ISSN 0012-2661, Differential Equations, 2006, Vol. 42, No. 3, pp. 380–390. © Pleiades Publishing, Inc., 2006.
Original Russian Text © S.V. Mukhigulashvili, 2006, published in Differentsial'nye Uravneniya, 2006, Vol. 42, No. 3, pp. 356–365.

ORDINARY DIFFERENTIAL EQUATIONS

On the Solvability of the Periodic Problem for Nonlinear Second-Order Function-Differential Equations

S. V. Mukhigulashvili

Razmadze Mathematical Institute, Georgian Academy of Sciences, Tbilisi, Georgia

Received August 30, 2004

DOI: 10.1134/S0012266106030086

1. STATEMENT OF THE PROBLEM AND MAIN RESULTS

1.1. Statement of the Problem and Basic Notation

Let ω be a positive number. On the interval $[0, \omega]$, we consider the functional-differential equation

$$u''(t) = f(u)(t) \tag{1.1}$$

with the periodic boundary conditions

$$u^{(i)}(0) = u^{(i)}(\omega) \qquad (i = 0, 1). \tag{1.2}$$

Problem (1.1), (1.2) has been studied quite comprehensively (e.g., see [1–11, 13–15] and the bibliography therein) for the case in which f is a Nemytskii operator, i.e., $f(u)(t) = f_0(t, u(t), u'(t))$, and is little studied in the general case.

In the present paper, we use the method of *a priori* estimates to derive effective sufficient conditions for the solvability of problem (1.1), (1.2). These conditions are in some sense optimal. On the one hand, they generalize the well-known Lasota–Opial theorem [14], and on the other hand, they supplement the results in [12, 16–20] on the solvability of the periodic problem for functional-differential equations.

We use the following notation: $R =]-\infty, +\infty[$; $R_+ = [0, +\infty[$; $C([a, b]; R)$ is the space of continuous functions $u : [a, b] \to R$ with the norm $\|u\|_C = \max\{|u(t)| : \ a \le t \le b\}$; $C'([a, b]; R)$ is the space of functions $u : [a, b] \to R$ continuous together with their first derivatives, equipped with the norm $\|u\|_{C'} = \|u\|_C + \|u'\|_C$; $\tilde{C}'([a, b]; R)$ is the set of functions $u : [a, b] \to R$ absolutely continuous together with their first derivatives; $L([a, b]; R)$ is the space of functions $q : [a, b] \to R$ Lebesgue integrable on $[a, b]$, equipped with the norm $\|q\|_L = \int_a^b |q(s)|ds$. We set $[x]_+ = (|x| + x)/2$ and $[x]_- = (|x| - x)/2$ for each $x \in R$.

Throughout the following, we assume that $f : C'([0, \omega]; R) \to L([0, \omega]; R)$ is a continuous operator satisfying the condition

$$\sup\{|f(x)(\cdot)| : \ \|x\|_{C'} \le r\} \in L([0, \omega]; R_+) \quad \text{if} \quad r > 0.$$

A *solution* of problem (1.1), (1.2) is understood as a function $u \in \tilde{C}'([0, \omega]; R)$ satisfying condition (1.2) and Eq. (1.1) almost everywhere on $[0, \omega]$.

Definition 1.1. An operator $p : C([a, b]; R) \to L([a, b]; R)$ is said to belong to the set P_{ab} if $p(x)(t) \ge 0$ almost everywhere on $[a, b]$ for each function $x \in C([a, b]; R_+)$.

Definition 1.2. Let $A \subseteq [a, b]$ be a nonempty set. An operator $\ell : C([a, b]; R) \to L([a, b]; R)$ is said to belong to the set $K_{ab}(A)$ if $p(x)(t) = 0$ almost everywhere on $[a, b]$ for each function $x \in C([a, b]; R)$ such that $x(t) = 0$ for $t \in A$.

Remark 1.1. Let $A \subseteq [0,\omega]$ be a nonempty set, and let $\ell(x)(t) = p(t)x(\tau(t))$, where $p \in L([0,\omega];R)$ and $\tau : [0,\omega] \to [0,\omega]$ is a measurable function. Moreover, suppose that either $\tau(t) \in A$ for $0 \le t \le \omega$ or $p(t) = 0$ for $\tau(t) \in [0,\omega]\backslash A$. Then $\ell \in K_{0\omega}(A)$.

Definition 1.3. A function $\eta : R \times R_+ \to R_+$ is said to belong to the set M_ω if $\eta(\cdot, r)$ belonging to $L([0,\omega];R_+)$ for $r \in R_+$, $\eta(t,\cdot)$ is nondecreasing for almost all $t \in [0,\omega]$, and

$$\lim_{r\to+\infty} \frac{1}{r}\int_0^\omega \eta(s,r)ds = 0. \tag{1.3}$$

1.2. Statement of the Main Results

For each nonempty set $A \subseteq R$, we set $\varrho_A(t) = \inf\{|t-s| : \ s \in A\}$ and $\sigma_A(t) = \varrho_A(t)+\varrho_A(t+\omega/2)$.

Theorem 1.1. *Suppose that there exist operators*

$$g_0 : C'([0,\omega];R) \to L([0,\omega];R), \qquad p_0 : C'([0,\omega];R) \times C([0,\omega];R) \to L([0,\omega];R)$$

and functions $p, g \in L([0,\omega];R_+)$ *and* $\eta \in M_\omega$ *such that the conditions*

$$(f(x)(t) - p_0(x,x)(t) - g_0(x)(t)x'(t)) \operatorname{sgn} x(t) \ge -\eta(t, \|x\|_{C'}), \tag{1.4}$$

$$|g_0(x)(t)| \le g(t), \qquad p_0(x,1)(t) \le p(t) \tag{1.5}$$

are satisfied almost everywhere on $[0,\omega]$ *for each* $x \in C'([0,\omega];R)$. *Moreover, suppose that*

$$\int_0^\omega p_0(x,1)(s)ds \ge \alpha_0 \quad \text{for} \quad x \in C'([0,\omega];R), \tag{1.6}$$

$$p_0(x,\cdot) \in P_{0\omega} \cap K_{0\omega}(A) \quad \text{for} \quad x \in C'([0,\omega];R), \tag{1.7}$$

where $A \subseteq [0,\omega]$ *is a nonempty set,* $\alpha_0 > 0$, *and*

$$\left(1 - 4\left(\frac{\delta}{\omega}\right)^2\right)\int_0^\omega p(s)ds < \frac{16}{\omega}\exp\left\{-\frac{1}{2}\int_0^\omega g(s)ds\right\}, \quad \delta = \min\{\sigma_A(t) : \ 0 \le t \le \omega/2\}. \tag{1.8}$$

Then problem (1.1), (1.2) *is solvable.*

Remark 1.2. The minimum of σ_A can readily be computed for some special sets $A \subseteq [0,\omega]$. For example, if $\alpha, \beta \in [0,\omega]$, $\alpha \le \beta$, and $A = [\alpha, \beta]$ (or $A = [0,\alpha] \cup [\beta,\omega]$), then $\delta = [\omega/2 - (\beta-\alpha)]_+$ (respectively, $\delta = [\omega/2 - (\beta-\alpha)]_-$).

Remark 1.3. An example constructed in [18] shows that condition (1.8) is optimal in the sense that it cannot be replaced by the condition

$$\left(1 - 4\left(\frac{\delta}{\omega}\right)^2\right)\int_0^\omega p(s)ds < \frac{16+\varepsilon}{\omega}\exp\left\{-\frac{1}{2}\int_0^\omega g(s)ds\right\}$$

with a constant $\varepsilon > 0$, however small.

Consider the case in which Eq. (1.1) has the form

$$u''(t) = \ell(u)(t) + f_1(u)(t), \tag{1.9}$$

where $\ell : C([0,\omega];R) \to L([0,\omega];R)$ is a nonnegative linear operator and

$$f_1 : C'([0,\omega];R) \to L([0,\omega];R)$$

is a continuous operator such that $\sup\{|f_1(x)(\cdot)| : \ \|x\|_{C'} \le r\} \in L([0,\omega];R_+)$ for $r > 0$.