Georgian Mathematical Journal
Volume 14 (2007), Number 3, 533–542

ON A PERIODIC BOUNDARY VALUE PROBLEM FOR FOURTH ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

SULKHAN MUKHIGULASHVILI

Abstract. The unimprovable sufficient conditions are established for the unique solvability of the periodic problem

$$u^{(4)}(t)=\sum_{i=0}^{3}\ell_i(u^{(i)})(t)+q(t),\quad u^{(j)}(0)=u^{(j)}(\omega)+c_j\quad (j=\overline{0,3}),$$

where $\ell_i : C([0,\omega])\to L([0,\omega])$ are linear bounded operators, $q\in L([0,\omega])$ and $c_j\in R$.

2000 Mathematics Subject Classification: 34K06, 34K10.
Key words and phrases: Linear functional differential equation, periodic boundary value problem, existence and uniqueness.

Introduction

Consider the problem on the existence and uniqueness of a solution of the equation

$$u^{(4)}(t)=\sum_{i=0}^{3}\ell_i(u^{(i)})(t)+q(t)\quad \text{for}\quad 0\le t\le\omega \tag{0.1}$$

satisfying the periodic boundary conditions

$$u^{(j)}(0)=u^{(j)}(\omega)+c_j\quad (j=\overline{0,3}), \tag{0.2}$$

where $\ell_i : C([0,\omega])\to L([0,\omega])$ are linear bounded operators, $q\in L([0,\omega])$ and $\omega>0,\ c_j\in R$.

By a solution of the problem (0.1), (0.2) we understand a function $u\in\widetilde{C}^3([0,\omega])$ that satisfies the equation (0.1) almost everywhere on $[0,\omega]$ and satisfies the conditions (0.2).

It is well-known that if linear operators $\ell_i : C([a,b])\to L([a,b])$ $(i=\overline{0,3})$ are strongly bounded, i.e., there exist summable functions $\eta_i : [a,b]\to [0,+\infty[$ such that

$$|\ell_i(x)(t)|\le\eta_i(t)\|x\|_C\quad\text{for}\quad 0\le t\le\omega,\ x\in C([a,b]),$$

then the following theorem on the Fredholm property is valid (see, e.g., [8, 9])

ISSN 1072-947X / \$8.00 / © Heldermann Verlag *www.heldermann.de*

Theorem 0.1. *The problem* (0.1), (0.2) *is uniquely solvable if and only if the corresponding homogeneous problem*

$$v^{(4)}(t) = \sum_{i=0}^{3} \ell_i(v^{(i)})(t), \quad v^{(j)}(0) = v^{(j)}(\omega) \quad (j = \overline{0,3}) \tag{0.3}$$

has only a trivial solution.

In 1972, H. H. Schaefer [16, Theorem 4] proved that there exists a linear bounded operator $\ell : C([a,b]) \to L([a,b])$ which is not strongly bounded. The same (Fredholm) property for functional differential equations with a nonstrongly bounded linear operator had not been investigated till 2000. The first important step in this direction was made by Bravyi (see, [2]), where the Fredholm property was proved for the boundary value problem for functional differential equations with a nonstrongly bounded linear operator (see,e.g., [3]). Those results were generalized for the n-th order functional differential systems in [5] and with such a generalization Theorem 1.1 is also valid if ℓ_i $(i = 0, 1, 2)$ are nonstrongly bounded linear operators.

A great deal of interesting work was carried out and many interesting results obtained on the existence and uniqueness of a solution for a periodic boundary value problem for higher order ordinary differential equations (see, e.g., [1, 6, 7, 10–12, 15] and the references therein). But an analogous problem for functional differential equations, even in the case of linear equations remains little investigated.

Thus, in the present paper, we study the problem (0.1), (0.2) under the assumptions that ℓ_0 is monotone (see Definition 1.1) and ℓ_i $(i = 1, 2, 3)$ are bounded, not necessarily strongly bounded, operators. We establish new nonimprovable, integral, sufficient conditions of the unique solvability of the problem (0.1), (0.2). On the other hand, these conditions generalize the well-known nonimprovable results of A. Lasota and Z. Opial (see [11,Theorem 6, p. 88]) for second order linear ordinary differential equations.

The obtained results are also new if (0.1) is an ordinary differential equation of the form

$$u^{(4)}(t) = \sum_{i=0}^{3} p_i(t) u^{(i)}(t) + q(t) \quad \text{for} \quad 0 \le t \le \omega. \tag{0.4}$$

The method used for the investigation of the considered problem is based on the method developed in our previous papers (see [13, 14]).

The following notation is used throughout:

N is the set of all natural numbers.

R is the set of all real numbers, $R_+ = [0, +\infty[$.

$C([a,b])$ is the Banach space of continuous functions $u : [a,b] \to R$ with the norm $\|u\|_C = \max\{|u(t)| : a \le t \le b\}$.

$\widetilde{C}^n([a,b])$, where $n \in N$, is the set of functions $u : [a,b] \to R$ which are absolutely continuous together with their first $n-1$ derivatives.

$L([a,b])$ is the Banach space of Lebesgue integrable functions $p:[a,b]\to R$ with the norm $\|p\|_L=\int_a^b|p(s)|ds$.

If $\ell: C([a,b])\to L([a,b])$ is a linear operator, then $\|\ell\|=\sup_{\|x\|_C\le 1}\|\ell(x)\|_L$.

Definition 0.1. We say that a linear operator $\ell: C([a,b])\to L([a,b])$ is *nonnegative* (*nonpositive*), if the inequality

$$\ell(x)(t)\ge 0\quad (\ell(x)(t)\le 0),\quad a\le t\le b,$$

is satisfied for any nonnegative $x\in C([a,b])$.

We say that an operator ℓ is *monotone* if it is nonnegative or nonpositive.

1. Main Results

Theorem 1.1. *Let $\ell_0: C([0,\omega])\to L([0,\omega])$ be a linear monotone operator,*

$$\int\limits_0^\omega \ell_0(1)(s)ds\ne 0, \tag{1.1}$$

and

$$1-\sum_{i=1}^3\frac{\omega^{3-i}}{d_{3-i}}\|\ell_i\|>0, \tag{1.2}$$

$$\int\limits_0^\omega|\ell_0(1)(s)|\,ds\le\frac{768}{\omega^3}\left(1-\sum_{i=1}^3\frac{\omega^{3-i}}{d_{3-i}}\|\ell_i\|\right), \tag{1.3}$$

where $d_0=1, d_1=4, d_2=32$. Then the problem (0.1), (0.2) *has a unique solution.*

Example 1.1. The example below shows that condition (1.3) in Theorem 1.1 is optimal and cannot be replaced by the condition

$$\int\limits_0^\omega|\ell_0(1)(s)|\,ds\le\frac{768}{\omega^3}\left(1-\sum_{i=1}^3\frac{\omega^{3-i}}{d_{3-i}}\|\ell_i\|\right)+\varepsilon, \tag{6_ε}$$

no matter how small $\varepsilon\in]0,1]$ is. Define the functions $W_k\in\widetilde{C}^3([0,1]),\ k\in N$ on $[0,1/4],\ [1/4,1/2]$, and $[1/2,1]$ by the equalities

$$W_k(t)=\begin{cases}\dfrac{1}{32}-\dfrac{\pi^2-8}{128\pi^2k^2}-\dfrac{t^2}{2} & \text{for } 0\le t\le 1/4-1/8k,\\[2mm] \dfrac{1}{16}-\dfrac{1}{32k}-\dfrac{2k-1}{8k}t+\dfrac{\sin\pi k(1-4t)}{16\pi^2k^2} & \text{for } 1/4-1/8k<t\le 1/4,\end{cases}$$

$$W_k(t)=-W_k\left(\frac12-t\right)\ \text{ for } \frac14\le t\le\frac12,\quad W_k(t)=W_k(1-t)\ \text{for } \frac12\le t\le 1.$$

respectively. Let $u_k(t)=\int_0^t W_k(s)ds$ for $0\le t\le 1$. Then for any natural $k>1$

$$u_k'\left(\frac{1}{4}\right)=0,\quad u_k''(0)=0,\quad u_k'''\left(\frac{1}{4}\right)=0, \tag{1.4}$$

$$u_k^{(i)}(0)=u_k^{(i)}(1)\quad (j=\overline{0,3}), \tag{1.5}$$

and there exist constants $\beta_1,\ \beta_2$ independent of k, such that

$$-u_k\left(\frac{3}{4}\right)=u_k\left(\frac{1}{4}\right)=\frac{1}{192}-\frac{1}{128k}+\frac{\beta_1}{k^2}+\frac{\beta_2}{k^3}>0. \tag{1.6}$$

In view of (1.6), there exist $r\in N$, such that $192<u_k^{-1}(1/4)$ if $k\ge r$ and $\lim\limits_{k\to\infty}u_k^{-1}(1/4)=192$. Then for arbitrary $\varepsilon>0$, there exist constants $k_0\ge r$ and $\lambda\ge 0$ such that

$$0<1-\frac{41\lambda}{32}\int_0^1|u_{k_0}^{(4)}(s)|ds, \tag{1.7}$$

$$u_{k_0}^{-1}\left(\frac{1}{4}\right)\le 192\left(1-\frac{41\lambda}{32}\int_0^1|u_{k_0}^{(4)}(s)|ds\right)+\frac{\varepsilon}{4}. \tag{1.8}$$

Now, define the functions $\tau_i:[0,1]\to[0,1]$ and the operators $\ell_i:C([0,1])\to L([0,1])$ $(j=\overline{0,3})$ by the equalities

$$\tau_1(t)\equiv\tau_3(t)\equiv\frac{1}{4},\quad \tau_2(t)\equiv 0,\quad \tau_0(t)=\begin{cases}1/4 & \text{for } u_{k_0}^{(4)}(t)\ge,0\\ 3/4 & \text{for } u_{k_0}^{(4)}(t)<0,\end{cases}$$

$$\ell_0(x)(t)=\frac{|u_{k_0}^{(4)}(t)|}{u_{k_0}(1/4)}x(\tau_0(t)),\ \ \ell_i(x)(t)=\lambda u_{k_0}^{(4)}(t)x(\tau_i(t))\ \ (i=1,2,3).$$

Then it is not difficult to verify that ℓ_0 is a monotone (nonnegative) operator and

$$\frac{1}{32}\|\ell_1\|+\frac{1}{4}\|\ell_2\|+\|\ell_3\|=\frac{41\lambda}{32}\int_0^1|u_{k_0}^{(4)}(s)|ds,$$

$$\int_0^1|\ell_0(1)(s)|\,ds=4u_{k_0}^{-1}\left(\frac{1}{4}\right)\int_0^{1/8k_0}(\sin 4\pi k_0 s)'\,ds=4u_{k_0}^{-1}\left(\frac{1}{4}\right).$$

These inequalities and (1.7), (1.8) imply that all the assumptions of Theorem 1.1 are satisfied except (1.3) and instead of (1.3) the condition (6_ε) is fulfilled with $\omega=1$. On the other hand, from the definition of the operator ℓ_0 and (1.4), (1.5) and (1.6) follows that

$$u_{k_0}^{(4)}(t)=|u_{k_0}^{(4)}(t)|\frac{u_{k_0}(\tau_0(t))}{u_{k_0}(1/4)}=\ell_0(u_{k_0})(t),\quad \sum_{i=1}^3\ell_i(u_{k_0}^{(i)})(t)=0,$$

i.e., u_{k_0} and $u_1(t) \equiv 0$ are different solutions of the problem (0.1),(0.2) with $\omega = 1$, $q(t) \equiv 0$, which contradicts the conclusion of Theorem 1.1.

Consider, on $[0, \omega]$, the equation with deviating arguments

$$u^{(4)}(t) = \sum_{i=0}^{3} p_i(t) u^{(i)}(\tau_i(t)) + q(t), \tag{1.9}$$

where $q, p_i \in L([0, \omega])$ and $\tau_i : [0, \omega] \to [0, \omega]$ are measurable functions.

Corollary 1.1. *Let*

$$0 \leq \sigma p_0(t) \not\equiv 0, \tag{1.10}$$

where $\sigma \in \{-1, 1\}$, $d_0 = 1$, $d_1 = 4$, $d_2 = 32$, *and*

$$1 - \sum_{i=1}^{3} \frac{\omega^{3-i}}{d_{3-i}} \int_0^\omega |p_i(s)| ds > 0, \tag{1.11}$$

$$\int_0^\omega |p_0(s)| ds \leq \frac{768}{\omega^3} \left(1 - \sum_{i=1}^{3} \frac{\omega^{3-i}}{d_{3-i}} \int_0^\omega |p_i(s)| ds \right). \tag{1.12}$$

Then the problem (0.4), (0.2) ((1.9), (0.2)) *has a unique solution.*

Remark 1.1. For the two term equation

$$u^{(4)}(t) = p_0(t) u(\tau(t)) + q(t), \tag{1.13}$$

Corollary 1.1 implies that the problem (1.13),(0.2) is uniquely solvable if (1.10) and the inequality

$$\int_0^\omega |p_0(s)| ds \leq \frac{768}{\omega^3} \tag{1.14}$$

hold. On the other hand, for $\tau(t) \equiv t$, in [6] (see [6], Proposition 1.1) it was proved that only the condition (1.10) with $\sigma = -1$ guarantees the unique solvability of the problem (1.13), (0.2). However if $\tau(t) \not\equiv t$, then for the unique solvability of the problem (1.13), (0.2), the condition (1.14) is not only essential but also cannot be replaced by

$$\int_0^\omega |p_0(s)| ds \leq \frac{768}{\omega^3} + \varepsilon, \tag{1.15}$$

no matter how small $\varepsilon \in]0, 1]$ is. Indeed, let $\varepsilon \in]0, 1]$, $\omega = 1$, the functions u_{k_0} and τ_0 be defined as in Example 1.1, $p_0(t) = |u_{k_0}^4(t)| u_{k_0}^{-1}(1/4)$. Then by (1.8) it is clear that

$$\int_0^1 |p_0(s)|\, ds = 4 u_{k_0}^{-1} \left(\frac{1}{4} \right) < 768 + \varepsilon.$$

Therefore the condition (1.10) is satisfied and instead of (1.14) the condition (1.15) is fulfilled. On the other hand, from the definition of functions u_0, τ, and p_0 it follows that

$$u_{k_0}^{(4)}(t) = |u_{k_0}^{(4)}(t)| \operatorname{sign} u_0^{(4)}(t) = |u_{k_0}^{(4)}(t)| \frac{u_{k_0}(\tau_0(t))}{u_{k_0}(1/4)} = p_0(t) u_{k_0}(\tau_0(t)).$$

Thus u_{k_0} is a nontrivial solution of the problem (1.13), (0.2) which, according to Theorem 0.1, implies that the problem (1.13), (0.2) is not uniquely solvable.

2. Proofs

Let $a \in R$, and define the functional $\Delta : C([a, a+\omega]) \to R_+$ by the equality

$$\Delta(x) = \max\{x(t) : a \le t \le a+\omega\} + \max\{-x(t) : a \le t \le a+\omega\}. \qquad (2.1)$$

To prove Theorem 1.1 we need two auxiliary propositions; the first is a consequence of more general Theorem 1.1 with $n = 3$ obtained in [4], while the second is rather trivial and we omit its proof.

Lemma 2.1. *Let $z \in \widetilde{C}^3([a, a+\omega])$, and*

$$z(t) \neq \text{Const}, \ \ z^{(j)}(a) = z^{(j)}(a+\omega) \quad (j = \overline{0,3}). \qquad (2.2)$$

Then if $d_1 = 4, d_2 = 32, d_3 = 192$, the estimates

$$\Delta(z^{(i)}) < \frac{\omega^{3-i}}{d_{3-i}} \Delta(z^{(3)}) \quad (i = 0, 1, 2) \qquad (2.3)$$

are satisfied.

Lemma 2.2. *Let $\sigma \in \{-1, 1\}$ and $\sigma\ell : C([a,b]) \to L([a,b])$ be a nonnegative linear operator. Then for an arbitrary $v \in C([a,b])$ the inequalities*

$$-m|\ell(1)(t)| \le \sigma\ell(v)(t) \le M|\ell(1)(t)| \quad \textit{for} \quad a \le t \le b$$

hold, where $m = \max\{-v(t) : a \le t \le b\}$, $M = \max\{v(t) : a \le t \le b\}$.

Proof of Corollary 1.1. According to Theorem 0.1, it is sufficient to show that the homogeneous problem (0.3) has only a trivial solution.

Now, assume the contrary that the problem (0.3) has a nontrivial solution v. Put

$$M_j = \max\left\{v^{(j)}(t) : 0 \le t \le \omega\right\}, \quad m_j = \max\left\{-v^{(j)}(t) : 0 \le t \le \omega\right\},$$

for $j = \overline{0,3}$, $a \in [0, \omega[$ is such that $v^{(3)}(a) = M_3$, and

$$C_\omega([a, a+\omega]) \overset{def}{\equiv} \{x \in C([a, a+\omega]) : x(a) = x(a+\omega)\}.$$

Then define the continuous operators $\gamma : L([0,\omega]) \to L([a, a+\omega])$, $\widetilde{\ell}_i : C_\omega([a, a+\omega]) \to L([a, a+\omega])$ $(i = \overline{0,3})$, and the function $v_0 \in C_\omega([a, a+\omega])$ by the equalities

$$\gamma(x)(t) = \begin{cases} x(t) & \text{for } \ a \le t \le \omega, \\ x(t-\omega) & \text{for } \ \omega < t \le a+\omega, \end{cases} \tag{2.4}$$

$$v_0(t) = \gamma(v)(t), \ \widetilde{\ell}_i(x)(t) = \gamma\left(\ell_i\left(\gamma^{-1}(x)\right)\right)(t) \quad \text{for} \quad a \le t \le a+\omega. \tag{2.5}$$

From these definitions it follows that

$$\begin{aligned} M_j &= \max\left\{v_0^{(j)}(t) : a \le t \le a+\omega\right\}, \\ m_j &= \max\left\{-v_0^{(j)}(t) : a \le t \le a+\omega\right\}, \end{aligned} \tag{2.6}$$

if ℓ_i is nonnegative (nonpositive). Then $\widetilde{\ell}_i$ is also nonnegative (nonpositive),

$$\begin{aligned} \int\limits_a^{a+\omega} \widetilde{\ell}_0(1)(s)ds &= \int\limits_0^{\omega} \ell_0(1)(s)ds, \\ \int\limits_a^{a+\omega} \widetilde{\ell}_i(v_0^{(i)})(s)ds &= \int\limits_0^{\omega} \ell_i(v^{(i)})(s)ds, \end{aligned} \tag{2.7}$$

if $i = 1,2,3$, and in view of (0.3)

$$v_0^{(4)}(t) = \sum_{i=0}^{3} \widetilde{\ell}_i(v_0^{(i)})(t) \quad \text{for} \quad a \le t \le a+\omega, \tag{2.8}$$

$$v_0^{(j)}(a) = v_0^{(j)}(a+\omega) \quad (j = \overline{0,3}). \tag{2.9}$$

From the condition (1.1) and (0.3) we get

$$v(t) \ne \text{Const}, \quad M_i > 0, \quad m_i > 0 \quad (i = 1,2,3), \tag{2.10}$$

and it follows from Lemma 2.2 that

$$\int\limits_0^{\omega} \left|\ell_i(v^{(i)})(s)\right| ds \le \|\ell_i\|\left(M_i + m_i\right) = \|\ell_i\|\Delta(v^{(i)}), \tag{2.11}$$

where $i = 1,2,3$, and the functional Δ is defined by (2.1). Thus from (2.7), (2.11) and Lemma 2.1 with $i = 1,2$ and $z(t) = v(t)$, we obtain

$$\begin{aligned} \int\limits_a^{a+\omega} \left|\widetilde{\ell}_3(v_0^{(3)})(s)\right| ds &\le \Delta(v^{(3)})\|\ell_3\|, \\ \int\limits_a^{a+\omega} \left|\widetilde{\ell}_i(v_0^{(i)})(s)\right| ds &\le \frac{\omega^{3-i}}{d_{3-i}}\Delta(v^{(3)})\|\ell_i\| \end{aligned} \tag{2.12}$$

if $(i = 1,2)$. On the other hand, from the definitions of v_0 and a, by (2.6) and (2.9), it follows that $M_3 = v_0^{(3)}(a) = v_0^{(3)}(a+\omega)$, and also the existence of

$b \in]a, a+\omega[$ such that $v_0^{(3)}(b) = -m_3$. Thus we get

$$0 < M_3 = v_0^{(3)}(a) = v_0^{(3)}(a+\omega), \quad 0 < m_3 = -v_0^{(3)}(b). \tag{2.13}$$

Put $\alpha = 1 - \sum\limits_{i=1}^{3} \frac{\omega^{3-i}}{d_{3-i}} \|\ell_i\|$. Then the integration of (2.8) from a to b and from b to $a+\omega$, respectively, in view of (2.12), (2.13) and (1.2) yields

$$\begin{aligned} 0 < \alpha\Delta(v^{(3)}) &\leq -\int\limits_a^b \widetilde{\ell}_0(v_0)(s)ds, \\ 0 < \alpha\Delta(v^{(3)}) &\leq \int\limits_b^{a+\omega} \widetilde{\ell}_0(v_0)(s)ds. \end{aligned} \tag{2.14}$$

Now suppose that v_0 does not change its sign on $[a, a+\omega]$. Then obviously, either $-\int_a^b \widetilde{\ell}_0(v_0)(s)ds \leq 0$ or $\int_b^{a+\omega} \widetilde{\ell}_0(v_0)(s)ds \leq 0$, which contradicts one of the inequalities of (2.14). Thus, our assumption is invalid and v_0 changes its sign on $[a, a+\omega]$. Next, let ℓ_0 ($\widetilde{\ell}_0$) be a nonpositive operator. Then from the definition of the constants M_0 and m_0 it is clear that $M_0 > 0$, $m_0 > 0$ and thus, according to Lemma 2.2, from (2.14) it follows that

$$\begin{aligned} 0 < \alpha\Delta(v^{(3)}) &\leq M_0 \int\limits_a^b \left|\widetilde{\ell}_0(1)(s)\right| ds, \\ 0 < \alpha\Delta(v^{(3)}) &\leq m_0 \int\limits_b^{a+\omega} \left|\widetilde{\ell}_0(1)(s)\right| ds. \end{aligned}$$

By multiplying these inequalities and applying the numerical inequality $4AB \leq (A+B)^2$, in view of (2.7) we get

$$0 < \alpha\Delta(v^{(3)}) \leq \frac{\Delta(v)}{4} \int\limits_0^{\omega} |\ell_0(1)(s)|\, ds. \tag{2.15}$$

By an analogous reasoning we can see that this estimate is also valid in the case where ℓ_0 ($\widetilde{\ell}_0$) is nonnegative. Then, in view of Lemma 2.1 with $i = 0$ and $z(t) = v(t)$, the inequality (2.15) contradicts (1.3). Consequently, v_0 cannot change its sign, i.e., our assumption is invalid and $v(t) \equiv 0$. □

Proof of Corollary 1.1. Let $\ell_i(x)(t) = p_i(t)x(\tau_i(t))$ ($\ell_i(x)(t) = p_i(t)x(t)$) ($i = \overline{0,3}$). According to (1.10), it is clear that ℓ_0 is a monotone operator, ℓ_i ($i = 1,2,3$) are bounded operators, and $\int_0^\omega |p_0(s)|ds \neq 0$, $\|\ell_i\| \leq \int_0^\omega |p_i(s)|ds$ ($i = 1,2,3$). Then the conditions (1.11) and (1.12) yield the conditions (1.2) and (1.3). Thus, all the assumptions of Theorem 0.2 are fulfilled. □

Acknowledgement

This research was supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. A V0Z10190503.

References

1. P. W. Bates and J. R. Ward, Jr., Periodic solutions of higher order systems. *Pacific J. Math.* **84**(1979), No. 2, 275–282.
2. E. Bravyi, A note on the Fredholm property of boundary value problems for linear functional differential equations. *Mem. Differential Equations Math. Phys.* **20**(2000), 133–135.
3. R. Hakl, A. Lomtatidze, and I. P. Stavroulakis, On a boundary value problem for scalar linear functional differential equations. *Abstr. Appl. Anal.* **2004**, No. 1, 45–67.
4. R. Hakl and S. Mukhigulashvili, On one estimate for periodic functions. *Georgian Math. J.* **12**(2005), No. 1, 97–114.
5. R. Hakl and S. Mukhigulashvili, On a boundary value problem for n-th order linear functional differential systems. *Georgian Math. J.* **12**(2005), No. 2, 229–236.
6. I. Kiguradze and T. Kusano, On periodic solutions of higher-order nonautonomous ordinary differential equations. (Russian) *Differ. Uravn.* **35**(1999), No. 1, 72–78, 142; English transl.: *Differential Equations* **35**(1999), No. 1, 71–77.
7. I. Kiguradze and T. Kusano, On conditions for the existence and uniqueness of a periodic solution of nonautonomous differential equations. (Russian) *Differ. Uravn.* **36**(2000), No. 10, 1301–1306; English transl.: *Differ. Equ.* **36**(2000), No. 10, 1436–1442.
8. I. Kiguradze and B. Půža, On boundary value problems for systems of linear functional-differential equations. *Czechoslovak Math. J.* **47(122)**(1997), No. 2, 341–373.
9. I. Kiguradze and B. Půža, Boundary value problems for systems of linear functional differential equations. *Folia Facultatis Scientiarium Naturalium Universitatis Masarykianae Brunensis. Mathematica,* 12. *Masaryk University, Brno,* 2003.
10. *M. Krasnosel'skiĭ* and A. Perov, On a certain principle of existence of bounded, periodic and almost periodic solutions of systems of ordinary differential equations. (Russian) *Dokl. Akad. Nauk SSSR* **123**(1958), 235–238.
11. A. Lasota and Z. Opial, Sur les solutions péeriodiques des éequations difféerentielles ordinaires. *Ann. Polon. Math.* **16**(1964), 69–94.
12. A. Lasota and F. Szafraniec, Sur les solutions péeriodiques d'une équation différentielle ordinaire d'ordre n. *Ann. Polon. Math.* **18**1966 339–344.
13. A. Lomtatidze and S. Mukhigulashvili, On periodic solutions of second order functional differential equations. *Mem. Differential Equations Math. Phys.* **5**(1995), 125–126.
14. S. Mukhigulashvili, On a periodic boundary value problem for second-order linear functional differential equations. *Bound. Value Probl.* **2005**, No. 3, 247–261.
15. A. N. Ronto and A. N. Samoĭlenko, On the unique solvability of some boundary value problems for linear functional-differential equations in a Banach space. (Russian) *Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki* **2002**, No. 9, 42–46.
16. H. H. Schaefer, Normed tensor products of Banach lattices. *Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces* (*Jerusalem*, 1972). *Israel J. Math.* **13**(1972), 400–415.

(Received 4.09.2006)

Author's addresses:

Current address:
Mathematical Institute
Academy of Sciences of the Czech Republic
Žižkova 22, 616 62 Brno
Czech Republic

Permanent address:
A. Razmadze Mathematical Institute
1, M. Aleksidze St., Tbilisi 0193
Georgia
E-mail: mukhig@ipm.cz