ON A PERIODIC BOUNDARY VALUE PROBLEM FOR SECOND-ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Unimprovable efficient sufficient conditions are established for the unique solvability of the periodic problem $u''(t) = \ell(u)(t) + q(t)$ for $0 \le t \le \omega$, $u^{(i)}(0) = u^{(i)}(\omega)$ $(i = 0,1)$, where $\omega > 0$, $\ell : C([0,\omega]) \to L([0,\omega])$ is a linear bounded operator, and $q \in L([0,\omega])$.

1. Introduction

Consider the equation

$$u''(t) = \ell(u)(t) + q(t) \quad \text{for } 0 \le t \le \omega \tag{1.1}$$

with the periodic boundary conditions

$$u^{(i)}(0) = u^{(i)}(\omega) \quad (i = 0,1), \tag{1.2}$$

where $\omega > 0$, $\ell : C([0,\omega]) \to L([0,\omega])$ is a linear bounded operator and $q \in L([0,\omega])$.

By a solution of the problem (1.1), (1.2) we understand a function $u \in \widetilde{C}'([0,\omega])$, which satisfies (1.1) almost everywhere on $[0,\omega]$ and satisfies the conditions (1.2).

The periodic boundary value problem for functional differential equations has been studied by many authors (see, for instance, [1, 2, 3, 4, 5, 6, 8, 9] and the references therein). Results obtained in this paper on the one hand generalise the well-known results of Lasota and Opial (see [7, Theorem 6, page 88]) for linear ordinary differential equations, and on the other hand describe some properties which belong only to functional differential equations. In the paper [8], it was proved that the problem (1.1), (1.2) has a unique solution if the inequality

$$\int_0^\omega |\ell(1)(s)|\, ds \le \frac{d}{\omega} \tag{1.3}$$

with $d = 16$ is fulfilled. Moreover, there was also shown that the condition (1.3) is nonimprovable. This paper attempts to find a specific subset of the set of linear monotone operators, in which the condition (1.3) guarantees the unique solvability of the problem

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(1.1), (1.2) even for $d \geq 16$ (see Corollary 2.3). It turned out that if A satisfies some conditions dependent only on the constants d and ω, then $K_{[0,\omega]}(A)$ (see Definition 1.2) is such a subset of the set of linear monotone operators.

The following notation is used throughout.

N is the set of all natural numbers.

R is the set of all real numbers, $R_+ = [0,+\infty[$.

$C([a,b])$ is the Banach space of continuous functions $u:[a,b] \to R$ with the norm $\|u\|_C = \max\{|u(t)| : a \leq t \leq b\}$.

$\widetilde{C}'([a,b])$ is the set of functions $u:[a,b] \to R$ which are absolutely continuous together with their first derivatives.

$L([a,b])$ is the Banach space of Lebesgue integrable functions $p:[a,b] \to R$ with the norm $\|p\|_L = \int_a^b |p(s)|ds$.

If $x \in R$, then $[x]_+ = (|x|+x)/2$, $[x]_- = (|x|-x)/2$.

Definition 1.1. We will say that an operator $\ell : C([a,b]) \to L([a,b])$ is *nonnegative (nonpositive)*, if for any nonnegative $x \in C([a,b])$ the inequality

$$\ell(x)(t) \geq 0 \quad (\ell(x)(t) \leq 0) \quad \text{for } a \leq t \leq b \tag{1.4}$$

is satisfied.

We will say that an operator ℓ is *monotone* if it is nonnegative or nonpositive.

Definition 1.2. Let $A \subset [a,b]$ be a nonempty set. We will say that a linear operator $\ell : C([a,b]) \to L([a,b])$ belongs to the set $K_{[a,b]}(A)$ if for any $x \in C([a,b])$, satisfying

$$x(t) = 0 \quad \text{for } t \in A, \tag{1.5}$$

the equality

$$\ell(x)(t) = 0 \quad \text{for } a \leq t \leq b \tag{1.6}$$

holds.

We will say that $K_{[a,b]}(A)$ is the set of operators *concentrated* on the set $A \subset [a,b]$.

2. Main results

Define, for any nonempty set $A \subseteq R$, the continuous (see Lemma 3.1) functions:

$$\rho_A(t) = \inf\{|t-s| : s \in A\}, \quad \sigma_A(t) = \rho_A(t) + \rho_A\left(t + \frac{\omega}{2}\right) \quad \text{for } t \in R. \tag{2.1}$$

Theorem 2.1. *Let $A \subset [0,\omega]$, $A \neq \emptyset$ and a linear monotone operator $\ell \in K_{[0,\omega]}(A)$ be such that the conditions*

$$\int_0^\omega \ell(1)(s)ds \neq 0, \tag{2.2}$$

$$\left(1 - 4\left(\frac{\delta}{\omega}\right)^2\right)\int_0^\omega |\ell(1)(s)|ds \leq \frac{16}{\omega} \tag{2.3}$$