ORDINARY DIFFERENTIAL EQUATIONS

On the Solvability of the Periodic Problem for Nonlinear Second-Order Function-Differential Equations

S. V. Mukhigulashvili
Razmadze Mathematical Institute, Georgian Academy of Sciences, Tbilisi, Georgia
Received August 30, 2004

DOI: 10.1134/S0012266106030086

1. STATEMENT OF THE PROBLEM AND MAIN RESULTS

1.1. Statement of the Problem and Basic Notation

Let $\omega$ be a positive number. On the interval $[0, \omega]$, we consider the functional-differential equation

$$u''(t) = f(u)(t)$$

(1.1)

with the periodic boundary conditions

$$u^{(i)}(0) = u^{(i)}(\omega) \quad (i = 0, 1).$$

(1.2)

Problem (1.1), (1.2) has been studied quite comprehensively (e.g., see [1–11, 13–15] and the bibliography therein) for the case in which $f$ is a Nemytskii operator, i.e., $f(u)(t) = f_0(t, u(t), u'(t))$, and is little studied in the general case.

In the present paper, we use the method of a priori estimates to derive effective sufficient conditions for the solvability of problem (1.1), (1.2). These conditions are in some sense optimal. On the one hand, they generalize the well-known Lasota–Opial theorem [14], and on the other hand, they supplement the results in [12, 16–20] on the solvability of the periodic problem for functional-differential equations.

We use the following notation: $R = (-\infty, +\infty]; R_+ = [0, +\infty]; C([a, b]; R)$ is the space of continuous functions $u : [a, b] \to R$ with the norm $\|u\|_C = \max\{|u(t)| : a \leq t \leq b\}; C'([a, b]; R)$ is the space of functions $u : [a, b] \to R$ continuous together with their first derivatives, equipped with the norm $\|u\|_{C'} = \|u\|_C + \|u'\|_C; C'([a, b]; R)$ is the set of functions $u : [a, b] \to R$ absolutely continuous together with their first derivatives; $L([a, b]; R)$ is the space of functions $q : [a, b] \to R$ Lebesgue integrable on $[a, b]$, equipped with the norm $\|q\|_L = \int_a^b |q(s)| ds$. We set $|x|_+ = (|x| + x)/2$ and $|x|_+ = (|x| - x)/2$ for each $x \in R$.

Throughout the following, we assume that $f : C'([0, \omega]; R) \to L([0, \omega]; R)$ is a continuous operator satisfying the condition

$$\sup \{|f(x)(\cdot)| : \|x\|_{C'} \leq r\} \in L([0, \omega]; R_+) \quad \text{if} \quad r > 0.$$

A solution of problem (1.1), (1.2) is understood as a function $u \in C'([0, \omega]; R)$ satisfying condition (1.2) and Eq. (1.1) almost everywhere on $[0, \omega]$.

Definition 1.1. An operator $p : C([a, b]; R) \to L([a, b]; R)$ is said to belong to the set $P_{ab}$ if $p(x)(t) \geq 0$ almost everywhere on $[a, b]$ for each function $x \in C([a, b]; R_+)$.  

Definition 1.2. Let $A \subseteq [a, b]$ be a nonempty set. An operator $\ell : C([a, b]; R) \to L([a, b]; R)$ is said to belong to the set $K_{ab}(A)$ if $p(x)(t) = 0$ almost everywhere on $[a, b]$ for each function $x \in C([a, b]; R)$ such that $x(t) = 0$ for $t \in A$. 

389
Remark 1.1. Let $A \subseteq [0, \omega]$ be a nonempty set, and let $\ell(x)(t) = p(t)x(\tau(t))$, where $p \in L([0, \omega]; R)$ and $\tau : [0, \omega] \to [0, \omega]$ is a measurable function. Moreover, suppose that either $\tau(t) \in A$ for $0 \leq t \leq \omega$ or $p(t) = 0$ for $\tau(t) \in [0, \omega] \setminus A$. Then $\ell \in K_\omega(A)$.

Definition 1.3. A function $\eta : R \times R_+ \to R_+$ is said to belong to the set $M_\omega$ if $\eta(\cdot, r)$ belonging to $L([0, \omega]; R_+)$ for $r \in R_+$, $\eta(t, \cdot)$ is nondecreasing for almost all $t \in [0, \omega]$, and
\begin{equation}
\lim_{r \to +\infty} \frac{1}{r} \int_0^r \eta(s, r) ds = 0. \tag{1.3}
\end{equation}

1.2. Statement of the Main Results

For each nonempty set $A \subseteq R$, we set $\varrho_A(t) = \inf\{t-s : s \in A\}$ and $\sigma_A(t) = \varrho_A(t) + \varrho_A(t+\omega/2)$.

Theorem 1.1. Suppose that there exist operators
\[ g_0 : C'([0, \omega]; R) \to L([0, \omega]; R), \quad p_0 : C'([0, \omega]; R) \times C([0, \omega]; R) \to L([0, \omega]; R) \]
and functions $p, g \in L([0, \omega]; R)$ and $\eta \in M_\omega$ such that the conditions
\begin{align}
(f(x)(t) - p_0(x, x)(t) - g_0(x)(t)x'(t)) & \leq -\eta(t, ||x||_{C'}) \tag{1.4} \\
|g_0(x)(t)| & \leq g(t), \quad p_0(x, 1)(t) \geq p(t) \tag{1.5}
\end{align}
are satisfied almost everywhere on $[0, \omega]$ for each $x \in C'([0, \omega]; R)$. Moreover, suppose that
\begin{equation}
\int_0^\omega p_0(x, 1)(s) ds \geq \alpha_0 \quad \text{for} \quad x \in C'([0, \omega]; R), \tag{1.6}
\end{equation}
\begin{equation}
p_0(x, \cdot) \in P_\omega \cap K_\omega(A) \quad \text{for} \quad x \in C'([0, \omega]; R), \tag{1.7}
\end{equation}
where $A \subseteq [0, \omega]$ is a nonempty set, $\alpha_0 > 0$, and
\begin{equation}
\left(1 - 4 \frac{\delta}{\omega}\right) \int_0^\omega p(s) ds < \frac{16}{\omega} \exp\left\{ -\frac{1}{2} \int_0^\omega g(s) ds \right\}, \tag{1.8}
\end{equation}
where $A \subseteq [0, \omega]$ is a nonempty set, $\alpha_0 > 0$, and
\begin{equation}
\delta = \min\{\sigma_A(t) : 0 \leq t \leq \omega/2\}. \tag{1.8}
\end{equation}

Then problem (1.1), (1.2) is solvable.

Remark 1.2. The minimum of $\sigma_A$ can readily be computed for some special sets $A \subseteq [0, \omega]$. For example, if $\alpha, \beta \in [0, \omega], \alpha \leq \beta$, and $A = [\alpha, \beta]$ (or $A = [\alpha, \omega] \cup [\beta, \omega]$), then $\delta = [\omega/2 - (\beta - \alpha)]_+$ (respectively, $\delta = [\omega/2 - (\beta - \alpha)]_-.$).

Remark 1.3. An example constructed in [18] shows that condition (1.8) is optimal in the sense that it cannot be replaced by the condition
\begin{equation}
\left(1 - 4 \frac{\delta}{\omega}\right) \int_0^\omega p(s) ds < \frac{16 + \varepsilon}{\omega} \exp\left\{ -\frac{1}{2} \int_0^\omega g(s) ds \right\},
\end{equation}
with a constant $\varepsilon > 0$, however small.

Consider the case in which Eq. (1.1) has the form
\[ u''(t) = \ell(u)(t) + f_1(u)(t), \tag{1.9}\]
where $\ell : C([0, \omega]; R) \to L([0, \omega]; R)$ is a nonnegative linear operator and
\[ f_1 : C'([0, \omega]; R) \to L([0, \omega]; R) \]
is a continuous operator such that $\sup \{|f_1(x)(\cdot)| : \|x\|_{C'} \leq r \} \in L([0, \omega]; R_+)$ for $r > 0$. 

Differential Equations Vol. 42 No. 3 2006