0, \quad v(t_2) < 0. \tag{3.8}$$

On the other hand, from (3.5) and (3.6) the estimates

$$0 < \beta\Delta(v'') \le v(t_1) \int\limits_{I_1} |\ell_0(1)(s)|ds, \quad 0 < \beta\Delta(v'') \le -v(t_2) \int\limits_{I_2} |\ell_0(1)(s)|ds,$$

it follow. By multiplying these estimates and applying the numerical inequality (2.8) we get

$$0 < \beta\Delta(v'') \le \frac{v(t_1) - v(t_2)}{4} \int_0^\omega |\ell_0(1)(s)|\, ds. \tag{3.9}$$

Reasoning analogously, we can see that this estimate is valid also in the case where ℓ_0 is nonnegative. Then, in view of (3.9) and Lemma 5, exist $a \in [0, t_1[$ and $c \in]t_1, t_2[$, such that

$$\beta < \frac{\omega}{16}\left(\frac{(c-t_1)(t_1-a)(a+\omega-t_2)(t_2-c)}{(c-a)(a+\omega-c)}\right)^{1/2} \int_0^\omega |\ell_0(1)(s)|\, ds. \tag{3.10}$$

Now, let the set B be defined as in the lemma 1 and $D = B \cap [a, a+\omega]$. Then from lemma 1 and the condition (1.5) it follows $\sigma_B(t) \ge \delta$ and in view of the relation $D \subset B$ it is clear that $\sigma_D(t) \ge \sigma_B(t)$ for $a \le t \le a + \omega/2$. Consequently

$$\sigma_D(t) \ge \delta \quad \text{for} \quad a \le t \le a + \frac{\omega}{2}. \tag{3.11}$$

On the other hand from definition of a, c, t_1, t_2 we gat

$$a < t_1 < c < t_2 < a + \omega, \quad t_1 \in D \cap [a, c], \; t_2 \in D \cap [c, a+\omega]. \tag{3.12}$$

In view (3.11) and (3.12), all the assumptions of Lemma 3 are satisfied. Thus from (3.10) by (2.1) we get the contradiction to (1.4). Consequently, the problem (3.1), (3.2) has only the trivial solution.

Proof of Corollary 1. Let $\delta = \frac{\omega}{2}\sqrt{1 - \frac{128}{d}\left(1 - \frac{\omega}{4}||\ell_1|| - ||\ell_2||\right)}$. Then, on account of (1.8) – (1.10), we obtain that the conditions (1.3) and (1.5) of Theorem 1 are fulfilled. Consequently, all the assumptions of Theorem 1 are satisfied.

Proof of Corollary 2. It is not difficult to verify that if $A = [\alpha, \beta]$ ($A = [0, \alpha] \cup [\beta, \omega]$), then

$$\sigma_A(t) \ge \left[\frac{\omega}{2} - \beta + \alpha\right]_+ \quad \left(\sigma_A(t) \ge \left[\frac{\omega}{2} - \beta + \alpha\right]_-\right) \quad \text{for} \quad 0 \le t \le \frac{\omega}{2}. \tag{3.13}$$

Consequently, in view of the condition (1.11_1) $((1.11_2))$, all the assumptions of Theorem 1 are satisfied.

Proof of Corollary 3. Let $\ell_j(u)(t) \equiv p_j(t)u(\tau_j(t))$ $(j = 0, 1, 2)$. On account of (1.13)– (1.15) we see that the operator ℓ_0 is monotone and the conditions (1.2), (1.3), and (1.4) are satisfied.

a) It is not difficult to verify that from the condition (1.16) it follows that $\ell_0 \in K_{[0,\omega]}(A)$. Consequently, all the assumptions of Theorem 1 are satisfied.

b) Let $A = [\alpha, \beta]$. Then in view of the condition (1.17) the inclusion $\ell_0 \in K_{[0,\omega]}(A)$ is satified. The inequality (3.13) obtained in the proof of Corollary 2, by virtue of (1.18), implies the inequality (1.5). Consequently, all the assumptions of Theorem 1 are satisfied.

Proof of Corollary 4. The validity of this assertion follows immediately from Corollary 3 a).

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