ON PERIODIC SOLUTIONS OF SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. The sufficient conditions are established for the existence and uniqueness of an ω-periodic solution of the functional differential equation

$$u''(t) = f(u)(t),$$

where f is a continuous operator acting from the space of continuously differentiable ω-periodic functions to the space of ω-periodic and Lebesgue integrable on $[0, \omega]$ functions.

1. Statement of the main results

1.1. Main notation and definitions

In the present paper, for linear and nonlinear functional differential equations

$$u''(t) = p_0(u)(t) + p_1(u')(t) + q(t) \tag{1.1}$$

and

$$u''(t) = f(u)(t) \tag{1.2}$$

we investigate the problem of the existence and uniqueness of a periodic solution with a preassigned period $\omega > 0$. Here $p_i : \mathbb{C}_\omega \to L_\omega$ $(i = 1, 2)$ are linear bounded operators, $q \in L_\omega$ and $f : \mathbb{C}'_\omega \to L_\omega$ is a nonlinear continuous operator. In the case if $p_i(y)(t) \equiv p_{0i}(t)y(t)$ $(i = 1, 2)$ and $f(u)(t) \equiv f_0(t, u(t), u'(t))$, i.e. if equations (1.1) and (1.2) have respectively the forms

$$u''(t) = p_{01}(t)u(t) + p_{02}(t)u'(t) + q(t) \tag{1.1$'$}$$

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and

$$u''(t) = f_0(t, u(t), u'(t)), \tag{1.2'}$$

to the above-mentioned problem is devoted an ample literature (see, for example, [1]–[10], [12]–[16], [18]–[20], [22]–[28] and references therein) and this problem is studied with sufficient thoroughness. The problem in a general case remains still little investigated. It is just this case we will consider in the present paper. More exactly, for equations (1.1) and (1.2) we have established nonimprovable sufficient conditions of the existence and uniqueness of an ω-periodic solution. The obtained results are new for equations (1.1′) and (1.2′) as well.

Throughout the paper we use the following notation.

$\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ = [0, +\infty[$, $\mathbb{R}^n$ is the n-dimensional real Euclidean space.

If $x \in \mathbb{R}$, then

$$[x]_+ = (|x| + x)/2, \quad [x]_- = (|x| - x)/2.$$

$\mathbb{C}_\omega$ is the space of ω-periodic continuous functions $x : \mathbb{R} \to \mathbb{R}$ with the norm

$$\|x\|_{\mathbb{C}_\omega} = \max\{|x(t)| : 0 \le t \le \omega\}.$$

$\mathbb{C}'_\omega$ is the space of ω-periodic continuously differentiable functions $x : \mathbb{R} \to \mathbb{R}$ with the norm

$$\|x\|_{\mathbb{C}'_\omega} = \max\{|x(t)| + |x'(t)| : 0 \le t \le \omega\}.$$

$\widetilde{\mathbb{C}}'_\omega$ is the space of ω-periodic and absolutely continuous together with their first derivative functions $x : \mathbb{R} \to \mathbb{R}$.

L_ω is the space of ω-periodic and Lebesgue integrable on $[0, \omega]$ functions $y : \mathbb{R} \to \mathbb{R}$ with the norm

$$\|y\|_{L_\omega} = \int_0^\omega |y(t)|\, dt.$$

The linear operator $p : \mathbb{C}_\omega \to L_\omega$ is said to be *non-negative* (*non-positive*) if for any non-negative function $x \in \mathbb{C}_\omega$ almost everywhere on $\mathbb{R}$ the inequality

$$p(x)(t) \ge 0 \quad (p(x)(t) \le 0)$$

is satisfied.

The linear operator $p : \mathbb{C}_\omega \to L_\omega$ is said to be *monotone* if it is either positive or negative.

Under an ω-periodic solution of equation (1.1) (equation (1.2)) is understood the function $u \in \widetilde{\mathbb{C}}'_\omega$ which almost everywhere on $\mathbb{R}$ satisfies this equation.

1.2. Periodic solutions of linear equations

Theorem 1.1. *Let for the operator* $p_1 : \mathbb{C}_\omega \to L_\omega$ *the representation*

$$p_1(x)(t) = p_1^+(x)(t) - p_1^-(x)(t) \quad \text{for } x \in \mathbb{C}_\omega,\ t \in \mathbb{R} \tag{1.3}$$

be valid, where $p_1^+, p_1^- : \mathbb{C}_\omega \to L_\omega$ *are non-negative linear operators. Moreover, let* $p_0 : \mathbb{C}_\omega \to L_\omega$ *be the monotone operator,*

$$\int_0^\omega p_0(1)(s)\, ds \neq 0 \tag{1.4}$$

and

$$\int_0^\omega |p_0(1)(s)|\, ds \le \frac{16}{\omega}\left(1 - \int_0^\omega [p_1^+(1)(s) + p_1^-(1)(s)]\, ds\right). \tag{1.5}$$

Then equation (1.1) *has one and only one* ω*-periodic solution.*

For $p_1(x)(t) \equiv 0$, from Theorem 1.1 follows Theorem 1 given in [21] without proof.

Example 1.1. An example below shows that condition (1.5) in Theorem 1.1 is optimal and it cannot be replaced by the condition

$$\int_0^\omega |p_0(1)(s)|\, ds \le \frac{16+\varepsilon}{\omega}\left(1 - \int_0^\omega [p_1^+(1)(s) + p_1^-(1)(s)]\, ds\right), \tag{1.5_ε}$$

no matter how small $\varepsilon \in]0, 1]$ is. Let α, β, u_0 and τ be the numbers and functions given by the equalities

$$\alpha = \frac{\varepsilon}{16(16+\varepsilon)}\ , \quad \beta = \frac{\pi\alpha}{\pi - 2}\ ,$$

$$u_0(t) = \begin{cases} 4t & \text{for } t \in \left[0,\ \frac{1}{4} - \beta\right[\\ 1 - 4\beta + \frac{8\beta}{\pi}\sin\left(\frac{\pi}{2\beta}\left(\beta + t - \frac{1}{4}\right)\right) & \text{for } t \in \left[\frac{1}{4} - \beta,\ \frac{1}{4} + \beta\right] \\ 2 - 4t & \text{for } t \in \left]\frac{1}{4} + \beta,\ \frac{3}{4} - \beta\right[\\ 4\beta - 1 - \frac{8\beta}{\pi}\sin\left(\frac{\pi}{2\beta}\left(\beta + t - \frac{3}{4}\right)\right) & \text{for } t \in \left[\frac{3}{4} - \beta,\ \frac{3}{4} + \beta\right] \\ 4(t-1) & \text{for } t \in \left]\frac{3}{4} + \beta,\ 1\right], \end{cases}$$

$$u_0(t) = u_0(t+1) \quad \text{for} \quad t \in \mathbb{R}$$

and

$$\tau(t)=\begin{cases} 1/4 & \text{for} \quad u_0''(t)\geq 0 \\ 3/4 & \text{for} \quad u_0''(t)<0. \end{cases}$$

Let $\omega=1$. For any $y\in \mathbb{C}_\omega$ we put

$$p_0(y)(t)=|u_0''(t)|\,y(\tau(t)), \quad p_1(y)(t)=\alpha\, u_0''(t)\,y(0). \tag{1.6}$$

It is evident that $p_0:\mathbb{C}_\omega\to L_\omega$ and $p_1:\mathbb{C}_\omega\to L_\omega$ are the linear bounded operators satisfying all the requirements of Theorem 1.1, except (1.5). On the other hand,

$$\int_0^\omega |p_0(1)(s)|\,ds=\int_0^1 |u_0''(s)|\,ds=16$$

and

$$1-\int_0^\omega [p_1^+(1)(s)+p_1^-(1)(s)]\,ds=1-\alpha\int_0^1 |u_0''(s)|\,ds=1-16\alpha=\frac{16}{16+\varepsilon}\,\cdot$$

Thus instead of (1.5) inequality (1.5_ε) with $\omega=1$ is fulfilled.

Note that the easily verifiable equalities

$$u_0'(0)=4,$$
$$1-u_0(\tau(t))\ \operatorname{sign} u_0''(t)=4\alpha \quad \text{for} \quad t\in\mathbb{R}$$

result in

$$\begin{aligned} u_0''(t)-|u_0''(t)|u_0(\tau(t)) &= u_0''(t)(1-u_0(\tau(t))\operatorname{sign} u_0''(t)) \\ &= \alpha\, u_0''(t)\, u_0'(0) \ \text{ for } \ t\in\mathbb{R}. \end{aligned}$$

Taking into account (1.6), the above equality implies that u_0 is a nontrivial $\omega=1$ periodic solution of the homogeneous equation

$$u''(t)=p_0(u)(t)+p_1(u')(t).$$

Consider now the equation with deviating arguments

$$u''(t)=\sum_{i=1}^m p_{0i}(t)\,u(\tau_{0i}(t))+\sum_{j=1}^n p_{1j}(t)\,u'(\tau_{1j}(t))+q(t), \tag{1.7}$$

where $q, p_{0i}, p_{1j}\in L_\omega$ and $\tau_{oi}, \tau_{1j}:\mathbb{R}\to\mathbb{R}$ are measurable functions such that

$$\tau_{0i}(t+\omega)=\nu_{0i}(t)\omega+\tau_{0i}(t), \quad \tau_{1j}(t+\omega)=\nu_{1j}(t)\omega+\tau_{1j}(t) \ \text{ for } \ t\in\mathbb{R}$$

if the functions ν_{0i} and ν_{1j} take only integral values.

Corollary 1.1. *Let there exist $\sigma\in\{-1,1\}$ such that*

$$\sigma\, p_{0i}(t)\geq 0 \quad (i=1,\dots,n) \ \textit{ for } \ t\in\mathbb{R}. \tag{1.8}$$

Moreover, let

$$\int_0^\omega p_{0i}(s)\,ds \neq 0 \quad (i = 1, \dots, n) \tag{1.9}$$

and

$$\sum_{i=1}^{m} \int_0^\omega |p_{0i}(s)|\,ds \leq \frac{16}{\omega}\left(1 - \sum_{j=1}^{n} \int_0^\omega |p_{1j}(s)|\,ds\right). \tag{1.10}$$

Then equation (1.7) *has one and only one* ω*-periodic solution.*

For equation (1.1′), where $p_{0i} \in L_\omega$ $(i = 1, 2)$ and $q \in L_\omega$, Corollary 1.1 results in

Corollary 1.1′. *Let*

$$p_{01}(t) \leq 0 \quad \textit{for} \quad t \in \mathbb{R}$$

and

$$0 < \int_0^\omega |p_{01}(s)|ds \leq \frac{16}{\omega}\left(1 - \int_0^\omega |p_{02}(s)|ds\right).$$

Then equation (1.1′) *has one and only one* ω*-periodic solution.*

For $p_{02}(t) \equiv 0$, this corollary coincides with the well-known result of Lasota-Opial.

Remark. It is clear that if $p_{01} \in L_\omega$ is non-negative, then (1.1′) has one and only one solution, no matter whatever $p_{02} \in L_\omega$ and $q \in L_\omega$ are. As we can see from Example 1.1, the similar statement is invalid for equation (1.7), i.e. the fact that the functions p_{0i} $(i = 1, \dots, n)$ are non-negative does not guarantee the existence of the unique ω-periodic solution of equation (1.7). Moreover, condition (1.10) in Corollary 1.1 is unimprovable even in the case in which

$$p_{0i}(x) \geq 0 \quad \text{for} \quad t \in \mathbb{R} \quad (i = 1, \dots, n).$$

1.3. Periodic solutions of nonlinear equations

Below, along with the differential equation (1.1) we have to consider the differential inequality

$$|v''(t) - g_0(v)(t)| \leq h_1(|v|)(t) + h_2(|v'|)(t), \tag{1.11}$$

where $g_0, h_1, h_2 : \mathbb{C}_\omega \to L_\omega$ are linear operators.

Definition 1.1. The inclusion

$$(h_{01}, h_{02}, h_1, h_2) \in \mathbb{O}_\omega$$

denotes that:

(i) $h_{0i}, h_i : \mathbb{C}_\omega \to L_\omega$ $(i = 1, 2)$ are linear non-negative operators;

(ii) the differential inequality (1.11) has no non-trivial solution if only $g_0 : \mathbb{C}_\omega \to L_\omega$ is a linear monotone operator such that

$$|h_{01}(x)(t)| \le |g_0(x)(t)| \le h_{02}(|x|)(t) \quad \text{for} \quad x \in \mathbb{C}_\omega, \ \ t \in \mathbb{R}. \tag{1.12}$$

Definition 1.2. We say that the function $\delta : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the set M_ω if

$$\delta(\cdot\,, \rho) \in L_\omega \quad \text{for} \quad \rho \in \mathbb{R}, \tag{1.13}$$

$\delta(t, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing function for almost all $t \in \mathbb{R}$, and

$$\lim_{\rho \to +\infty} \frac{1}{\rho} \int_0^\omega \delta(s, \rho)\, ds = 0. \tag{1.14}$$

Theorem 1.2. *Let*

$$|f(x)(t) - g(x, x)(t)| \le h_1(|x|)(t) + h_2(|x'|)(t) + \delta(t, \|x\|_{\mathbb{C}'_\omega}) \quad \textit{for } x \in \mathbb{C}'_\omega, \ \ t \in \mathbb{R}, \tag{1.15}$$

$$|h_{01}(y)(t)| \le |g(x, y)(t)| \le h_{02}(|y|)(t) \ \textit{for } x \in \mathbb{C}'_\omega, \ \ y \in \mathbb{C}_\omega, \ \ t \in \mathbb{R}, \tag{1.16}$$

where

$$(h_{01}, h_{02}, h_1, h_2) \in \mathbb{O}_\omega, \tag{1.17}$$

$\delta \in M_\omega$ and $g : \mathbb{C}'_\omega \times \mathbb{C}'_\omega \to L_\omega$ is a continuous operator such that $g(x, \cdot) : \mathbb{C}'_\omega \to L_\omega$ is a linear monotone operator for an arbitrary $x \in \mathbb{C}'_\omega$. Then equation (1.2) *has at least one ω-periodic solution.*

Corollary 1.2. *Let conditions* (1.15), (1.16) *be satisfied, where $\delta \in M_\omega$, $h_{0i}, h_i : \mathbb{C}_\omega \to L_\omega$ are linear non-negative operators and $g : \mathbb{C}'_\omega \times \mathbb{C}'_\omega \to L_\omega$ is a continuous operator such that $g(x, \cdot) : \mathbb{C}'_\omega \to L_\omega$ is a linear monotone operator for an arbitrary $x \in \mathbb{C}'_\omega$. Moreover, let*

$$\int_0^\omega (h_{02}(1)(s) + 4h_1(1)(s))ds \le \frac{16}{\omega} \left(1 - \int_0^\omega h_2(1)(s)\, ds\right) \tag{1.18}$$

and

$$h_1\left(|y|\right)(t) \le h_{01}\left(|y|\right)(t) \ \textit{for } y \in \mathbb{C}_\omega, \ \ t \in \mathbb{R}, \tag{1.10}$$

$$\operatorname{mes}\{t \in [0, \omega] : h_1(1)(t) < h_{01}(1)(t)\} > 0. \tag{1.19$'$}$$

Then equation (1.2) *has at least one ω-periodic solution.*

Consider now the case, where equation (1.2) has the form

$$u''(t) = f_0(t, u(\tau_1(t)), u'(\tau_2(t))) \quad \text{for} \quad t \in \mathbb{R}. \tag{1.20}$$

Here $f_0 : \mathbb{R}^3 \to \mathbb{R}$ belongs to the Carathéodory class, $\tau_1, \tau_2 : \mathbb{R} \to \mathbb{R}$ are measurable functions such that

$$\tau_i(t+\omega) = \nu_i(t)\omega + \tau_i(t) \ \ (i = 1, 2) \ \text{ for } \ t \in \mathbb{R}$$

if the functions ν_i $(i = 1, 2)$ take only integral values, and

$$f_0(t+\omega, x, y) = f_0(t, x, y) \quad \text{for} \quad (t, x, y) \in \mathbb{R}^3. \tag{1.21}$$

Then the following corollary is valid.

Corollary 1.3. *Let*

$$|f_0(t, x_1, x_2) - g_0(t, x_1, x_2)x_1| \le p_1(t)|x_1| + p_2(t)|x_2| + \delta(t, |x_1| + |x_2|) \quad \text{for } (t, x_1, x_2) \in \mathbb{R}^3, \tag{1.22}$$

$$p_{01}(t) \le \sigma g_0(t, x_1, x_2) \le p_{02}(t) \quad \text{for } (t, x_1, x_2) \in \mathbb{R}^3, \tag{1.23}$$

where $\sigma \in \{-1, 1\}$, $\delta \in M_\omega$, the functions $p_{0i}, p_i \in L_\omega$ $(i = 1, 2)$ are non-negative, and $g_0 : \mathbb{R}^3 \to \mathbb{R}$ is ω-periodic in the first argument and belongs to the Carathéodory class. Moreover, let

$$\int_0^\omega (p_{02}(s) + 4p_1(s))\, ds \le \frac{16}{\omega}\left(1 - \int_0^\omega p_2(s)\, ds\right) \tag{1.24}$$

and

$$\operatorname{mes}\{t \in [0, \omega] : p_1(t) < p_{01}(t)\} > 0, \ p_1(t) \le p_{01}(t) \ \textit{for}\ t \in \mathbb{R}. \tag{1.25}$$

Then equation (1.20) has at least one ω-periodic solution.

Theorem 1.3. *Let*

$$|f(x)(t) - f(\overline{x})(t) - g(x, \overline{x}, x - \overline{x})(t)| \le h_1(|x - \overline{x}|)(t) + h_2(|x' - \overline{x}\,'|)(t) \quad \text{for } x, \overline{x} \in \mathbb{C}'_\omega, \ t \in \mathbb{R}, \tag{1.26}$$

$$|h_{01}(y)(t)| \le |g(x, \overline{x}, y)(t)| \le h_{02}(|y|)(t) \ \text{for } x, \overline{x} \in \mathbb{C}'_\omega, \ y \in \mathbb{C}_\omega, \ t \in \mathbb{R}, \tag{1.27}$$

where

$$(h_{01}, h_{02}, h_1, h_2) \in \mathbb{O}_\omega,$$

and $g : \mathbb{C}'_\omega \times \mathbb{C}'_\omega \times \mathbb{C}_\omega \to L_\omega$ is a continuous operator such that $g(x, \overline{x}, \cdot) : \mathbb{C}_\omega \to L_\omega$ is a linear monotone operator for arbitrary $x, \overline{x} \in \mathbb{C}'_\omega$. Then equation (1.2) has one and only one ω-periodic solution.

Corollary 1.4. *Let conditions* (1.26) *and* (1.27) *be satisfied, where* $h_{0i}, h_i : \mathbb{C}_\omega \to L_\omega$ $(i = 1, 2)$ *are the linear non-negative operators and* $g : \mathbb{C}'_\omega \times \mathbb{C}'_\omega \times \mathbb{C}_\omega \to L_\omega$ *is the continuous operator such that* $g(x, \overline{x}, \cdot) : \mathbb{C}_\omega \to L_\omega$ *is a linear monotone operator for arbitrary* $x, \overline{x} \in \mathbb{C}'_\omega$. *Moreover, let inequalities* (1.18), (1.19) *and* (1.19′) *be satisfied. Then equation* (1.2) *has one and only one* ω*-periodic solution.*

Corollary 1.5. *Let*

$$p_{01}(t) \le \sigma \, \frac{\partial f_0(t, x, y)}{\partial x} \le p_{02}(t) \quad \text{for } (t, x, y) \in \mathbb{R}^3, \tag{1.28}$$

$$\left| \frac{\partial f_0(t, x, y)}{\partial y} \right| \le p_2(t) \qquad \text{for } (t, x, y) \in \mathbb{R}^3, \tag{1.29}$$

where $\sigma \in \{-1, 1\}$ *and* $p_{01}, p_{02}, p_2 \in L_\omega$ *are non-negative functions. Moreover, let*

$$\int_0^\omega p_{02}(s)\, ds \le \frac{16}{\omega} \left(1 - \int_0^\omega p_2(s)\, ds \right) \tag{1.30}$$

and

$$\int_0^\omega p_{01}(s)\, ds > 0. \tag{1.31}$$

Then equation (1.20) *has one and only one* ω*-periodic solution.*

This corollary is new as well as in the case, where $\tau_i(t) \equiv t$ $(i = 1, 2)$, i.e. when equation (1.20) coincides with (1.2′).

2. Auxiliary propositions

2.1. On one property of periodic functions

Define the functional $\Delta : \mathbb{C}_\omega \to \mathbb{R}_+$ by the equality

$$\Delta(x) = \max\{x(t) : 0 \le t \le \omega\} - \min\{x(t) : 0 \le t \le \omega\} \quad \text{for } x \in \mathbb{C}_\omega$$

and prove the following proposition.

Lemma 2.1. *Let*

$$v_0 \in \mathbb{C}'_\omega, \quad v_0(t) \not\equiv \text{const}. \tag{2.1}$$

Then

$$\Delta(v_0) < \frac{\omega}{4}\, \Delta(v'_0). \tag{2.2}$$

Proof. Since the function v_0 is ω-periodic, there exist $a \in [0,\omega[$ and $b \in]a, a+\omega[$ such that

$$v_0(a) = \max\{v_0(t) : 0 \le t \le \omega\}, \quad v_0(b) = \min\{v_0(t) : a \le t \le a+\omega\}.$$

Therefore

$$\Delta(v_0) = -\int_a^b v_0'(s)\,ds, \quad \Delta(v_0) = \int_b^{a+\omega} v_0'(s)\,ds.$$

From these equations, in view of conditions (2.1) we get respectively the estimates

$$\begin{aligned} &\Delta(v_0) < -(b-a)\min\{v_0'(t) : 0 \le t \le \omega\}, \\ &\Delta(v_0) < (a+\omega-b)\max\{v_0'(t) : 0 \le t \le \omega\}, \end{aligned} \tag{2.3}$$

where

$$\min\{v_0'(t) : 0 \le t \le \omega\} < 0, \quad \max\{v_0'(t) : 0 \le t \le \omega\} > 0.$$

Multiplying inequalities (2.3) and applying twice the numerical inequality

$$\lambda_1 \cdot \lambda_2 \le \frac{1}{4}(\lambda_1 + \lambda_2)^2 \quad \text{for} \quad \lambda_1 \ge 0, \quad \lambda_2 \ge 0, \tag{2.4}$$

we conclude that the lemma is valid. ■

2.2. The principle of a priori boundedness

Definition 2.1. We say that an operator $g : \mathbb{C}'_\omega \times \mathbb{C}'_\omega \to L_\omega$ belongs to the class $\mathbf{V}_\omega$ if it is continuous and satisfies the following three conditions:

(i) $g(x,\cdot) : \mathbb{C}'_\omega \to L_\omega$ is a linear monotone operator for any arbitrarily fixed $x \in \mathbb{C}'_\omega$;

(ii) There exists a non-decreasing in the second argument function $\alpha : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}_+$ such that $\alpha(\cdot\,,\rho) \in L_\omega$ for $\rho \in]0,+\infty[$, and for any $x, y \in \mathbb{C}'_\omega$ and almost all $t \in \mathbb{R}$ the inequality

$$|g(x,y)(t)| \le \alpha(t, \|x\|_{\mathbb{C}'_\omega})\|y\|_{\mathbb{C}'_\omega}$$

holds;

(iii) There exists a positive number ρ_1 such that for any $x \in \mathbb{C}'_\omega$ and $q \in L_\omega$ an arbitrary ω-periodic solution v of the differential equation

$$v''(t) = g(x,v)(t) + q(t) \tag{2.5}$$

admits the estimate

$$\|v\|_{\mathbb{C}'_\omega} \le \rho_1 \|q\|_{L_\omega}. \tag{2.6}$$

From Theorem 2.1 of [17] follows

Lemma 2.2. *Let there exist a positive number* ρ_2 *and an operator* $g \in \mathbf{V}_\omega$ *such that for any* $\lambda \in]0,1[$ *an arbitrary* ω*-periodic solution* v *of the differential equation*

$$v''(t) = (1-\lambda)g(v,v)(t) + \lambda f(v)(t) \tag{2.7}$$

admits the estimate

$$\|v\|_{\mathbb{C}'_\omega} \le \rho_2. \tag{2.8}$$

Then equation (1.2) *has at least one* ω*-periodic solution.*

2.3. Lemmas on the a priori estimate

Let us consider the differential inequality

$$|v''(t) - g_0(v)(t)| \le h_1(|v|)(t) - h_2(|v'|)(t) + q(t). \tag{2.9}$$

Then the following lemma is valid.

Lemma 2.3. *Let*

$$(h_{01}, h_{02}, h_1, h_2) \in \mathbb{O}_\omega. \tag{2.10}$$

Then there exists a constant ρ_0 *such that for any non-negative function* $q \in L_\omega$ *and linear monotone operator* $g_0 : \mathbb{C}_\omega \to L_\omega$ *which satisfies the condition*

$$|h_{01}(x)(t)| \le |g_0(x)(t)| \le h_{02}(|x|)(t) \ \ \textit{for}\ \ x \in \mathbb{C}_\omega, \ \ t \in \mathbb{R}, \tag{2.11}$$

every ω*-periodic solution* v *of inequality* (2.9) *admits the estimate*

$$\|v\|_{\mathbb{C}'_\omega} \le \rho_0 \|q\|_{L_\omega}. \tag{2.12}$$

Proof. Assume the contrary that the lemma is invalid. Then for any natural k there exist the monotone operator $g_k : \mathbb{C}_\omega \to L_\omega$ and the function $v_k \in \mathbb{C}'_\omega$ such that almost everywhere on $\mathbb{R}$ the inequality

$$|v_k''(t) - g_k(v_k)(t)| \le h_1(|v_k|)(t) + h_2(|v_k'|)(t) + q(t) \tag{2.13}$$

hold and

$$|h_{01}(x)(t)| \le |g_k(x)(t)| \le h_{02}(|x|)(t) \quad \text{for } x \in \mathbb{C}_\omega, \ t \in \mathbb{R}, \tag{2.14}$$

$$\sigma\, g_k(x)(t) \ge 0 \qquad \text{for } t \in \mathbb{R}, \tag{2.15}$$

where $\sigma \in \{-1, 1\}$ and

$$\|v_k\|_{\mathbb{C}'_\omega} > k\|q\|_{L_\omega}. \tag{2.16}$$

Now let us show that from the sequence $(g_k)_{k=1}^{\infty}$ we can choose a subsequence of monotone operators $(g_{kk})_{k=1}^{\infty}$ such that

$$\lim_{k\to+\infty}\int_0^t g_{kk}(x)(s)ds = \int_0^t \widetilde{g}_0(x)(s)ds \ \text{ uniformly on } \ [0,\omega], \tag{2.17}$$

where $\widetilde{g}_0 : \mathbb{C}_\omega \to L_\omega$ is the monotone linear operator,

$$|h_{01}(x)(t)| \le |\widetilde{g}_0(x)(t)| \le h_{02}(|x|)(t) \ \text{ for } \ x \in \mathbb{C}_\omega, \ \ t \in \mathbb{R} \tag{2.18}$$

and for some $\sigma \in \{-1, 1\}$

$$\sigma\, \widetilde{g}_0(1)(t) \ge 0 \quad \text{for} \quad t \in \mathbb{R}. \tag{2.19}$$

Towards this end, we consider some set $\{y_1, y_2, \ldots\} \in \mathbb{C}_\omega$ which is everywhere dense in $\mathbb{C}_\omega$. Taking into account inequality (2.14) and the fact that the operator h_{02} is monotone, the sequence $w_k(y_1)(t) = \int_0^t g_k(y_1)(s)ds$ $(k = 1, 2, \ldots)$ will be uniformly bounded and equicontinuous on $[0, \omega]$. According to the Arzela–Ascoli lemma, we can choose from $(g_k)_{k=1}^{\infty}$ a subsequence $(g_{1k})_{k=1}^{\infty}$ such that the sequence $w_{1k}(y_1)(t) = \int_0^t g_{1k}(y_1)(s)ds$ uniformly converges on $[0, \omega]$. Similarly, from $(g_{1k})_{k=1}^{\infty}$ one can choose a subsequence $(g_{2k})_{k=1}^{\infty}$ such that $w_{2k}(y_2)(t) = \int_0^t g_{2k}(y_2)(s)ds$ will uniformly converge on $[0, \omega]$. If we continue this process infinitely, we will get a system of sequences $(g_{ik})_{k=1}^{\infty}$ $(i = 1, 2, \ldots)$ such that $(g_{jk})_{k=1}^{\infty}$ is a subsequence of the sequence $(g_{ik})_{k=1}^{\infty}$ for any natural i and $j > i$, the sequence $w_{jk}(y_i)(t) = \int_0^t g_{jk}(y_i)(s)ds$ uniformly converges on $[0, \omega]$ and

$$|h_{01}(x)(t)| \le |g_{kk}(x)(t)| \le h_{02}(|x|)(t) \ \text{ for } x \in \mathbb{C}_\omega, \ t \in \mathbb{R} \tag{2.20}$$

and

$$|w_{kk}(x)(t) - w_{kk}(x)(s)| \le \|x\|_{\mathbb{C}} \int_s^t h_{02}(1)(\xi)d\xi \ \text{for } x \in \mathbb{C}_\omega, \ 0 \le s < t \le \omega. \tag{2.21}$$

Therefore by virtue of the Banach-Steinhaus theorem ([11], Ch.VII, §1, Theorem 3), there exists a continuous linear operator w_0 acting from $\mathbb{C}_\omega$ to the space of continuous on $[0, \omega]$ functions such that

$$\lim_{k\to\infty} w_{kk}(x)(t) = w_0(x)(t) \quad \text{uniformly on} \quad [0, \omega]. \tag{2.22}$$

It is clear from (2.21) and (2.22) that $w_0(x)(0) = 0$, $|w_0(x)(t) - w_0(x)(s)| \leq \leq \|x\|_{\mathbb{C}_\omega} \int_s^t h_{02}(1)(\xi)d\xi$ for $0 \leq s < t \leq \omega$. Hence the function $w_0(x) : [0,\omega] \to \mathbb{R}$ is absolutely continuous and

$$w_0(x)(t) = \int_0^t \widetilde{g}_0(x)(s)\,ds \quad \text{for} \quad 0 \leq t \leq \omega, \tag{2.23}$$

where $\widetilde{g}_0(x)(t) = \dfrac{d}{dt}[w_0(x)(t)]$, i.e. in view of (2.22) equality (2.17) is satisfied. Integrating inequality (2.20) from s to t $(0 \leq s < t \leq \omega)$, dividing it by $t - s$ and passing to the limit as $k \to \infty$ and then as $s \to t$ and taking into account equalities (2.22) and (2.23), we can see that $\widetilde{g}_0$ is the linear operator acting from $\mathbb{C}_\omega$ to the space of functions, Lebesgue summable on $[0,\omega]$ and satisfying on $[0,\omega]$ inequality (2.18). Further, extending $\widetilde{g}_0$ to the entire $\mathbb{R}$ by the equality

$$\widetilde{g}_0(x)(t+\omega) = \widetilde{g}_0(x)(t) \quad \text{for} \quad x \in \mathbb{C}_\omega, \;\; t \in \mathbb{R},$$

we find that $\widetilde{g}_0$ is a linear operator acting from $\mathbb{C}_\omega$ to L_ω and satisfying inequality (2.18). From equality (2.17), the fact that the operators g_k are monotone and inequalities (2.15) it becomes clear that $\widetilde{g}_0$ is a monotone operator and inequality (2.19) is satisfied.

By equality (2.17), without loss of generality we can suppose that for the sequence $(g_k)_{k=1}^{\infty}$ the equality

$$\lim_{k\to\infty} \int_0^t g_k(x)(s)\,ds = \int_0^t \widetilde{g}_0(x)(s)\,ds \;\; \text{uniformly on} \;\; [0,\omega] \tag{2.24}$$

is satisfied.

Suppose now that $z_k(t) = \dfrac{v_k(t)}{\|v_k\|_{\mathbb{C}'_\omega}}$. Then for any $k \in \mathbb{N}$

$$\|z_k\|_{\mathbb{C}'_\omega} = 1 \tag{2.25}$$

and due to inequalities (2.13) and (2.16), we arrive at

$$\left| z'_k(t) - z'_k(s) - \int_s^t g_k(z_k)(\xi)\,d\xi \right| \leq \int_s^t (h_1(|z_k|)(\xi) + h_2(|z'_k|)(\xi))d\xi + \frac{1}{k} \quad \text{for} \quad 0 \leq s < t \leq \omega. \tag{2.26}$$

It follows from (2.14), (2.25) and (2.26) that for any natural k

$$\|z_k\|_{\mathbb{C}_\omega} \leq 1, \quad \|z'_k\|_{\mathbb{C}_\omega} \leq 1$$

and

$$|z'_k(t) - z'_k(s)| \leq \int_s^t (h_{02}(1)(\xi) + h_1(1)(\xi) + h_2(1)(\xi))d\xi + \frac{1}{k} \quad \text{for} \quad 0 \leq s < t \leq \omega,$$

i.e., the sequences $(z_k^{(i)})_{k=1}^{\infty}$ $(i=0,1)$ are uniformly bounded and equicontinuous on $[0,\omega]$. Taking into account the Arzela–Ascoli lemma and the fact that the functions z_k $(k=1,2,\dots)$ are ω-periodic, we can choose a subsequence $(z_{k_m})_{m=1}^{\infty}$ such that $\lim\limits_{m\to\infty} z_{k_m}^{(i)}(t)=z_0^{(i)}(t)$ $(i=0,1)$ uniformly on $\mathbb{R}$. Therefore without loss of generality we can assume that for the sequence $(z_k)_{k=1}^{\infty}$ the equality

$$\lim_{k\to+\infty} z_k^{(i)}(t)=z_0^{(i)}(t) \ \ (i=0,1) \ \text{ uniformly on } \ [0,\omega] \tag{2.27}$$

is satisfied, where $z_0\in\mathbb{C}'_\omega$. Then from (2.14), (2.24), (2.25) and (2.27) we obtain

$$\|z_0\|_{\mathbb{C}'_\omega}=1 \tag{2.28}$$

and

$$\begin{aligned}\left|\int_s^t (g_k(z_k)(\xi)-\widetilde{g}_0(z_0)(\xi))\,d\xi\right| &\le \int_s^t\left| g_k(z_k-z_0)(\xi)\right|\,d\xi+\\ +\int_s^t |g_k(z_0)(\xi)-\widetilde{g}_0(z_0)(\xi)|\,d\xi &\le \int_s^t h_{02}(1)(\xi)\,d\xi\,\|z_k-z_0\|_{\mathbb{C}'_\omega}+\\ +\int_s^t |g_k(z_0)(\xi)-\widetilde{g}_0(z_0)(\xi)|\,d\xi \quad &\text{for} \quad 0\le s<t\le\omega,\end{aligned}$$

i.e.

$$\lim_{k\to\infty}\int_s^t g_k(z_k)(\xi)\,d\xi=\int_s^t \widetilde{g}_0(z_0)(\xi)\,d\xi \ \text{ for } \ 0\le s<t\le\omega. \tag{2.29}$$

Passing in equality (2.26) to the limit as $k\to\infty$ and taking into account (2.27) and (2.29), we obtain

$$\begin{aligned}\left|z_0'(t)-z_0'(s)-\int_s^t \widetilde{g}_0(z_0)(\xi)\,d\xi\right| \le \int_s^t (h_1(|z_0|)(\xi)+h_2(|z_0'|)(\xi))\,d\xi&\\ \text{for} \quad 0\le s<t\le\omega.&\end{aligned} \tag{2.30}$$

It is clear from (2.30) that $z_0\in\widetilde{\mathbb{C}}'_\omega$. Analogously, if we divide both parts of (2.30) by $t-s$ and pass to the limit as $s\to t$, we will find that z_0 is an ω-periodic solution of inequality (1.11) for $g_0=\widetilde{g}_0$. Then due to inclusion (2.10) we have $z_0(t)\equiv 0$, but this contradicts (2.28). The obtained contradiction proves the lemma. ■

Lemma 2.4. *Let the linear non-negative operators $h_{0i},h_i:\mathbb{C}_\omega\to L_\omega$ $(i=1,2)$ be such that conditions* (1.18), (1.19) *and* (1.19′) *are satisfied. Then inclusion* (2.10) *is valid.*

Proof. To prove the lemma, it suffices to establish that condition (ii) of Definition 1.1 is satisfied.

Assume the contrary that for some monotone operator $g_0:\mathbb{C}_\omega\to L_\omega$ which satisfies inequality (1.12) the differential inequality (1.11) has a non-trivial ω-periodic solution v_0. Consider first the case $v_0\equiv\text{const}$. From (1.11) it follows that

$$|g_0(1)(t)|\le h_1(1)(t) \quad \text{for} \quad t\in\mathbb{R},$$

which by condition (1.19′) contradicts (1.12). Hence $v_0 \not\equiv \text{const}$, and v_0' is of alternating signs. In this case we introduce the functional $\Delta : \mathbb{C}_\omega \to \mathbb{R}_+$ and the numbers $a \in [0, \omega[$, $b \in]a, a+\omega[$ as in proving Lemma 2.1. Since v_0' is of alternating signs, we have the estimate

$$|v_0'(t)| \le \Delta(v_0') \quad \text{for} \quad t \in \mathbb{R},$$

from which due to the fact that h_2 is monotone, we obtain the estimate

$$(-1)^k h_2(|v_0'|)(t) \le h_2(1)(t)\,\Delta(v_0') \ \ (k=1,2) \ \text{ for } \ t \in \mathbb{R}. \tag{2.31}$$

On the other hand, it is clear from (1.11) that

$$-h_2(|v_0'|)(t) - h_1(|v_0|)(t) \le v_0''(t) - g_0(v_0)(t) \le h_1(|v_0|)(t) + h_2(|v_0'|)(t).$$

Thus integrating from a to b and from b to $a+\omega$, the above inequality results, respectively in the following inequalities

$$\Delta(v_0') + \int_a^b g_0(v_0)(s)\,ds \le \int_a^b h_1(|v_0|)(s)\,ds + \int_a^b h_2(|v_0'|)(s)\,ds \tag{2.32}$$

and

$$\Delta(v_0') - \int_b^{a+\omega} g_0(v_0)(s)\,ds \le \int_b^{a+\omega} h_1(|v_0|)(s)\,ds + \int_b^{a+\omega} h_2(|v_0'|)(s)\,ds. \tag{2.32$_2$}$$

For the sake of definiteness, consider the case, where the operator g_0 is non-negative. Assuming that v_0 is of constant signs, in case $v_0(t) \ge 0$ from inequality (2.32$_1$) and in case $v_0(t) \le 0$ from inequality (2.32$_2$) with regard for (1.12), (1.19) and (2.31) we get

$$1 \le \int_a^{a+\omega} h_2(1)(s)\,ds. \tag{2.33}$$

On the other hand, owing to (2.19′) and (1.12) and taking into account that the operator h_1 is non-negative, we have

$$\int_0^\omega h_{02}(1)(s)\,ds > 0. \tag{2.34}$$

In view of this inequalities, from condition (1.18) we conclude that

$$\int_0^\omega h_2(1)(s)\,ds < 1.$$

By inclusion $h_2(1) \in L_\omega$, the last inequality contradicts (2.33), i.e. v_0 is of alternating signs and

$$\begin{aligned} &\min\{v_0(t) : 0 \le t \le \omega\} < 0, \\ &\max\{v_0(t) : 0 \le t \le \omega\} > 0. \end{aligned} \tag{2.35}$$

From these inequalities, in view of the fact that the operators g_0 and h_1 are non-negative, we obtain the estimates

$$g_0(1)(t)\min\{v_0(t):0\le t\le\omega\}\le$$
$$\le g_0(v_0)(t)\le g_0(1)(t)\max\{v_0(t):0\le t\le\omega\}\quad\text{for}\ \ t\in\mathbb{R}$$

and

$$h_1(|v_0|)(t)\le h_1(1)(t)\,\Delta(v_0)\quad\text{for}\quad t\in\mathbb{R}.$$

Taking into account the above estimates and (2.35), from inequalities (2.32_1) and (2.32_2) we find respectively

$$\Delta(v_0')\left(1-\int_a^{a+\omega}h_2(1)(s)\,ds\right)-\Delta(v_0)\int_a^{a+\omega}h_1(1)(s)\,ds\le$$
$$\le-\min\{v_0(t):0\le t\le\omega\}\cdot\int_a^b g_0(1)(s)\,ds$$

and

$$\Delta(v_0')\left(1-\int_a^{a+\omega}h_2(1)(s)\,ds\right)-\Delta(v_0)\int_a^{a+\omega}h_1(1)(s)\,ds\le$$
$$\le\max\{v_0(t):0\le t\le\omega\}\cdot\int_b^{a+\omega}g_0(1)(s)\,ds.$$

Multiplying these inequalities by each other, with regard for (2.2) and for inclusions $h_1(1),h_2(1)\in L_\omega$ we obtain

$$\Delta^2(v_0')\left(1-\int_0^\omega h_2(1)(s)\,ds-\frac{\omega}{4}\int_0^\omega h_1(1)(s)\,ds\right)^2\le$$
$$\le-\min\{v_0(t):0\le t\le\omega\}\cdot\max\{v_0(t):0\le t\le\omega\}\times$$
$$\times\int_a^b g_0(1)(s)\,ds\int_b^{a+\omega}g_0(1)(s)\,ds.$$

Applying twice inequality (2.4) to the last estimate (and this is quite possible by conditions (2.35)) and taking into account the inclusion $g_0(1)\in L_\omega$, we obtain

$$\Delta(v_0')\left(1-\int_0^\omega h_2(1)(s)\,ds-\frac{\omega}{4}\int_0^\omega h_1(1)(s)\,ds\right)\le\frac{\Delta(v_0)}{4}\int_0^\omega g_0(1)(s)\,ds.$$

This inequality on account of (1.12), (2.2) and (2.34) contradicts condition (1.18). The obtained contradiction proves that $v_0\equiv0$, i.e. condition (ii) is satisfied. Reasoning analogously, we can see that condition (ii) is valid in the case, where the operator g_0 is non-positive. Thus the lemma is proved. ■

3. Proof of the main results

Proof of Theorem 1.1. Since operators p_0, p_1^+, p_1^- are monotone, equation (1.1) has a unique ω-periodic solution if the corresponding homogeneous equation

$$v''(t) = p_0(v)(t) + p_1(v')(t) \tag{3.1}$$

has only the trivial ω-periodic solution.

Since the operator p_0 is monotone, there exists a constant $\sigma \in \{-1, 1\}$ such that

$$\sigma\, p_0(1)(t) \geq 0 \quad \text{for} \quad t \in \mathbb{R}. \tag{3.2}$$

Let the non-negative operators $h_{0i}, h_i : \mathbb{C}_\omega \to L_\omega$ $(i = 1, 2)$ be defined by the equalities

$$h_{0i}(x)(t) = \sigma\, p_0(x)(t) \;\; (i = 1, 2), \;\; h_1(x)(t) = 0,$$
$$h_2(x)(t) = p_1^+(x)(t) + p_1^-(x)(t) \;\; \text{for } x \in \mathbb{C}_\omega, \;\; t \in \mathbb{R}.$$

Then the inequality

$$|h_{01}(x)(t)| \leq |p_0(x)(t)| \leq h_{02}(|x|)(t) \quad x \in \mathbb{C}_\omega, \;\; t \in \mathbb{R} \tag{3.3}$$

is valid, and conditions (1.4), (1.5) and (3.2) imply that conditions (1.18), (1.19) and (1.19′) are valid. From Lemma 2.4 it follows that inclusion (2.10) is valid. This and inequality (3.3) for $g_0 = p_0$ imply that (1.11), which in our notation takes the form

$$|v''(t) - p_0(v)(t)| \leq p_1^+(|v'|)(t) + p_1^-(|v'|)(t), \tag{3.4}$$

has only the trivial ω-periodic solution.

It remains to note that by representation (1.3), every solution of equation (3.1) satisfies inequality (3.4). Hence equation (3.1) has also only the trivial ω-periodic solution. Thus the theorem is proved. ∎

Proof of Corollary 1.1. Let us introduce the following linear operators:

$$\begin{aligned} p_0(x)(t) &= \sum_{i=1}^{m} p_{0i}(t)\, x(\tau_{0i}(t)), \\ p_1(x)(t) &= \sum_{j=1}^{n} p_{1j}(t)\, x(\tau_{1j}(t)) \;\; \text{for} \;\; t \in \mathbb{R} \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} p_1^+(x)(t) &= \sum_{j=1}^{n} [p_{1j}(t)]_+\, x(\tau_{1j}(t)), \\ p_1^-(x)(t) &= \sum_{j=1}^{n} [p_{1j}(t)]_-\, x(\tau_{1j}(t)) \;\; \text{for} \;\; t \in \mathbb{R}. \end{aligned} \tag{3.6}$$

From equalities (3.5), (3.6) and from the definition of functions τ_{0i}, τ_{1j} it is clear that the operators p_0, p_1, p_1^-, p_1^+ act from the space $\mathbb{C}_\omega$ to L_ω. Moreover, by conditions (1.8) and (1.9), the operator p_0 is monotone and inequality (1.4) is satisfied. The fact that the operators p_1^+, p_1^- are non-negative follows from inequalities $[p_{1j}(t)]_+ \geq 0$ and $[p_{1j}(t)]_- \geq 0$. It is not difficult to notice that condition (1.10) in our notation can be rewritten in the form (1.5). Consequently, all the requirements of Theorem 1.1 are satisfied and hence equation (1.7) has one and only one ω-periodic solution. Thus the corollary is proved. ∎

Proof of Theorem 1.2. Let us first prove that the operator g mentioned in the theorem satisfies the inclusion

$$g \in \mathbf{V}_\omega. \tag{3.7}$$

Indeed, condition (ii) of Definition 2.1 follows from inequality (1.16), where $\alpha(t,\rho) = h_{02}(1)(t)$ for $t \in \mathbb{R}$, $\rho \in \mathbb{R}_+$. We fix $x_0 \in \mathbb{C}'_\omega$ arbitrarily and assume $g_0(\cdot)(t) = g(x_0, \cdot)(t)$. Taking into account inequality (1.16) and the fact that the operator $g_0(x_0, \cdot)$ is monotone, $g_0 : \mathbb{C}_\omega \to L_\omega$ will be the monotone linear operator satisfying inequality (2.11). Further, every ω-periodic solution v of equation (2.5) will be simultaneously the solution of inequality (2.9). Therefore by inclusion (1.17), from Lemma 2.3 it follows that estimate (2.6) is valid, where $\rho_1 = \rho_0$, and ρ_0 does not depend on the choice of the function $x_0 \in \mathbb{C}'_\omega$. Thus condition (iii) is also satisfied and inclusion (3.7) is valid.

Let us now show that every ω-periodic solution of equation (2.7) satisfies estimate (2.8) for any $\lambda \in]0,1[$. Indeed, if v is the ω-periodic solution of equation (2.7), then

$$|v''(t) - g(v,v)(t)| = \lambda |f(v)(t) - g(v,v)(t)|$$

and by virtue of condition (1.15), v satisfies the inequality

$$|v''(t) - g_0(v)(t)| \leq h_1(|v|)(t) + h_2(|v'|)(t) + \delta(t, \|v\|_{\mathbb{C}'_\omega}),$$

where $g_0(\cdot)(t) = g(v, \cdot)(t)$. Therefore from conditions (1.13), (1.16) and (1.17) it follows that all the requirements of Lemma 2.3 are fulfilled and hence the estimate

$$\|v\|_{\mathbb{C}'_\omega} \leq \rho_0 \int_0^\omega \delta(s, \|v\|_{\mathbb{C}'_\omega})\, ds \tag{3.8}$$

is valid. On the other hand, since $\delta \in M_\omega$, from equation (1.14) follows the existence of the constant $\rho_2 \in \mathbb{R}_+$ such that if we assume that $\|v\|_{\mathbb{C}'_\omega} > \rho_2$, then

$$\frac{\rho_0}{\|v\|_{\mathbb{C}'_\omega}} \int_0^\omega \delta(s, \|v\|_{\mathbb{C}'_\omega})\, ds < 1,$$

which contradicts (3.8), i.e. our assumption is invalid and estimate (2.8) is true. Consequently, all the requirements of Lemma 2.2 are satisfied and hence the theorem is valid. ∎

Proof of Corollary 1.2. In this corollary all the conditions of Theorem 1.2 are immediately required, except inclusion (1.17) whose validity follows directly from Lemma 2.4 by virtue of conditions (1.18), (1.19) and (1.19′). Consequently, all the requirements of Theorem 1.2 are satisfied and hence the corollary is valid. ■

Proof of Corollary 1.3. We define the operators h_{0i}, h_i $(i = 1, 2)$ and f, g by the equalities

$$h_{0i}(x)(t) = p_{0i}(t)\, x(\tau_1(t)), \;\; h_i(x)(t) = p_i(t)\, x(\tau_i(t)) \;\; (i = 1, 2) \text{ for } t \in \mathbb{R} \tag{3.9}$$

and

$$\begin{aligned} &f(x)(t) = f_0(t, x(\tau_1(t)), x'(\tau_2(t))),\\ &g(x, y)(t) = g_0(t, x(\tau_1(t)), x'(\tau_2(t)))\, y(\tau_1(t)) \;\text{ for }\; t \in \mathbb{R}. \end{aligned} \tag{3.10}$$

From (3.9) and the fact that p_{0i}, p_i $(i = 1, 2)$ are positive and also from the definition of the functions τ_1, τ_2 it is clear that h_{0i} and h_i are non-negative linear operators acting from the space $\mathbb{C}_\omega$ to L_ω. Analogously, from (3.10) on account of (1.21) and the definition of the functions g_0, τ_1, τ_2 it is clear that f and g are the continuous operators acting respectively from the spaces $\mathbb{C}'_\omega$ and $\mathbb{C}'_\omega \times \mathbb{C}_\omega$ to L_ω. It remains to note that in our notation from conditions (1.22)–(1.24) and (1.25) it follows that conditions (1.15), (1.16), (1.18) and (1.19), (1.19′) are valid, respectively. That is all the requirements of Corollary 1.2 are satisfied and hence the corollary is valid. ■

Proof of Theorem 1.3. According to (1.26) and (1.27), the inequalities

$$|f(x)(t) - \widetilde{g}(x, x)(t)| \le h_1(|x|)(t) h_2(|x'|)(t) + f(0)(t) \;\text{ for } x \in \mathbb{C}'_\omega, \; t \in \mathbb{R}$$

and

$$|h_{01}(y)(t)| \le |\widetilde{g}(x, y)(t)| \le h_{02}(|y|)(t) \;\text{ for }\; x \in \mathbb{C}'_\omega, \; y \in \mathbb{C}_\omega, \; t \in \mathbb{R}$$

are satisfied, where $\widetilde{g}(x, y)(t) = g(x, 0, y)(t)$. Then from Theorem 1.2 it follows that equation (1.2) has at least one ω-periodic solution. Let us now prove the uniqueness of the solution.

Let x and $\overline{x}$ be ω-periodic solutions of equation (1.2). Assume that $y = x - \overline{x}$ and $g_0(\cdot)(t) = \widetilde{g}(x, \overline{x}, \cdot)(t)$. Then according to conditions (1.26) and (1.27) we obtain

$$|y'(t) - g_0(y)(t)| \le h_1(|y|)(t) + h_2(|y'|)(t) \;\text{ for }\; t \in \mathbb{R}$$

and

$$|h_{01}(z)(t)| \le |g_0(z)(t)| \le h_{02}(|z|)(t) \;\text{ for }\; z \in \mathbb{C}_\omega, \;\; t \in \mathbb{R}.$$

By virtue of inclusion (1.17) the above inequalities imply that $y \equiv 0$, i.e. $x(t) \equiv \overline{x}(t)$. Thus the theorem is proved. ■

Proof of Corollary 1.4. By Lemma 2.4, from the non-negativeness of the operators h_{0i}, h_i $(i = 1, 2)$ and also from conditions (1.18), (1.19) and (1.19′) it follows that inclusion (1.17) is valid. That is all the requirements of Theorem 1.3 are satisfied and hence the corollary is true. ∎

Proof of Corollary 1.5. According to the theorem on a finite increment, the equality

$$f_0(t,x,y) - f_0(t,\overline{x},\overline{y}) =$$
$$= \int_0^1 \left[\frac{\partial f_0(t, x{+}(1{-}s)\overline{x}, y{+}(1{-}s)\overline{y})}{\partial x}(x{-}\overline{x}) + \frac{\partial f_0(t, x{+}(1{-}s)\overline{x}, y{+}(1{-}s)\overline{y})}{\partial y}(y{-}\overline{y}) \right] ds$$
$$\text{for } (t,x,\overline{x},y,\overline{y}) \in \mathbb{R}^5$$

is valid. If we introduce the notation

$$g_1(t,x,\overline{x},y,\overline{y}) = \int_0^1 \frac{\partial f_0(t, x + (1-s)\overline{x}, y + (1-s)\overline{y})}{\partial x}\, ds,$$

then from (3.11) and conditions (1.28) and (1.29) we get

$$|f_0(t,x,y) - f_0(t,\overline{x},\overline{y}) - g_1(t,x,\overline{x},y,\overline{y})(x-\overline{x})| \le p_2(t)|y-\overline{y}| \quad \text{for } (t,x,\overline{x},y,\overline{y}) \in \mathbb{R}^5,$$

$$p_{01}(t) \le \sigma\, g_1(t,x,\overline{x},y,\overline{y}) \le p_{02}(t) \qquad \text{for } (t,x,\overline{x},y,\overline{y}) \in \mathbb{R}^5.$$

Now we define the operators $h_2, h_{0i} : \mathbb{C}_\omega \to L_\omega$ $(i = 1,2)$, $f : \mathbb{C}'_\omega \to L_\omega$ by using equalities (3.9), (3.10) and the operator $g : \mathbb{C}'_\omega \times \mathbb{C}'_\omega \times \mathbb{C} \to L_\omega$ by using the equality

$$g(x,\overline{x},y)(t) = g_1(t, x(\tau_1(t)), \overline{x}(\tau_1(t)), y(\tau_2(t)), \overline{y}(\tau_2(t)))\, y(t).$$

Then from inequalities (3.12) and (3.13) it follows that conditions (1.26) and (1.27) for $h_1 \equiv 0$ are satisfied. In the same manner, in our notation from (1.30) and (1.31) it follows that conditions (1.18), (1.19) and (1.19′) are valid. Consequently, all the requirements of Corollary 1.4 are satisfied and hence the corollary is true. ∎

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