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On the Solvability of the Dirichlet Problem for Nonlinear Second-Order Functional-Differential Equations

S. V. Mukhigulashvili and J. Šremr

Mathematical Institute, Academy of Sciences, Tbilisi, Georgia
Mathematical Institute, Academy of Sciences of the Czech Republic, Brno, Czech Republic

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1. STATEMENT OF THE PROBLEM AND MAIN RESULTS

1.1. Statement of the Problem and Basic Notation

On the closed interval $[a, b]$, consider the functional-differential equation

$$u''(t) = f(u)(t) \tag{1.1}$$

with the boundary conditions

$$u(a) = 0, \qquad u(b) = 0. \tag{1.2}$$

Numerous papers deal with problem (1.1), (1.2) (see [1–8] and the bibliography therein). However, it has been studied comprehensively only for the case in which $f(u)(t) = f_0(t, u(t), u'(t))$ (e.g., see [9–13]).

In the present paper, we obtain in a sense optimal sufficient solvability conditions for problem (1.1), (1.2). We use the method of *a priori* estimates and impose one-sided constraints on the operator f. Special attention is paid to the case in which Eq. (1.1) is an equation with deviating argument or an ordinary differential equation.

We use the following notation: $R =]-\infty, +\infty[$, $R_+ = [0, +\infty[$, $C([a, b]; R)$ is the space of continuous functions $u : [a, b] \to R$ with the norm $\|u\|_C = \max\{|u(t)| : \ a \le t \le b\}$, $C'([a, b]; R)$ is the space of functions $u : [a, b] \to R$ continuous together with their first derivatives with the norm $\|u\|_{C'} = \|u\|_C + \|u'\|_C$, $\tilde{C}'([a, b]; R)$ is the set of functions $u : [a, b] \to R$ absolutely continuous together with their first derivatives, and $L([a, b]; R)$ is the space of functions $q : [a, b] \to R$ Lebesgue integrable on $[a, b]$ with the norm $\|q\|_L = \int_a^b |q(s)|ds$. We set $[x]_+ = (|x|+x)/2$ and $[x]_- = (|x|-x)/2$ for any $x \in R$.

Throughout the following, we assume that $f : C'([a, b]; R) \to L([a, b]; R)$ is a continuous operator satisfying the condition $\sup\{|f(x)(\cdot)| : \ \|x\|_{C'} \le r\} \in L([a, b]; R_+)$ for $r > 0$. A *solution* of problem (1.1), (1.2) is defined as a function $u \in \tilde{C}'([a, b]; R)$ such that condition (1.2) holds and Eq. (1.1) is satisfied almost everywhere on $[a, b]$.

Definition 1.1. An operator $p : C([a, b]; R) \to L([a, b]; R)$ is said *to belong to the set* P_{ab} if it is linear and the inequality $p(x)(t) \ge 0$ holds almost everywhere on $[a, b]$ for each function $x \in C([a, b]; R_+)$.

Definition 1.2. Let $A \subseteq [a, b]$ be a nonempty set. An operator $\ell : C([a, b]; R) \to L([a, b]; R)$ is said *to belong to the set* $K_{ab}(A)$ if $p(x)(t) = 0$ almost everywhere on $[a, b]$ for each function $x \in C([a, b]; R)$ such that $x(t) = 0$ for $t \in A$.

Remark 1.1. Let $A \subseteq [a, b]$ be a nonempty set, and let $\ell(x)(t) = p(t)x(\tau(t))$, where $p \in L([a, b]; R)$ and $\tau : [a, b] \to [a, b]$ is a measurable function. Moreover, suppose that either $\tau(t) \in A$ for $a \le t \le b$ or $p(t) = 0$ for $\tau(t) \in [a, b]\backslash A$. Then $\ell \in K_{ab}(A)$.

Definition 1.3. A function $\eta : R \times R_+ \to R_+$ is said to belong to the set M_{ab} if $\eta(\cdot, r)$ belonging to $L([a,b]; R_+)$ for $r \in R_+$, $\eta(t, \cdot)$ is nondecreasing for almost all $t \in [a,b]$, and

$$\lim_{r \to +\infty} \frac{1}{r} \int_a^b \eta(s, r) ds = 0. \tag{1.3}$$

1.2. Statement of the Main Results

We set

$$\varrho_A(t) = \inf\{|t-s| : \ s \in A\}, \qquad \sigma_A(t) = \varrho_A(t) + \varrho_A(t + (b-a)/2)$$

for each nonempty set $A \subseteq R$.

Theorem 1.1. *Suppose that there exist operators*

$$g_0 : C'([a,b]; R) \to L([a,b]; R), \qquad p_0 : C'([a,b]; R) \times C([a,b]; R) \to L([a,b]; R)$$

and functions $p, g \in L([a,b]; R_+)$ *and* $\eta \in M_{ab}$ *such that the conditions*

$$(f(x)(t) - p_0(x,x)(t) - g_0(x)(t)x'(t)) \operatorname{sgn} x(t) \geq -\eta\left(t, \|x\|_{C'}\right), \tag{1.4}$$

$$|g_0(x)(t)| \leq g(t), \qquad p_0(x,1)(t) \leq p(t) \tag{1.5}$$

are satisfied almost everywhere on $[a,b]$ *for each* $x \in C'([a,b]; R)$. *Furthermore, suppose that*

$$p_0(x, \cdot) \in P_{ab} \cap K_{ab}(A) \quad \text{for} \quad x \in C'([a,b]; R), \tag{1.6}$$

where $A \subseteq [a,b]$ *is a nonempty set, and*

$$\left(1 - 4\left(\frac{\delta}{b-a}\right)^2\right) \int_a^b p(s) ds < \frac{16}{b-a} \exp\left\{-\frac{1}{2}\int_a^b g(s) ds\right\}, \quad \text{where} \quad \delta = \min\left\{\sigma_A(t) : \ a \leq t \leq \frac{b+a}{2}\right\}. \tag{1.7}$$

Then problem (1.1), (1.2) *is solvable.*

Remark 1.2. One can readily compute the minimum of the function σ_A for some special sets $A \subseteq [a,b]$. For example, if $\alpha, \beta \in [a,b]$, $\alpha \leq \beta$, and $A = [\alpha, \beta]$ (respectively, $A = [a, \alpha] \cup [\beta, b]$), then $\delta = [(b-a)/2 - (\beta - \alpha)]_+$ (respectively, $\delta = [(b-a)/2 - (\beta - \alpha)]_-$).

Remark 1.3. There is an example in [8] showing that condition (1.7) is optimal in the sense that it cannot be replaced by the condition

$$\left(1 - 4\left(\frac{\delta}{b-a}\right)^2\right) \int_a^b p(s) ds < \frac{16 + \varepsilon}{b-a} \exp\left\{-\frac{1}{2}\int_a^b g(s) ds\right\},$$

however small the constant $\varepsilon > 0$ is.

Consider the case in which Eq. (1.1) has the form

$$u''(t) = \ell(u)(t) + f_1(u)(t), \tag{1.8}$$

where $\ell : C([a,b]; R) \to L([a,b]; R)$ is a linear nonnegative operator and $f_1 : C'([a,b]; R) \to L([a,b]; R)$ is a continuous operator such that $\sup\{|f_1(x)(\cdot)| : \ \|x\|_{C'} \leq r\} \in L([a,b]; R_+)$ for $r > 0$.