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On Nonlinear Boundary Value Problems for Two-Dimensional Differential Systems

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1. STATEMENT OF THE MAIN RESULTS

1.1. Statement of the Problems

We study the boundary value problem

$$\frac{du_i}{dt} = f_i\,(t, u_1, u_2) \qquad (i = 1, 2), \tag{1.1}$$

$$\varphi_i\,(u_1(a), u_2(a), u_1(b), u_2(b)) = 0 \qquad (i = 1, 2), \tag{1.2}$$

where the $f_i : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ $(i = 1, 2)$ are functions satisfying the local Carathéodory conditions and the $\varphi_i : \mathbb{R}^4 \to \mathbb{R}$ $(i = 1, 2)$ are continuous functions satisfying one of the following two inequalities in $\mathbb{R}^4$:

$$(\varphi_1\,(x_1, x_2, x_3, x_4) - x_1)\,x_2 - (\varphi_2\,(x_1, x_2, x_3, x_4) - x_3)\,x_4 \le \gamma, \tag{1.3}$$

$$(\varphi_1\,(x_1, x_2, x_3, x_4) - x_1)\,x_2 - (\varphi_2\,(x_1, x_2, x_3, x_4) - x_4)\,x_3 \le \gamma. \tag{1.4}$$

Here $\gamma = \text{const} \ge 0$.

We separately consider the case in which $f_i\,(t, x_1, x_2) \equiv f_i\,(t, x_{3-i})$ $(i = 1, 2)$ and either

$$\varphi_1\,(x_1, x_2, x_3, x_4) = x_1 - \mu x_4 + \psi_1\,(x_2)\,,$$
$$\varphi_2\,(x_1, x_2, x_3, x_4) \equiv x_3 - \mu x_2 - \psi_2\,(x_4)$$

or

$$\varphi_1\,(x_1, x_2, x_3, x_4) = x_1 - \mu x_3 + \psi_1\,(x_2)\,,$$
$$\varphi_2\,(x_1, x_2, x_3, x_4) \equiv x_4 - \mu x_2 - \psi_2\,(x_3)\,,$$

that is, the case in which system (1.1) has the form

$$\frac{du_1}{dt} = f_1\,(t, u_2)\,, \qquad \frac{du_2}{dt} = f_2\,(t, u_1)\,, \tag{1.5}$$

and the boundary conditions (1.2) have one of the following two forms:

$$u_1(a) = \mu u_2(b) - \psi_1\,(u_2(a))\,, \qquad u_1(b) = \mu u_2(a) + \psi_2\,(u_2(b))\,, \tag{1.2$_1$}$$

$$u_1(a) = \mu u_1(b) - \psi_1\,(u_2(a))\,, \qquad u_2(b) = \mu u_2(a) + \psi_2\,(u_1(b))\,, \tag{1.2$_2$}$$

where μ is an arbitrary real number and the $\psi_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ are continuous functions such that

$$x\psi_1(x) + y\psi_2(y) \le \gamma \quad \text{for} \quad (x, y) \in \mathbb{R}^2. \tag{1.6}$$

This class of boundary conditions includes, for example, well-known two-point, periodic, and antiperiodic boundary conditions

$$u_1(a) = 0, \qquad u_1(b) = 0, \tag{1.2$_3$}$$

$$u_1(a) = 0, \qquad u_2(b) = 0, \tag{1.2$_4$}$$

$$u_1(a) = u_1(b), \qquad u_2(a) = u_2(b), \tag{1.2$_5$}$$

$$u_1(a) = -u_1(b), \qquad u_2(a) = -u_2(b). \tag{1.2$_6$}$$