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ON ONE ESTIMATE FOR PERIODIC FUNCTIONS

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Abstract. For $v \in \widetilde{C}_\omega^n$ ($n \in N$, $\omega > 0$), the estimate

$$\triangle(v) < \frac{\omega^n}{d_n} \triangle\left(v^{(n)}\right)$$

is derived, where

$$\triangle\left(v^{(i)}\right) = \max\left\{v^{(i)}(t) : t \in R\right\} - \min\left\{v^{(i)}(t) : t \in R\right\} \quad (i = \overline{0,n})$$

and d_n are defined by a certain recurrent formula.

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Introduction

The following notation is used throughout the paper:

N is the set of all natural numbers.

R is the set of all real numbers, $R_+ = [0, +\infty[$.

$\widetilde{C}_\omega^n$, where $\omega > 0$, is a set of ω-periodic functions $u : R \to R$, which are absolutely continuous together with their n-th derivative.

$[k]$, where $k \in R$, is an integer part of k.

In many fields of mathematics, inequalities are used, in which a function is estimated by its derivatives, e.g., Wirtinger inequality (see [1]– [4]), Kolmogorov–Hardy inequality (see [5]), Sobolev inequality, generalized Poincaré inequality, etc. (see [6]). Inequalities of this type are frequently used in investigating boundary value problems for differential equations (see, e.g., [2–4]). In this paper, the difference of maximal and minimal values of an ω-periodic function is estimated by using the difference of maximal and minimal values of its n-th derivative. This inequality can be successfully applied in the investigation of a periodic problem for functional differential equations of higher order.

1. The Main Result

In the sequel, the following notation is used:

$$A_0 = 1, \quad A_1 = \frac{1}{15}, \quad A_j = A_1 \sum_{m_1=1}^{2} \sum_{m_2=1}^{m_1+1} \cdots \sum_{m_{j-1}=1}^{m_{j-2}+1} \frac{1}{\eta(m_1) \ldots \eta(m_{j-1})},$$

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$$B_1 = \frac{1}{8}, \quad B_j = A_1 \sum_{m_1=1}^{2} \sum_{m_2=1}^{m_1+1} \dots \sum_{m_{j-1}=1}^{m_{j-2}+1} \frac{1}{\eta(m_1)\dots\eta(m_{j-1})} \prod_{i=1}^{m_{j-1}+1} \left(1 + \frac{1}{2i}\right)$$

for $j \geq 2$, where

$$\eta(t) = (2t+1)(2t+3).$$

Let $d_1 = 4,\ d_2 = 32,\ d_3 = 192$, and for $p \in N$ put

$$\begin{gathered} d_{2p+2} = \frac{1}{\max\left\{(h_p(t)h_p(1-t))^{1/2} : 0 \leq t \leq 1\right\}}, \\ d_{2p+3} = \frac{1}{\max\left\{(f_p(s,t)f_p(1-s,1-t))^{1/2} : 0 \leq s \leq 1,\ 0 \leq t \leq 1\right\}}, \end{gathered} \tag{1.1}$$

where the functions $f_p : [0,1] \times [0,1] \to R_+$, $h_p : [0,1] \to R_+$ are defined as follows:

$$f_p(s,t) = \sum_{j=0}^{p-1} \alpha_{pj} t^{2(j+1)} + \alpha_{pp} t^{2p+3} s, \qquad h_p(t) = \sum_{j=0}^{p} \beta_{pj} t^{2(j+1)}, \tag{1.2}$$

and

$$\begin{gathered} \alpha_{pj} = \frac{A_j}{3 \cdot 4^{j+1} d_{2(p-j)+1}}, \quad \beta_{pj} = \frac{A_j}{3 \cdot 4^{j+1} d_{2(p-j)}} \quad (j = \overline{0, p-1}), \\ \alpha_{pp} = \frac{A_p}{3 \cdot 4^{p+1}}, \quad \beta_{pp} = \frac{B_p}{3 \cdot 4^{p+1}}. \end{gathered} \tag{1.3}$$

Theorem 1.1. *Let $n \in N$, $v \in \widetilde{C}_\omega^n$, $d_1 = 4$, $d_2 = 32$, $d_3 = 192$, and d_n (if $n \geq 4$) be given by equalities* (1.1)*. Let, moreover,*

$$v(t) \not\equiv const. \tag{1.4}$$

Then

$$\triangle(v) < \frac{\omega^n}{d_n} \triangle\left(v^{(n)}\right), \tag{1.5}$$

where

$$\triangle\left(v^{(i)}\right) = \max\left\{v^{(i)}(t) : t \in R\right\} - \min\left\{v^{(i)}(t) : t \in R\right\} \quad (i = \overline{0,n}). \tag{1.6}$$

Remark 1.1. From Theorem 1.1 it follows that the inequalities

$$\triangle\left(v^{(i)}\right) < \frac{\omega^{n-i}}{d_{n-i}} \triangle\left(v^{(n)}\right) \qquad \text{for} \quad i = 1, \dots, n-1$$

are also fulfilled.

Remark 1.2. An estimate

$$d_n < (2\pi)^n \qquad \text{for} \quad n \in N \tag{1.7}$$

holds.

Remark 1.3. In Theorem 1.1, the numbers d_n $(n = 1, \dots, 7)$ are nonimprovable in the sense that for every $\varepsilon > 0$ there exists $v_0 \in \widetilde{C}^n_\omega$ such that

$$\triangle (v_0) \geq \frac{\omega^n}{d_n + \varepsilon} \triangle \left(v_0^{(n)}\right), \tag{1.8}$$

where

$$\begin{gathered} d_4 = \frac{2^{11} \cdot 3}{5}, \qquad d_5 = 2^9 \cdot 3 \cdot 5, \\ d_6 = \frac{2^{16} \cdot 3^2 \cdot 5}{61}, \qquad d_7 = \frac{2^{14} \cdot 3^2 \cdot 5 \cdot 7}{17}. \end{gathered} \tag{1.9}$$

Remark 1.4. To prove the optimality of estimate (1.5) for $n \geq 8$ (in the sense of Remark 1.3) it is sufficient to show that for $p \geq 3$ we have

$$\max \left\{ h_p(t) \cdot h_p(1-t) : 0 \leq t \leq 1 \right\} = h_p^2(1/2), \tag{1.10}$$

$$\max \left\{ f_p(s,t) \cdot f_p(1-s, 1-t) : 0 \leq s, t \leq 1 \right\} = f_p^2(1/2, 1/2), \tag{1.11}$$

where the functions h_p and f_p are defined by (1.2). Equalities (1.10) and (1.11) are proved for $p = 1, 2$ (see On Remark 1.3 in Section 4). In the general case (started with $p = 3$), the proof of (1.10) and (1.11) is not known to the authors.

2. Auxiliary Propositions

Let $Q_m :]0, +\infty[\to]0, +\infty[$ $(m \in N)$ be the functions defined by the equality

$$Q_m(t) = \frac{2^m}{m! t^{2m}} \prod_{i=1}^{m} (2i+1).$$

Lemma 2.1. *Let $p \in N$, $\omega > 0$, and $v \in \widetilde{C}^{2p+3}_\omega$. Let, moreover, $a \in R$, $b \in]a, a+\omega[$, $\omega_1 = b - a$, (1.4) be fulfilled, and*

$$x(t) = (b-t)(t-a) \qquad \text{for} \quad a \leq t \leq b.$$

Then the following equalities hold:

$$\int_a^b x(s) v^{(3)}(s) ds = \frac{{\omega_1}^2}{6} \int_a^b v^{(3)}(s) ds - \frac{1}{12} \int_a^b x^2(s) v^{(5)}(s) ds \tag{2.1_1}$$

if $p = 1$, and

$$\begin{aligned} \int_a^b x(s) v^{(3)}(s) ds &= \frac{2}{3} \sum_{j=0}^{p-1} (-1)^j \left(\frac{\omega_1}{2}\right)^{2(j+1)} A_j \int_a^b v^{(2j+3)}(s) ds \\ &+ (-1)^p \frac{2}{45} \left(\frac{\omega_1}{2}\right)^{2(p+1)} \sum_{m_1=1}^{2} \sum_{m_2=1}^{m_1+1} \cdots \sum_{m_{p-1}=1}^{m_{p-2}+1} \frac{Q_{m_{p-1}+1}(\omega_1)}{\eta(m_1) \dots \eta(m_{p-1})} \\ &\times \int_a^b x^{m_{p-1}+1}(s) v^{(2p+3)}(s) ds \end{aligned} \tag{2.1_p}$$

if $p \geq 2$.

Proof. Let $m \in N$, $r \in \{1,2,\dots,2p-1\}$. Then the integration by parts, in view of (1.4), yields

$$\int_a^b x^m(s)v^{(r)}(s)ds = \frac{m}{m+1}\int_a^b (b-s)^{m+1}(s-a)^{m-1}v^{(r)}(s)ds$$
$$+\frac{1}{m+1}\int_a^b (b-s)^{m+1}(s-a)^m v^{(r+1)}(s)ds,$$
$$\int_a^b x^m(s)v^{(r)}(s)ds = \frac{m}{m+1}\int_a^b (b-s)^{m-1}(s-a)^{m+1}v^{(r)}(s)ds$$
$$-\frac{1}{m+1}\int_a^b (b-s)^{m}(s-a)^{m+1} v^{(r+1)}(s)ds.$$

Summing the last two equalities and adding to both sides the term

$$\frac{2m}{m+1}\int_a^b x^m(s)v^{(r)}(s)ds,$$

we obtain

$$\int_a^b x^m(s)v^{(r)}(s)ds = {\omega_1}^2\frac{m}{2(2m+1)}\int_a^b x^{m-1}(s)v^{(r)}(s)ds$$
$$-\frac{1}{2(m+1)(2m+1)}\int_a^b x^{m+1}(s)v^{(r+2)}(s)ds. \tag{2.2$_m$}$$

Now using the method of mathematical induction we will prove the following equality:

$$Q_m(\omega_1)\int_a^b x^m(s)v^{(r)}(s)ds = \int_a^b v^{(r)}(s)ds$$
$$-\left(\frac{\omega_1}{2}\right)^2\sum_{m_1=1}^{m}\frac{Q_{m_1+1}(\omega_1)}{(2m_1+1)(2m_1+3)}\int_a^b x^{m_1+1}(s)v^{(r+2)}(s)ds. \tag{2.3$_m$}$$

The validity of equality (2.3_1) immediately follows from (2.2_1). Now suppose that equality (2.3_{m-1}) holds and show that (2.3_m) is true. It is not difficult to verify that

$${\omega_1}^2\frac{m}{2(2m+1)}\,Q_m(\omega_1) = Q_{m-1}(\omega_1),$$

$$\frac{Q_m(\omega_1)}{2(2m+1)(m+1)} = \left(\frac{\omega_1}{2}\right)^2 \frac{Q_{m+1}(\omega_1)}{(2m+1)(2m+3)}\,.$$

Then from (2.2_m) we obtain

$$Q_m(\omega_1)\int_a^b x^m(s)v^{(r)}(s)ds = Q_{m-1}(\omega_1)\int_a^b x^{m-1}(s)v^{(r)}(s)ds$$
$$-\left(\frac{\omega_1}{2}\right)^2 \frac{Q_{m+1}(\omega_1)}{(2m+1)(2m+3)}\int_a^b x^{m+1}(s)v^{(r+2)}(s)ds.$$

Hence, applying (2.3_{m-1}), we get (2.3_m).

Now (2.3_1) with $r=3$ results in (2.1_1).

Further, using the method of mathematical induction we will show that (2.1_p) holds. From (2.3_2) with $r=5$ and (2.1_1) we have

$$\int_a^b x(s)v^{(3)}(s)ds = \frac{{\omega_1}^2}{6}\int_a^b v^{(3)}(s)ds - \frac{\omega_1^4}{360}\int_a^b v^{(5)}(s)ds$$
$$+\frac{1}{360}\sum_{i=1}^{2}\frac{\omega_1^{4-2i}}{3-i}\int_a^b x^{i+1}(s)v^{(7)}(s)ds,$$

and so equality (2.1_2) is valid. Suppose now that equality (2.1_{p-1}) holds and show that (2.1_p) is fulfilled. For this it is sufficient to use equality (2.3_m) with $m = m_{p-2}+1$, $r = 2p+1$ in equality (2.1_{p-1}). □

Lemma 2.2. *Let all the assumptions of Lemma* 2.1 *be fulfilled with* $v \in \widetilde{C}_{\omega}^{2p+2}$. *Then the following equalities hold:*

$$\int_a^b x(s)v^{(3)}(s)ds = \frac{{\omega_1}^2}{6}\int_a^b v^{(3)}(s)ds + \frac{1}{12}\int_a^b \left(x^2(s)\right)' v^{(4)}(s)ds \tag{2.4$_1$}$$

if $p=1$, *and*

$$\int_a^b x(s)v^{(3)}(s)ds = \frac{2}{3}\sum_{j=0}^{p-1}(-1)^j\left(\frac{\omega_1}{2}\right)^{2(j+1)} A_j\int_a^b v^{(2j+3)}(s)ds$$
$$+(-1)^{p+1}\frac{2}{45}\left(\frac{\omega_1}{2}\right)^{2(p+1)}\sum_{m_1=1}^{2}\sum_{m_2=1}^{m_1+1}\cdots\sum_{m_{p-1}=1}^{m_{p-2}+1}\frac{Q_{m_{p-1}+1}(\omega_1)}{\eta(m_1)\dots\eta(m_{p-1})}$$
$$\times\int_a^b \left(x^{m_{p-1}+1}(s)\right)' v^{(2p+2)}(s)ds \tag{2.4$_p$}$$

if $p \geq 2$.

Proof. From Lemma 2.1, by integration by parts, it follows that (2.4_1) and (2.4_p) hold for $v \in \widetilde{C}_\omega^{2p+3}$. Since $\widetilde{C}_\omega^{2p+3}$ is a dense subset of $\widetilde{C}_\omega^{2p+2}$, equalities (2.4_1) and (2.4_p) are fulfilled for every $v \in \widetilde{C}_\omega^{2p+2}$ as well. □

For $k \in N$ and $m \in N \cup \{0\}$, define the functions $W_{m,k} : [0,1] \to [-1,1]$ by the equalities

$$W_{0,k}(t) = \begin{cases} 1 & \text{for } 0 \le t \le \frac{1}{4} - \frac{1}{8k}\,, \\ \sin \pi k(1-4t) & \text{for } \frac{1}{4} - \frac{1}{8k} < t < \frac{1}{4} + \frac{1}{8k}\,, \\ -1 & \text{for } \frac{1}{4} + \frac{1}{8k} \le t \le \frac{1}{2}\,, \end{cases} \tag{2.5}$$

$$W_{0,k}\left(\frac{1}{2} + t\right) = W_{0,k}\left(\frac{1}{2} - t\right) \qquad \text{for} \quad 0 \le t \le \frac{1}{2}\,, \tag{2.6}$$

and

$$W_{m,k}(t) = \int_0^t W_{m-1,k}(s)ds - \delta_m \int_0^{1/4} W_{m-1,k}(s)ds \quad \text{for } \ t \in [0,1], \ \ m \in N, \tag{2.7}$$

where

$$\delta_m = \begin{cases} 0 & \text{if } m = 2\mu - 1, \\ 1 & \text{if } m = 2\mu, \end{cases} \qquad \mu \in N. \tag{2.8}$$

Lemma 2.3. *Let the functions $W_{m,k}$ be defined by (2.5)–(2.7). Then for every $p, k \in N$ and $m \in N \cup \{0\}$, the following equalities hold:*

$$W_{m,k}(0) = W_{m,k}(1), \tag{2.9}$$

$$\triangle\left(W_{2p,k}\right) = 2\left|W_{2p,k}\left(\frac{1}{2}\right)\right|, \tag{2.10}$$

$$\triangle\left(W_{2p-1,k}\right) = 2\left|W_{2p-1,k}\left(\frac{1}{4}\right)\right|, \tag{2.11}$$

and

$$W_{m,k}(t) = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} W_{0,k}(t_m)dt_m \ldots dt_1$$

$$+ \sum_{i=1}^{\left[\frac{m}{2}\right]} \frac{(-1)^i t^{m-2i}}{(m-2i)!}\left|W_{2i,k}\left(\frac{1}{2}\right)\right| \qquad \textit{for} \quad 0 < t \le 1, \quad m \ge 2. \tag{2.12_m}$$

Proof. First we show that the equalities

$$W_{m,k}\left(\frac{1}{2} - t\right) = (-1)^m W_{m,k}\left(\frac{1}{2} + t\right) \qquad \text{for} \quad 0 \le t \le \frac{1}{2} \tag{2.13_m}$$

and

$$W_{m,k}\left(\frac{1}{4} - t\right) = (-1)^{m-1} W_{m,k}\left(\frac{1}{4} + t\right) \qquad \text{for} \quad 0 \le t \le \frac{1}{4} \tag{2.14_m}$$

hold. It is not difficult to verify that (2.13_m) and (2.14_m) are valid for $m = 1, 2$. Now assume that (2.13_{m-1}) and (2.14_{m-1}) hold and show that (2.13_m) and (2.14_m) are fulfilled.

First note that

$$W_{m,k}\left(\frac{1}{2}-t\right) - (-1)^m W_{m,k}\left(\frac{1}{2}+t\right)$$
$$= \int_0^{1/2-t} W_{m-1,k}(s)ds - (-1)^m \int_0^{1/2+t} W_{m-1,k}(s)ds. \tag{2.15}$$

In view of (2.13_{m-1}) and (2.14_{m-1}) we have

$$\int_{1/2-t}^{1/2+t} W_{m-1,k}(s)ds = 0 \quad \text{if } m \text{ is even}$$

and

$$\int_0^{1/2-t} W_{m-1,k}(s)ds = -\int_0^{1/2+t} W_{m-1,k}(s)ds \quad \text{if } m \text{ is odd.}$$

From (2.15) and the last two equalities, the validity of (2.13_m) immediately follows. Analogously, we can prove equality (2.14_m).

It is clear that equalities (2.13_m) and (2.14_m) result in (2.9).

According to (2.14_m) and the definitions of the functions $W_{0,k}$ and $W_{m,k}$, using the method of mathematical induction, it is easy to show that

$$(-1)^p W_{2p,k}(t) > 0 \quad \text{for } 0 < t < \frac{1}{4}, \quad (-1)^p W_{2p-1,k}(t) < 0 \quad \text{for } 0 < t < \frac{1}{2}.$$

Consequently, in view of (2.13_m) and (2.14_m), for $p \in N$ we have

$$\begin{aligned} &(-1)^p W_{2p,k}(t) > 0 \quad \text{for } t \in]0, 1/4[\cup]3/4, 1[, \\ &(-1)^p W_{2p,k}(t) < 0 \quad \text{for } t \in]1/4, 3/4[, \end{aligned} \tag{2.16_p}$$

and

$$\begin{aligned} &(-1)^p W_{2p-1,k}(t) < 0 \quad \text{for } t \in]0, 1/2[, \\ &(-1)^p W_{2p-1,k}(t) > 0 \quad \text{for } t \in]1/2, 1[. \end{aligned} \tag{2.17_p}$$

From (2.16_p) and (2.17_p), in view of the relation

$$W_{m,k}^{(i)}(t) = W_{m-i,k}(t) \qquad \text{for} \quad 0 \le t \le 1, \quad i = 0, \dots, m, \tag{2.18}$$

we get

$$\begin{aligned} \min\left\{(-1)^p W_{2p,k}(t) : 0 \le t \le 1\right\} &= (-1)^p W_{2p,k}\left(\frac{1}{2}\right), \\ \max\left\{(-1)^p W_{2p,k}(t) : 0 \le t \le 1\right\} &= (-1)^p W_{2p,k}(0). \end{aligned} \tag{2.19}$$

On the other hand, from (2.14_m), (2.16_p), and (2.19) we obtain

$$\triangle(W_{2p,k}) = (-1)^p \left[W_{2p,k}(0) - W_{2p,k}\left(\frac{1}{2}\right)\right] = 2\left|W_{2p,k}\left(\frac{1}{2}\right)\right|. \tag{2.20}$$

Therefore, (2.10) is valid. Analogously (2.13_m), (2.16_p), (2.17_p), and (2.19) result in (2.11).

From (2.7) and (2.8) it immediately follows that

$$W_{m,k}(t) = \int_0^t \int_0^{t_1} \dots \int_0^{t_{m-1}} W_{0,k}(t_m)dt_m \dots dt_1$$

$$- \sum_{i=1}^{\left[\frac{m}{2}\right]} \frac{t^{m-2i}}{(m-2i)!} \int_0^{1/4} W_{2i-1,k}(s)ds \qquad \text{for} \quad 0 < t \le 1, \quad m \ge 2. \qquad (2.21_m)$$

However, in view of (2.14_m), (2.16_p), and (2.18), we have

$$\int_0^{1/4} W_{2i-1,k}(s)ds = -W_{2i,k}(0) = W_{2i,k}\left(\frac{1}{2}\right) = (-1)^{i-1} \left| W_{2i,k}\left(\frac{1}{2}\right) \right|,$$

and, consequently, (2.21_m) results in (2.12_m). □

Now define the functions $W_0 : [0,1] \to \{-1, 1\}$, $W_m : [0,1] \to R$, and positive constants $l_{m,k}, l_m$ $(m, k \in N)$ by the equalities

$$W_0(t) = \begin{cases} 1 & \text{for } t \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right], \\ -1 & \text{for } t \in \left]\frac{1}{4}, \frac{3}{4}\right[, \end{cases} \qquad (2.22)$$

$$W_m(t) = \int_0^t W_{m-1}(s)ds - \delta_m \int_0^{1/4} W_{m-1}(s)ds \qquad \text{for} \quad t \in [0,1], \qquad (2.23)$$

$$l_{2p-1,k} = \frac{1}{\left|W_{2p-1,k}\left(\frac{1}{4}\right)\right|}, \qquad l_{2p,k} = \frac{1}{\left|W_{2p,k}\left(\frac{1}{2}\right)\right|},$$
$$l_{2p-1} = \frac{1}{\left|W_{2p-1}\left(\frac{1}{4}\right)\right|}, \qquad l_{2p} = \frac{1}{\left|W_{2p}\left(\frac{1}{2}\right)\right|}, \qquad (2.24)$$

where $p \in N$ and δ_m are given by (2.8). Note that

$$\lim_{k \to +\infty} W_{0,k}(t) = W_0(t) \quad \text{almost everywhere on } [0,1], \qquad (2.25)$$

and by the Lebesgue Dominated Convergence Theorem we have that

$$\lim_{k \to +\infty} W_{m,k}(t) = W_m(t) \quad \text{uniformly on } [0,1], \quad m \in N. \qquad (2.26)$$

Therefore, on account of (2.24), we have

$$\lim_{k \to +\infty} l_{m,k} = l_m \qquad \text{for} \quad m \in N. \qquad (2.27)$$

Lemma 2.4. *Let $k \in N$ and let the functions W_{0k}, W_0, $W_{m,k}$, W_m, and the numbers $l_{m,k}$, l_m $(m \in N)$ be defined by* (2.5)–(2.7), *and* (2.22)–(2.24), *respectively. Then*

$$\triangle\left(W_{m,k}\right) = \frac{1}{l_{m,k}} \triangle\left(W_{0,k}\right) \qquad \text{for} \quad m \in N, \qquad (2.28)$$

$$\triangle\left(W_m\right) = \frac{1}{l_m}\triangle\left(W_0\right) \qquad for \quad m \in N, \tag{2.29}$$

and

$$l_{2p-1} = \frac{(-1)^{p+1}4^{2p-1}}{\sum\limits_{i=0}^{p-1} \frac{(-1)^i 16^i}{(2p-2i-1)!l_{2i}}}, \quad l_{2p} = \frac{(-1)^{p+1}4^{2p}}{\sum\limits_{i=0}^{p-1} \frac{(-1)^i 16^i}{(2p-2i)!l_{2i}}} \qquad for \quad p \in N, \tag{2.30}$$

where $l_0 = 1$*.*

Proof. Note that $\triangle\left(W_{0,k}\right) = 2$, and thus from (2.10), (2.11), and (2.24) we obtain (2.28), whence, in view of (2.25)–(2.27), we get (2.29).

By the definition of the functions $W_{0,k}$, we get

$$\lim_{k\to+\infty} \int\limits_0^{1/4} \int\limits_0^{t_1} \dots \int\limits_0^{t_{m-1}} W_{0,k}(t_m)dt_m \dots dt_1 = \frac{1}{m!4^m}, \tag{2.31}$$

and also, on account of (2.16_p), (2.17_p), we have

$$\left| W_{2p-i,k}\left(\frac{1}{2(i+1)}\right)\right| = (-1)^{p+1} W_{2p-i,k}\left(\frac{1}{2(i+1)}\right) \quad (i = 0, 1). \tag{2.32}$$

Then from (2.12_m) with $m = 2p-1$, $t = 1/4$, (2.24), (2.31) and (2.32) we get

$$l_{2p-1,k} = \frac{(-1)^{p+1}}{W_{2p-1,k}\left(\frac{1}{4}\right)} = \frac{(-1)^{p+1}}{\frac{1}{(2p-1)!4^{2p-1}} + \sum\limits_{i=1}^{p-1} \frac{(-1)^i}{(2p-2i-1)!4^{2p-2i-1}l_{2i,k}}}.$$

Hence, by virtue of (2.27), we obtain the first equality in (2.30).

Furthermore, note that from (2.14_{2p-1}) it follows that

$$\int\limits_0^{1/4} W_{2p-1,k}(s)ds = \int\limits_{1/4}^{1/2} W_{2p-1,k}(s)ds.$$

Therefore from (2.7), in view of (2.8), we have

$$W_{2p,k}\left(\frac{1}{2}\right) = \int\limits_0^{1/4} W_{2p-1,k}(s)ds$$

Hence, together with (2.12_{2p-1}), (2.24), (2.31) with $m = 2p$, and (2.32), we obtain

$$l_{2p,k} = \frac{(-1)^{p+1}}{W_{2p,k}\left(\frac{1}{2}\right)} = \frac{(-1)^{p+1}}{\frac{1}{(2p)!4^{2p}} + \sum\limits_{i=1}^{p-1} \frac{(-1)^i}{(2p-2i)!4^{2p-2i}l_{2i,k}}}.$$

Consequently, the last equality, by virtue of (2.27), results in the second equality in (2.30). $\square$

Lemma 2.5. *Let there exist $m \in N$ such that*

$$d_m = l_m. \tag{2.33}$$

Then for arbitrary $\varepsilon > 0$, there exists $v_0 \in \widetilde{C}^m_\omega$ such that

$$\triangle\,(v_0) \geq \frac{\omega^m}{d_m + \varepsilon} \triangle \left(v_0^{(m)} \right). \tag{2.34}$$

Proof. By virtue of (2.27) there exists $k_0 \in N$ such that

$$l_{m,k_0} \leq l_m + \varepsilon. \tag{2.35}$$

On the other hand, (2.28), in view of (2.18), yields

$$\triangle \left(\widetilde{W}_{m,k_0} \right) = \frac{\omega^m}{l_{m,k_0}} \triangle \left(\widetilde{W}_{0,k_0} \right),$$

where

$$\widetilde{W}_{m,k_0}(t) \stackrel{def}{=} W_{m,k_0}\left(\frac{t}{\omega} \right) \qquad \text{for} \quad t \in [0,\omega],$$

$$\widetilde{W}_{m-i,k_0}(t) \stackrel{def}{=} \widetilde{W}^{(i)}_{m,k_0}(t) \qquad \text{for} \quad t \in [0,\omega], \quad i = 1,\dots,m.$$

Now if we put

$$v_0(t) = \widetilde{W}_{m,k_0}(t) \qquad \text{for} \quad t \in [0,\omega],$$

then, on account of (2.18) and the fact that $W_{0,k_0} \in \widetilde{C}_\omega$, we get $v_0 \in \widetilde{C}^m_\omega$, and

$$\triangle\,(v_0) = \frac{\omega^m}{l_{m,k_0}} \triangle \left(v_0^{(m)} \right).$$

The last equality, together with (2.33) and (2.35), results in (2.34). $\square$

Lemma 2.6. *Let*

$$g(t) = \gamma_0 t^2 + \gamma_1 t^4 + \gamma_2 t^6 \qquad \textit{for} \quad 0 \leq t \leq 1 \tag{2.36}$$

and

$$\gamma_i \geq 0 \qquad (i = 0,1,2), \qquad \gamma_0 \geq \frac{\gamma_2}{2} - \frac{\gamma_1}{4}. \tag{2.37}$$

Then

$$\max\left\{ g(t)g(1-t) : 0 \leq t \leq 1 \right\} = g^2\left(\frac{1}{2} \right). \tag{2.38}$$

Proof. Since the function $g(t)g(1-t)$ is symmetric with respect to the point $t = \frac{1}{2}$, it is sufficient to show that

$$\frac{d}{dt}\big(g(t)g(1-t) \big) \geq 0 \qquad \text{for} \quad 0 \leq t \leq \frac{1}{2}. \tag{2.39}$$

First note that, in view of the equalities

$$t^2 + (1-t)^2 = 1 - 2x(t), \qquad t^4 + (1-t)^4 = 2x^2(t) - 4x(t) + 1,$$

where

$$x(t) = t(1-t),$$

we have

$$\begin{aligned} g(t)g(1-t) = {}& \gamma_1^2 x^4(t) + \gamma_2^2 x^6(t) \\ & + \gamma_0\big[(\gamma_0+\gamma_1+\gamma_2)x^2(t) - 2(\gamma_1+2\gamma_2)x^3(t)\big] \\ & + \gamma_2\big[(2\gamma_0+\gamma_1)x^4(t) - 2\gamma_1 x^5(t)\big]. \end{aligned} \tag{2.40}$$

On the other hand, on account of (2.37), we have

$$\frac{d}{x}\big((\gamma_0+\gamma_1+\gamma_2)x^2 - 2(\gamma_1+2\gamma_2)x^3\big) \geq 0 \qquad \text{for} \quad 0 \leq x \leq \frac{1}{4}\,, \tag{2.41}$$

$$\frac{d}{x}\big((2\gamma_0+\gamma_1)x^4 - 2\gamma_1 x^5\big) \geq 0 \qquad \text{for} \quad 0 \leq x \leq \frac{1}{4}\,. \tag{2.42}$$

Furthermore, it is obvious that

$$x\left(\frac{1}{2}\right) = \frac{1}{4}\,, \qquad x'(t) \geq 0 \qquad \text{for} \quad 0 \leq x \leq \frac{1}{2}\,. \tag{2.43}$$

Consequently, (2.40)–(2.43) result in (2.39). □

Lemma 2.7. *Let the function g be defined by* (2.36) *with*

$$\gamma_0 \geq 0, \qquad \gamma_1 \geq 0, \qquad \gamma_2 = 0, \tag{2.44}$$

and let

$$g_1(s,t) = g(t) + \gamma t^k s \qquad \textit{for} \quad 0 \leq s \leq 1, \quad 0 \leq t \leq 1, \tag{2.45}$$

where

$$\gamma > 0, \qquad k \geq 5. \tag{2.46}$$

Then

$$\max\big\{g_1(s,t)g_1(1-s,1-t) : 0 \leq s \leq 1, 0 \leq t \leq 1\big\} = g_1^2\left(\frac{1}{2}\,,\frac{1}{2}\right). \tag{2.47}$$

Proof. First note that

$$\begin{aligned} g_1(s,t)g_1(1-s,1-t) = {}& g(t)g(1-t) + \gamma q_0(s,t) \\ & + \gamma q_1(s,t)t^2(1-t)^2 + \gamma^2 t^k(1-t)^k s(1-s), \end{aligned} \tag{2.48}$$

where

$$q_j(s,t) = \gamma_j t^2(1-t)^2\left((1-t)^{k-2(j+1)}(1-s) + t^{k-2(j+1)}s\right) \quad (j=0,1).$$

It can be easily verified that if $q_j \not\equiv 0$, then

$$\Theta = \big\{(s,0),(s,1/2),(s,1) : 0 \leq s \leq 1\big\}$$

is a set of all zeros of the function $\frac{\partial}{\partial s}q_j(s,t)$. Moreover, since $k \geq 5$, the points

$$(1/2,1/2), \quad (s,0), \quad (s,1) \qquad \text{for} \quad 0 \leq s \leq 1$$

are the only zeros of the function $\frac{\partial}{\partial t}q_j(s,t)$ in Θ. Consequently, only at these point the functions q_j may take extremal values. Hence, on account of (2.44), (2.46), and the fact that $q_j(s,0) = q_j(s,1) = 0$ for $0 \leq s \leq 1$, we obtain

$$0 \leq q_j(s,t) \leq q_j\left(\frac{1}{2}\,,\frac{1}{2}\right) \quad \text{for} \quad 0 \leq s \leq 1, \quad 0 \leq t \leq 1 \quad (j=0,1). \tag{2.49}$$

Obviously, inequality (2.49) holds also in the case where $q_j \equiv 0$.

On the other hand, by virtue of (2.44), the assumptions of Lemma 2.6 are fulfilled. Consequently, from (2.48), in view of (2.38) and (2.49), it follows that (2.47) holds. □

3. Proof of the Main Result

Proof of Theorem 1.1. First we will show that the theorem is valid for $n = 1, 2, 3$, and then we will prove the theorem by the method of mathematical induction separately for the case where n is odd and for the case where n is even.

First we introduce some notations. Let $n \in N$, $v \in \widetilde{C}^n_\omega$, and for every $m \in N \cup \{0\}$ put

$$M_{i,m} = \max\left\{(-1)^m v^{(i)}(t) : 0 \le t \le \omega\right\} \qquad \text{for} \quad i = 0, 1, \dots, n. \tag{3.1}$$

Choose $a_1 \in R$, $a_2 \in]a_1, a_1 + \omega[$ such that

$$v(a_1) = M_{0,0}, \qquad v(a_2) = -M_{0,1}. \tag{3.2}$$

Let

$$\omega_1 = a_2 - a_1, \qquad \omega_2 = a_1 + \omega - a_2. \tag{3.3}$$

Obviously,

$$v(a_1 + \omega) = M_{0,0}.$$

It is not difficult to verify that for every $m_1, m_2 \in N \cup \{0\}$ and $i \in \{0, 1, \dots, n\}$ we have

$$M_{i,m_1} + M_{i,m_1+1} = M_{i,m_2} + M_{i,m_2+1},$$

and, consequently, from (3.1) and (1.6), we get

$$\triangle\left(v^{(i)}\right) = M_{i,m} + M_{i,m+1} \qquad \text{for} \quad i = 0, 1, \dots, n. \tag{3.4}$$

Moreover, in view of (3.1)–(3.3) it is clear that

$$v'(a_1) = 0, \quad v'(a_1 + \omega_1) = 0, \quad v'(a_2 + \omega_2) = 0. \tag{3.5}$$

From (3.1) and (3.2) we have

$$\triangle(v) = (-1)^r \int_{a_r}^{a_r+\omega_r} v'(s)ds, \qquad r = 1, 2. \tag{3.6}$$

Put

$$x_r(t) = (a_r + \omega_r - t)(t - a_r), \qquad r = 1, 2.$$

Then by integration by parts, from (3.6), in view of (3.5), for $v \in \widetilde{C}^2_\omega$ and $v \in \widetilde{C}^3_\omega$, we obtain

$$\triangle(v) = \frac{(-1)^r}{2} \int_{a_r}^{a_r+\omega_r} x_r'(s)v''(s)ds, \qquad r = 1, 2, \tag{3.7}$$

and

$$\triangle(v) = \frac{(-1)^{r-1}}{2} \int\limits_{a_r}^{a_r+\omega_r} x_r(s) v^{(3)}(s) ds, \qquad r = 1, 2, \tag{3.8$_r$}$$

respectively.

Now let $n \in \{1, 2, 3\}$. From the conditions $v \in \widetilde{C}^n_\omega$ and (1.4) it follows that

$$v^{(n)}(t) \not\equiv 0 \tag{3.9}$$

at least on one of the intervals $]a_r, a_r + \omega_r[$, $(r = 1, 2)$. Assume that (3.9) is fulfilled on the interval $]a_1, a_1 + \omega_1[$ (the case where (3.9) holds on $]a_2, a_2 + \omega_2[$ is similar). Then from equalities (3.6)–(3.8$_r$) we get the following estimates, respectively:

$$\triangle(v) < \omega_1 M_{1,1}, \qquad \triangle(v) \le \omega_2 M_{1,2}, \tag{3.10}$$

$$\triangle(v) < \frac{1}{2}\left(M_{2,1} \int\limits_{a_1}^{a_1+\frac{\omega_1}{2}} x_1'(s) ds + M_{2,2} \int\limits_{a_1+\frac{\omega_1}{2}}^{a_1+\omega_1} |x_1'(s)| ds \right)$$

$$= \frac{{\omega_1}^2}{8}(M_{2,1} + M_{2,2}), \qquad \triangle(v) \le \frac{{\omega_2}^2}{8}(M_{2,1} + M_{2,2}), \tag{3.11}$$

$$\triangle(v) < \frac{{\omega_1}^3}{12} M_{3,0}, \qquad \triangle(v) \le \frac{{\omega_2}^3}{12} M_{3,1}. \tag{3.12}$$

Multiplying the corresponding sides of the inequalities in (3.10) and applying the numerical inequality

$$4\lambda_1\lambda_2 \le (\lambda_1 + \lambda_2)^2 \qquad \text{for} \quad \lambda_1 \ge 0, \ \lambda_2 \ge 0, \tag{3.13}$$

we get

$$\triangle(v) < \frac{\omega}{d_1} \triangle(v'). \tag{3.14}$$

Analogously, from (3.11) and (3.12), in view of (3.4) and (3.13), we respectively have

$$\triangle(v) < \frac{\omega^2}{d_2} \triangle(v''), \quad \triangle(v) < \frac{\omega^3}{d_3} \triangle\left(v^{(3)}\right). \tag{3.15}$$

Thus (3.14) and (3.15) show that the theorem is valid for $n = 1, 2, 3$.

Now let $n = 2p + 3$, $p \in N$, $v \in \widetilde{C}^n_\omega$, and assume that (1.5) holds for $n = 2j+1$ $(j = 0, 1, \ldots, p)$. Then it is not difficult to see that $v^{(2j+2)} \in \widetilde{C}^{2(p-j)+1}_\omega$ $(j = 0, 1, \ldots, p)$,

$$\triangle\left(v^{(2j+2)}\right) < \frac{\omega^{2(p-j)+1}}{d_{2(p-j)+1}} \triangle\left(v^{(2p+3)}\right), \tag{3.16}$$

and

$$\left| \int\limits_{a_r}^{a_r+\omega_r} v^{(2j+3)}(s) ds \right| \le \triangle\left(v^{(2j+2)}\right) \qquad \text{for} \quad r = 1, 2; \quad j = 0, 1, \ldots p.$$

Hence, in view of (3.16), we get

$$\left|\int\limits_{a_r}^{a_r+\omega_r} v^{(2j+3)}(s)ds\right| < \frac{\omega^{2(p-j)+1}}{d_{2(p-j)+1}}\triangle\left(v^{(2p+3)}\right) \tag{3.17}$$

for $r = 1, 2$; $j = 0, 1, \dots p$. It can also be verified that

$$\int\limits_{a_r}^{a_r+\omega_r} {x_r}^m ds = \frac{m!m!}{(2m+1)!}{\omega_r}^{2m+1} \qquad \text{for} \quad r = 1, 2; \quad m = 0, 1, \dots,$$

and, consequently,

$$(-1)^r \int\limits_{a_r}^{a_r+\omega_r} {x_2}^2 v^{(5)}(s)ds \le \frac{A_1}{2}{\omega_r}^5 M_{5,r} \qquad \text{for} \quad r = 1, 2, \tag{3.18}$$

$$(-1)^{p+r-1}\left(\frac{\omega_r}{2}\right)^{2(p+1)} Q_{m_{p-1}+1}(\omega_r)\int\limits_{a_r}^{a_r+\omega_r} {x_r}^{m_{p-1}+1}(s)v^{(2p+3)}(s)ds$$

$$\le 2\left(\frac{\omega_r}{2}\right)^{2p+3} M_{2p+3,p+r-1} \qquad \text{for} \quad r = 1, 2; \quad p = 2, 3, \dots. \tag{3.19}$$

From (3.8_r), by virtue of (2.1_1), (2.1_p) with $a = a_r, b = a_r + \omega_r$ ($r = 1, 2$), and estimates (3.17), (3.18), and (3.19), we get

$$\triangle(v) < \triangle\left(v^{(2p+3)}\right)\frac{1}{3}\sum_{j=0}^{p-1}\left(\frac{\omega_r}{2}\right)^{2(j+1)}\frac{\omega^{2(p-j)+1}}{d_{2(p-j)+1}}A_j$$

$$+\frac{2}{3}A_p\left(\frac{\omega_r}{2}\right)^{2p+3} M_{2p+3,p+r-1} \qquad \text{for} \quad r = 1, 2; \quad p = 1, 2, \dots$$

whence we obtain

$$\triangle(v) < \omega^{2p+3}\triangle\left(v^{(2p+3)}\right) f_p(s_r, t_r) \quad (r = 1, 2), \tag{3.20_r}$$

where

$$t_r = \frac{\omega_r}{\omega}, \quad s_r = \frac{M_{2p+3,p+r-1}}{\triangle\left(v^{(2p+3)}\right)} \quad (r = 1, 2).$$

In view of (3.2) and (3.3),

$$t_1 + t_2 = 1, \quad s_1 + s_2 = 1.$$

Multiplying the corresponding sides in inequalities (3.20_1) and (3.20_2) we get

$$\triangle(v) < \omega^{2p+3}\triangle\left(v^{(2p+3)}\right)\left(f_p(s_1, t_1)\cdot f_p(1 - s_1, 1 - t_1)\right)^{1/2}$$

whence we get the validity of the theorem for $n = 2p + 3$.

Now let $n = 2p+2$, $p \in N$, $v \in \widetilde{C}^n_\omega$, and assume that (1.5) holds for $n = 2j$ $(j = 1, \dots, p)$. Then

$$(-1)^{r-1} \int_{a_r}^{a_r+\omega_r} \left(x_r{}^2(s)\right)' v^{(4)}(s)ds \le M_{4,r-1} \int_{a_r}^{a_r+\frac{\omega_r}{2}} \left(x_r{}^2(s)\right)' ds$$

$$+M_{4,r} \int_{a_r}^{a_r+\frac{\omega_r}{2}} \left(x_r{}^2(s)\right)' ds = \left(\frac{\omega_r}{2}\right)^4 \triangle\left(v^{(4)}\right), \tag{3.21}$$

$$(-1)^{p+r} Q_{m_{p-1}+1}(\omega_r) \int_{a_r}^{a_r+\omega_r} \left(x_r{}^{m_{p-1}+1}(s)\right)' v^{(2p+2)}(s)ds$$

$$\le Q_{m_{p-1}+1}(\omega_r) \left(M_{2p+2,p+r} \int_{a_r}^{a_r+\frac{\omega_r}{2}} \left(x_r{}^{m_{p-1}+1}(s)\right)' ds \right.$$

$$\left. + M_{2p+2,p+r+1} \int_{a_r}^{a_r+\frac{\omega_r}{2}} \left(x_r{}^{m_{p-1}+1}(s)\right)' ds \right)$$

$$= \triangle\left(v^{(2p+2)}\right) \prod_{i=1}^{m_{p-1}+1} \left(1 + \frac{1}{2i}\right) \qquad \text{for} \quad p = 2, 3, \dots . \tag{3.22}$$

Analogously, we get

$$\left| \int_{a_r}^{a_r+\omega_r} v^{(2j+3)}(s)ds \right| < \frac{\omega^{2(p-j)}}{d_{2(p-j)}} \triangle\left(v^{(2p+2)}\right). \tag{3.23}$$

Now from (3.8_r), by virtue of (2.4_1) and (2.4_p) with $a = a_r$, $b = a_r + \omega_r$ $(r = 1, 2)$, and estimates (3.21), (3.22), and (3.23), we obtain

$$\triangle(v) < \triangle\left(v^{(2p+2)}\right) \left(\frac{1}{3} \sum_{j=0}^{p-1} \left(\frac{\omega_r}{2}\right)^{2(j+1)} \frac{\omega^{2(p-j)} A_j}{d_{2(p-j)}} + \frac{B_p}{3} \left(\frac{\omega_r}{2}\right)^{2p+2} \right).$$

Hence we get

$$\triangle(v) < \omega^{2p+2} \triangle\left(v^{(2p+2)}\right) h_p(t_r) \qquad (r = 1, 2), \tag{3.24_r}$$

where $t_r = \omega_r/\omega$, $(r = 1, 2)$, and in view of (3.3) we have $t_2 = 1 - t_1$. Multiplying the corresponding sides in inequalities (3.24_1) and (3.24_2) we obtain

$$\triangle(v) < \omega^{2p+2} \triangle\left(v^{(2p+2)}\right) \left(h_p(t_1) \cdot h_p(1-t_1)\right)^{1/2},$$

and, consequently, the theorem is valid for $n = 2p+2$ as well. □

4. On the Remarks

On Remark 1.2. Estimate (1.7) can be derived from (1.5) if we put $\omega = 2\pi$, $v(t) = \cos t$ for $t \in [0, 2\pi]$.

On Remark 1.3. According to Lemma 2.5 it is sufficient to show that for $n \in \{1, 2, \dots, 7\}$ we have

$$d_n = l_n. \tag{4.1$_n$}$$

From (2.30) we immediately get

$$\begin{gathered} l_1 = 4, \qquad l_2 = 32, \qquad l_3 = 192, \qquad l_4 = \frac{2^{11} \cdot 3}{5}, \\ l_5 = 2^9 \cdot 3 \cdot 5, \qquad l_6 = \frac{2^{16} \cdot 3^2 \cdot 5}{61}, \qquad l_7 = \frac{2^{14} \cdot 3^2 \cdot 5 \cdot 7}{17}. \end{gathered} \tag{4.2}$$

For $n = 1, 2, 3$, the validity of (4.1$_n$) follows from (4.2) and the definition of d_n.

Let $n = 4$. From (1.2) and (1.3) we obtain $\beta_{10} = \beta_{11} = 1/384$ and

$$h_1^{-1}\left(\frac{1}{2}\right) = \frac{2^{11} \cdot 3}{5}. \tag{4.3}$$

Consequently, all the assumptions of Lemma 2.6 are fulfilled with $g(t) = h_1(t)$, $\gamma_j = \beta_{1j}$ $(j = 0, 1)$, $\gamma_2 = 0$, and thus, in view of (1.1) and (4.3) we have

$$d_4 = \frac{2^{11} \cdot 3}{5}. \tag{4.4}$$

Hence, on account of (4.2), we get (4.1$_4$).

Let $n = 6$. From (1.2), (1.3), and (4.4) we obtain $\beta_{20} = 5/(2^{13} \cdot 3^2)$, $\beta_{21} = 1/(2^9 \cdot 3 \cdot 5)$, $\beta_{22} = 1/(2^{10} \cdot 3 \cdot 5)$, and

$$h_2^{-1}\left(\frac{1}{2}\right) = \frac{2^{16} \cdot 3^2 \cdot 5}{61}. \tag{4.5}$$

Consequently, all the assumptions of Lemma 2.6 are fulfilled with $g(t) = h_2(t)$, $\gamma_j = \beta_{2j}$ $(j = 0, 1, 2)$, and thus, in view of (1.1), (4.3), and (4.5), we have (4.1$_6$).

Let $n = 5$. From (1.2) and (1.3) we obtain $\alpha_{10} = 1/2304$, $\alpha_{11} = 1/720$, and

$$f_1^{-1}\left(\frac{1}{2}, \frac{1}{2}\right) = 2^9 \cdot 3 \cdot 5. \tag{4.6}$$

Consequently, all the assumptions of Lemma 2.7 are fulfilled with $g_1(s, t) = f_1(s, t)$, $\gamma_0 = \alpha_{10}$, $\gamma_1 = 0$, $\gamma_2 = \alpha_{11}$, and thus, in view of (1.1) and (4.6) we have

$$d_5 = 2^9 \cdot 3 \cdot 5. \tag{4.7}$$

Hence, on account of (4.2), we get (4.1$_5$).

Let $n = 7$. From (1.2), (1.3), and (4.7) we obtain $\alpha_{20} = 1/(2^{11} \cdot 3^2 \cdot 5)$, $\alpha_{21} = 1/(2^{10} \cdot 3^5 \cdot 5)$, $\alpha_{22} = 1/(2^5 \cdot 3 \cdot 5 \cdot 7)$, and

$$f_2^{-1}\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{2^{14} \cdot 3^2 \cdot 5 \cdot 7}{17}.$$

Consequently, all the assumptions of Lemma 2.7 are fulfilled with $g_1(s,t) = f_2(s,t)$, $\gamma_j = \alpha_{2j}$ $(j = 0,1)$, $\gamma = \alpha_{22}$, and thus, analogously to the above we get (4.1_7).

On Remark 1.4. Assuming that (1.10) and (1.11) are valid, it remains to show that equality (1.10) (equality (1.11)) implies (4.1_n) for $n = 2p+2$ $(n = 2p+3)$, whence, in view of Lemma 2.5, the optimality of (1.5) follows.

The validity of (4.1_n) for $n = 1,\dots,7$ follows from Remark 1.3. Now assume that (4.1_j) holds for $j = 1,\dots,n-1$. We will show that (4.1_n) is valid under the hypothesis that (1.10) and (1.11) hold.

Let $n = 2p+2$. Then, on account of (2.20), the equalities (2.4_p) $(p \geq 3)$ and (3.8_1) with $v(t) = (-1)^{p+1} W_{2p+2,k}(t)$, $a = a_1 = 0$, $b = a_1 + \omega_1 = 1/2$ result in

$$\begin{aligned} \triangle\left(W_{2p+2,k}\right) &= \frac{2}{3}\sum_{j=0}^{p-1}\left(\frac{1}{4}\right)^{2(j+1)} A_j \left| W_{2(p-j),k}\left(\frac{1}{2}\right)\right| \\ &+ \frac{1}{45}\left(\frac{1}{4}\right)^{2(p+1)} \sum_{m_1=1}^{2}\sum_{m_2=1}^{m_1+1}\cdots\sum_{m_{p-1}=1}^{m_{p-2}+1} \frac{Q_{m_{p-1}+1}(1/2)}{\eta(m_1)\dots\eta(m_{p-1})} \\ &\times \int_0^{1/2} \left(x^{m_{p-1}+1}(s)\right)' W_{0,k}(s)ds. \end{aligned} \tag{4.8}$$

Now using (2.22), (2.24), (2.25), (2.27), and (4.1_n), from (4.8) we get

$$\triangle\left(W_{2p+2}\right) = h_p(1/2)\triangle\left(W_0\right). \tag{4.9_1}$$

Analogously, if $n = 2p+3$, we can show that the equalities (2.1_p) $(p \geq 3)$ and (3.8_1) with $v(t) = (-1)^{p+1} W_{2p+3,k}(t)$, $a = a_1 = 1/4$, $b = a_1 + \omega_1 = 3/4$, yield

$$\triangle\left(W_{2p+3}\right) = f_p(1/2,1/2)\triangle\left(W_0\right). \tag{4.9_2}$$

On the other hand, according to Lemma 2.4 we have

$$\triangle\left(W_{2p+2}\right) = \frac{1}{l_{2p+2}}\triangle\left(W_0\right), \quad \triangle\left(W_{2p+3}\right) = \frac{1}{l_{2p+3}}\triangle\left(W_0\right), \quad p \geq 3. \tag{4.10}$$

Consequently, if we prove that the maximal values of the polynomials $f_p(s,t)\cdot f_p(1-s,1-t)$ and $h_p(t)\cdot h_p(1-t)$ are achieved at the points $(s,t) = (1/2,1/2)$ and $t = 1/2$, respectively, from the definition of d_n we will get

$$d_{2p+2} = h_p^{-1}(1/2), \qquad d_{2p+3} = f_p^{-1}(1/2,1/2). \tag{4.11}$$

However, then, in view of (4.9_1)–(4.11), we will obtain $d_n = l_n$.

It remains to show that the polynomials $f_p(s,t)\cdot f_p(1-s,1-t)$ and $h_p(t)\cdot h_p(1-t)$ achieve their maximal values at the points $(s,t) = (1/2,1/2)$ and $t = 1/2$, respectively.

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