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ON A BOUNDARY VALUE PROBLEM FOR n-TH ORDER LINEAR FUNCTIONAL DIFFERENTIAL SYSTEMS

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Abstract. In this paper, theorems on the Fredholm alternative and well-posedness of the linear boundary value problem

$$u'(t) = \ell(u)(t) + q(t), \quad h(u) = c,$$

where $\ell : C([a, b]; R^n) \to L([a, b]; R^n)$ and $h : C([a, b]; R^n) \to R^n$ are linear bounded operators, $q \in L([a, b]; R^n)$, and $c \in R^n$, are established even when ℓ is not a *strongly bounded* operator.

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Statement of the Problem and Notation

The following notation is used throughout the paper:

N is the set of all natural numbers, $R =]-\infty, +\infty[$;

R^n is the space of n-dimensional column vectors $x = (x_i)_{i=1}^n$ with elements $x_i \in R$ $(i = 1, \dots, n)$ and the norm

$$\|x\| = \sum_{i=1}^n |x_i|;$$

$C([a, b]; R^n)$ is the Banach space of continuous vector functions $u : [a, b] \to R^n$ with the norm

$$\|u\|_C = \max\big\{\|u(t)\| : a \le t \le b\big\};$$

$\widetilde{C}([a, b]; R^n)$ is the set of absolutely continuous vector functions $u : [a, b] \to R^n$;

$L([a, b]; R^n)$ is the Banach space of Lebesgue integrable vector functions $p : [a, b] \to R^n$ with the norm $\|p\|_L = \int\limits_a^b \|p(s)\| ds$;

if $n = 1$, we write $C([a, b]; R)$ and $L([a, b]; R)$ instead of $C([a, b]; R^1)$ and $L([a, b]; R^1)$;

for any linear bounded operator $\ell : C([a, b]; R^n) \to L([a, b]; R^n)$ (linear bounded functional $h : C([a, b]; R^n) \to R^n$), denote by $|||\ell|||$ $(|||h|||)$ its norm, and let

$$M_\ell = \left\{ y \in \widetilde{C}([a, b]; R^n) : y(t) = z(a) + \int\limits_a^t \ell(z)(s) ds, \;\; z \in C([a, b]; R^n) \right\};$$

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$\mathcal{L}^n_{ab}$ is the set of linear bounded operators $\ell : C([a,b];R^n) \to L([a,b];R^n)$.

$\widetilde{\mathcal{L}}^n_{ab}$ is the set of linear strongly bounded operators, i.e., operators $\ell \in \mathcal{L}^n_{ab}$. For each of them there exists a nonnegative function $\eta \in L([a,b];R)$ such that

$$\|\ell(v)(t)\| \le \eta(t)\|v\|_C \quad \text{for} \quad a \le t \le b, \quad v \in C([a,b];R^n).$$

On the segment $[a,b]$ consider the boundary value problem

$$u'(t) = \ell(u)(t) + q(t), \tag{0.1}$$

$$h(u) = c, \tag{0.2}$$

where $\ell \in \mathcal{L}^n_{ab}$, $h : C([a,b];R^n) \to R^n$ is a linear bounded functional, $q \in L([a,b];R^n)$, and $c \in R^n$.

By a solution of problem (0.1), (0.2) we understand a function $u \in \widetilde{C}([a,b];R^n)$ satisfying equality (0.1) almost everywhere on $[a,b]$ and condition (0.2).

Together with (0.1), (0.2) we will consider the corresponding homogeneous problem

$$u'(t) = \ell(u)(t), \quad h(u) = 0 \tag{0.1_0}$$

and, for every $k \in N$, the perturbed boundary value problem

$$u'(t) = \ell_k(u)(t) + q_k(t), \quad h_k(u) = c_k, \tag{0.3_k}$$

where $\ell_k \in \mathcal{L}^n_{ab}$, $h_k : C([a,b];R^n) \to R^n$ is a linear bounded functional, $q_k \in L([a,b];R^n)$, and $c_k \in R^n$.

From the general theory of boundary value problems for functional differential equations it is known that if $\ell \in \widetilde{\mathcal{L}}^n_{ab}$, then problem (0.1), (0.2) has a Fredholm property (see, e.g., [5, 4]), i.e., problem (0.1), (0.2) is uniquely solvable if and only if the corresponding homogeneous problem (0.1_0) has only a trivial solution, and a well-posedness property (see, e.g., [5, 4]).

In 1972, H. H. Schaefer proved that there exists an operator $\ell \in \mathcal{L}^n_{ab}$ such that $\ell \notin \widetilde{\mathcal{L}}^n_{ab}$ (see [6, Theorem 4]). Therefore there naturally arises a necessity to study the Fredholm property and well-posedness of problem (0.1), (0.2) in the general case $\ell, \ell_k \in \mathcal{L}^n_{ab}$ $(k \in N)$.

The first important step in this direction was made by E. Bravyi in 2000 (see [1]), where the Fredholm property of (0.1), (0.2) was proved for $\ell \in \mathcal{L}^1_{ab}$ and $h : C([a,b];R) \to R$ (i.e. without an additional assumption that $\ell \in \widetilde{\mathcal{L}}^1_{ab}$). Bravyi's proof essentially uses Nikol'ski's theorem (see, e.g., [3, Ch. XIII, § 5, Theorem 2]) and concentrates on the question of Fredholm property.

In 2004, in the paper [2], for the scalar equation, i.e., for $\ell, \ell_k \in \mathcal{L}^1_{ab}$ $(k \in N)$, the Fredholm property (using the method slightly different from that in [1]) and the well-posedness of (0.1), (0.2) were proved.

In the present paper, the results obtained in [2] are generalized for systems of n-th order. More precisely, it is proved that problem (0.1), (0.2) has the Fredholm property and the property of well-posedness when $\ell, \ell_k \in \mathcal{L}^n_{ab}$ (see Theorems 1.1 and 1.2).

The proofs of the main results are based on the results obtained in [2] (see Proposition 2.1 therein) and [5] (see Theorems 1.1.1 and 1.4.1 therein).

1. Main Results

Theorem 1.1. *Let $\ell \in \mathcal{L}^n_{ab}$. Then problem* (0.1), (0.2) *is uniquely solvable if and only if the corresponding homogeneous problem* (0.1_0) *has only a trivial solution.*

Theorem 1.2. *Let problem* (0.1), (0.2) *have a unique solution u,*

$$\sup\left\{\left\|\int_a^t \big[\ell_k(y)(s)-\ell(y)(s)\big]ds\right\| : a \le t \le b, y \in M_{\ell_k}\right\} \to 0 \quad as \quad k \to +\infty, \tag{1.1}$$

and let for every $y \in \widetilde{C}([a,b];R^n)$

$$\lim_{k\to+\infty}(1+\||\ell_k\||)\int_a^t \big[\ell_k(y)(s)-\ell(y)(s)\big]ds = 0 \quad uniformly\ on\ [a,b]. \tag{1.2}$$

Let, moreover,

$$\lim_{k\to+\infty}(1+\||\ell_k\||)\int_a^t \big[q_k(s)-q(s)\big]ds = 0 \quad uniformly\ on\ [a,b], \tag{1.3}$$

$$\lim_{k\to+\infty} h_k(y) = h(y) \quad for \quad y \in C([a,b];R^n), \tag{1.4}$$

and

$$\lim_{k\to+\infty} c_k = c. \tag{1.5}$$

Then there exists $k_0 \in N$ such that for every $k > k_0$, problem (0.3_k) has a unique solution u_k and

$$\lim_{k\to+\infty} \|u_k - u\|_C = 0. \tag{1.6}$$

2. Auxiliary Propositions

In [2] (see Proposition 2.1 therein), the following assertion is proved.

Lemma 2.1. *Let $\ell \in \mathcal{L}^1_{ab}$. Then the operator $p : C([a,b];R) \to C([a,b];R)$ defined by*

$$p(x)(t) = \int_a^t \ell(x)(s)ds \quad for \quad a \le t \le b \tag{2.1}$$

is compact.

Lemma 2.2. *Let $\ell \in \mathcal{L}^n_{ab}$. Then the operator $p : C([a,b];R^n) \to C([a,b];R^n)$ defined by equality* (2.1) *is compact.*

Proof. Let $x = (x_j)_{j=1}^n$ be an element of $C([a,b];R^n)$, and $\widetilde{x}_j = (0,\dots,x_j,\dots,0)$. Then from the linearity of the operator ℓ follow the representations

$$\ell(x) = (\ell_i(x))_{i=1}^n, \quad \ell_i(x) = \sum_{j=1}^n \ell_{ij}(x_j) \quad (i = 1,\dots,n),$$

where $\ell_i : C([a,b];R^n) \to L([a,b];R)$, $\ell_{ij} : C([a,b];R) \to L([a,b];R)$, and $\ell_{ij}(x_j) = \ell_i(\widetilde{x}_j)$. Since $\ell \in \mathcal{L}^n_{ab}$, we have $\ell_{ij} \in \mathcal{L}^1_{ab}$ $(i,j = 1,\dots,n)$. Thus, according to Lemma 2.1, the operators

$$p_{ij}(x)(t) \overset{def}{=} \int_a^t \ell_{ij}(x_j)(s)ds \quad \text{for} \quad a \le t \le b \quad (i,j = 1,\dots,n)$$

are compact. Consequently, it is obvious that the operator p is also compact. □

From Lemma 2.2 we immediately obtain

Lemma 2.3. *Let $\ell \in \mathcal{L}^n_{ab}$, the operator $p : C([a,b];R^n) \to C([a,b];R^n)$ be defined by equality (2.1), and $\{\beta_k\}_{k=1}^{+\infty} \subset C([a,b];R^n)$ be a bounded sequence. Then the sequence $\{p(\beta_k)\}_{k=1}^{+\infty}$ contains a uniformly convergent subsequence.*

Lemma 2.4. *Let problem (0.1_0) has only the trivial solution and let the sequences of operators $\ell_k \in \mathcal{L}^n_{ab}$ and linear bounded functionals $h_k : C([a,b];R^n) \to R^n$ satisfy conditions (1.1) and (1.4). Then there exist $k_0 \in N$ and $r > 0$ such that an arbitrary $z \in \widetilde{C}([a,b];R^n)$ admits an estimate*

$$\|z\|_C \le r\rho_k(z) \quad \textit{for} \quad k > k_0,$$

where

$$\rho_k(z) = \|h_k(z)\| + \max\left\{(1 + |||\ell_k|||)\left\| \int_a^t \big[z'(s) - \ell_k(z)(s)\big]ds \right\| : a \le t \le b\right\}.$$

Proof. Note first that according to the Banach–Steinhaus theorem and condition (1.4), the sequence $\{|||h_k|||\}_{k=1}^{+\infty}$ is bounded, i.e., there exists $r_0 > 0$ such that

$$\|h_k(y)\| \le r_0\|y\|_C \quad \text{for} \quad y \in C([a,b];R^n), \quad k \in N. \tag{2.2}$$

Put for $y \in C([a,b];R^n)$

$$p(y)(t) = \int_a^t \ell(y)(s)ds, \quad p_k(y)(t) = \int_a^t \ell_k(y)(s)ds \quad \text{for} \quad a \le t \le b, \quad k \in N.$$

Obviously, $p : C([a,b];R^n) \to C([a,b];R^n)$ and $p_k : C([a,b];R^n) \to C([a,b];R^n)$ are linear bounded operators for $k \in N$ and

$$|||p_k||| \le |||\ell_k||| \quad \text{for} \quad k \in N. \tag{2.3}$$

By our notation, condition (1.1) can be rewritten as follows:

$$\sup\left\{\|p_k(y) - p(y)\|_C : y \in M_{\ell_k}\right\} \to 0 \quad \text{as} \;\; k \to +\infty. \tag{2.4}$$

Assume, on the contrary, that the lemma is not valid. Then there exist an increasing sequence of natural numbers $\{k_m\}_{m=1}^{+\infty}$ and a sequence of functions $z_m \in \widetilde{C}([a,b];R^n)$, $m \in N$, such that

$$\|z_m\|_C > m\rho_{k_m}(z_m) \quad \text{for} \quad m \in N. \tag{2.5}$$

Put

$$y_m(t) = \frac{z_m(t)}{\|z_m\|_C}, \quad v_m(t) = \int_a^t \big[y_m'(s) - \ell_{k_m}(y_m)(s)\big] ds \quad \text{for} \quad a \le t \le b, \tag{2.6}$$

$$y_{0m}(t) = y_m(t) - v_m(t) \quad \text{for} \quad a \le t \le b, \tag{2.7}$$

$$w_m(t) = p_{k_m}(y_{0m})(t) - p(y_{0m})(t) + p_{k_m}(v_m)(t) \quad \text{for} \quad a \le t \le b. \tag{2.8}$$

Obviously,

$$\|y_m\|_C = 1 \quad \text{for} \quad m \in N, \tag{2.9}$$

$$y_{0m}(t) = y_m(a) + p_{k_m}(y_m)(t) \quad \text{for} \quad a \le t \le b, \quad m \in N, \tag{2.10}$$

$$y_{0m}(t) = y_m(a) + p(y_{0m})(t) + w_m(t) \quad \text{for} \quad a \le t \le b, \quad m \in N. \tag{2.11}$$

On the other hand, from (2.3) and (2.6), by virtue of (2.5), we get

$$\|v_m\|_C \le \frac{\rho_{k_m}(z_m)}{\|z_m\|_C(1 + |||\ell_{k_m}|||)} < \frac{1}{m(1 + |||\ell_{k_m}|||)} \quad \text{for} \quad m \in N, \tag{2.12}$$

$$\|p_{k_m}(v_m)\|_C \le |||\ell_{k_m}||| \cdot \|v_m\|_C < \frac{1}{m} \quad \text{for} \quad m \in N. \tag{2.13}$$

From (2.9) and (2.10) it follows that $y_{0m} \in M_{\ell_{k_m}}$, and therefore, in view of (2.4), we have

$$\lim_{m \to +\infty} \|p_{k_m}(y_{0m}) - p(y_{0m})\|_C = 0. \tag{2.14}$$

On account of (2.13) and (2.14), equality (2.8) implies

$$\lim_{m \to +\infty} \|w_m\|_C = 0, \tag{2.15}$$

and, by virtue of (2.7), (2.9), and (2.12),

$$\|y_{0m}\|_C \le \|y_m\|_C + \|v_m\|_C \le 2 \quad \text{for} \quad m \in N. \tag{2.16}$$

According to Lemma 2.3, without loss of generality we can assume that

$$\lim_{m \to +\infty} y_{0m}(t) = y_0(t) \quad \text{uniformly on } [a, b].$$

By (2.7), (2.9), (2.11), (2.12), and (2.15),

$$\lim_{m \to +\infty} \|y_m - y_0\|_C = 0, \tag{2.17}$$

$$\|y_0\|_C = 1, \quad y_0(t) = y_0(a) + p(y_0)(t) \quad \text{for} \quad a \le t \le b.$$

Consequently, y_0 is a nontrivial solution of equation (0.1_0).

On the other hand, from (2.2) and (2.5) we get

$$\begin{aligned} \|h_{k_m}(y_0)\| &\le \|h_{k_m}(y_0 - y_m)\| + \|h_{k_m}(y_m)\| \\ &\le r_0\|y_0 - y_m\|_C + \frac{1}{\|z_m\|_C}\|h_{k_m}(z_m)\| \\ &\le r_0\|y_0 - y_m\|_C + \frac{1}{m} \quad \text{for} \quad m \in N. \end{aligned}$$

Hence, on account of (1.4) and (2.17), we obtain $h(y_0) = 0$. Thus y_0 is a nontrivial solution of problem (0.1_0), which contradicts the assumption of the lemma. □

3. Proofs of the Main Results

Proof of Theorem 1.1. Let $X = C([a,b];R^n)\times R^n$ be a Banach space containing elements $x = (u,\alpha)$, where $u \in C([a,b];R^n)$ and $\alpha \in R^n$, and having the norm $\|x\|_X = \|u\|_C + \|\alpha\|$. Define a linear operator $f : X \to X$ and $g \in X$ by

$$f(x) = \left(\alpha + u(a) + \int_a^t \ell(u)(s)ds,\ \alpha - h(u)\right), \quad g = \left(\int_a^t q(s)ds, c\right).$$

Obviously, problem (0.1), (0.2) is equivalent to the operator equation

$$x = f(x) + g \tag{3.1}$$

in the space X in the following sense: if $x = (u,\alpha) \in X$ is a solution of (3.1), then $\alpha = 0$, $u \in \widetilde{C}([a,b];R^n)$, and u is a solution of (0.1), (0.2), and vice versa: if $u \in \widetilde{C}([a,b];R^n)$ is a solution of (0.1), (0.2), then $x = (u,0)$ is a solution of (3.1).

According to Lemma 2.2, we have that the operator f is compact. From the Riesz–Schauder theory it follows that the equation (3.1) is uniquely solvable if and only if the corresponding homogeneous equation

$$x = f(x) \tag{3.1_0}$$

has only a trivial solution (see, e.g., [3, Ch. XIII, § 1, Theorem 2]). On the other hand, equation (3.1_0) is equivalent to problem (0.1_0) in the above-mentioned sense. □

Proof of Theorem 1.2. Let r and k_0 be numbers, the existence of which is guaranteed by Lemma 2.4. Then, obviously, for every $k > k_0$ the problem

$$u'(t) = \ell_k(u)(t), \quad h_k(u) = 0$$

has only a trivial solution. According to Theorem 1.1, for every $k > k_0$ problem (0.3_k) is uniquely solvable.

We will show that if u and u_k are solutions of problems (0.1), (0.2) and (0.3_k), respectively, then (1.6) holds. Put $v_k(t) = u_k(t) - u(t)$ for $a \le t \le b$. Then for every $k > k_0$

$$v_k'(t) = \ell_k(v_k)(t) + \widetilde{q}_k(t) \quad \text{for} \quad a \le t \le b, \quad h_k(v_k) = \widetilde{c}_k, \tag{3.2}$$

where

$$\widetilde{q}_k(t) = \ell_k(u)(t) - \ell(u)(t) + q_k(t) - q(t) \quad \text{for} \quad a \le t \le b, \quad \widetilde{c}_k = c_k - h_k(u).$$

Now by virtue of (1.2)–(1.5) we have

$$\delta_k = (1 + |||\ell_k|||) \max\left\{ \left\| \int_a^t \widetilde{q}_k(s)ds \right\| : a \le t \le b \right\} \to 0 \quad \text{as} \quad k \to +\infty, \tag{3.3}$$

$$\lim_{k\to+\infty} \widetilde{c}_k = 0. \tag{3.4}$$

According to Lemma 2.4, (3.2), and (3.3),

$$\|v_k\|_C \le r\big(\|\widetilde{c}_k\| + \delta_k\big) \quad \text{for} \quad k > k_0.$$

Hence, in view of (3.3) and (3.4) we obtain $\lim_{k\to+\infty} \|v_k\|_C = 0$, and, consequently, (1.6) holds. □

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