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ORDINARY DIFFERENTIAL EQUATIONS

On the Unique Solvability of the Dirichlet Problem for a Second-Order Linear Functional-Differential Equation

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1. STATEMENT OF THE PROBLEM AND BASIC NOTATION

Consider the linear functional-differential equation

$$u''(t) = p(u)(t) + q(t) \tag{1.1}$$

with the boundary conditions

$$u(a) = c_1, \qquad u(b) = c_2, \tag{1.2}$$

where $a, b, c_1, c_2 \in \mathbb{R}$, $p : \mathbb{C}([a,b]) \to L([a,b])$ is a linear operator, and $q \in L([a,b])$.

A *solution* of problem (1.1), (1.2) is a function $u \in \tilde{\mathbb{C}}'([a,b])$ satisfying condition (1.2) and satisfying Eq. (1.1) almost everywhere on $[a,b]$.

There is a wide literature on problem (1.1), (1.2) (see [1–9] and the bibliography therein). However, the problem has been studied comprehensively only for the case $p(u)(t) \equiv p_0(t)u(t)$ (e.g., see [3]).

In the present paper, we study the unique solvability of problem (1.1), (1.2) for the case in which p is a nonnegative operator (see Definition 2.1). We obtain sufficient conditions for the unique solvability of problem (1.1), (1.2), which generalize some earlier known results and are in a sense sharp. Special attention is paid to the case in which Eq. (1.1) is an equation with deviating arguments, i.e., has the form

$$u''(t) = \sum_{j=1}^{m} p_j(t) u\left(\tau_j(t)\right) + q(t), \tag{1.3}$$

where $q, p_j \in L([a,b])$ and the $\tau_j : [a,b] \to [a,b]$ are measurable functions.

Throughout the paper, we use the following notation: $\mathbb{R} = \,]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$; $\mathbb{C}([a,b])$ is the space of continuous functions $u : [a,b] \to \mathbb{R}$ equipped with the norm

$$\|u\|_C = \max\{|u(t)| : a \le t \le b\},$$

$\tilde{\mathbb{C}}'([a,b])$ is the set of functions $u : [a,b] \to \mathbb{R}$ absolutely continuous together with the first derivative, $L([a,b])$ is the set of functions $q : [a,b] \to \mathbb{R}$ Lebesgue integrable on $[a,b]$, and $[x]_+ = (|x| + x)/2$ for all $x \in \mathbb{R}$.

2. STATEMENT OF THE MAIN RESULTS

We introduce the following definitions.

Definition 2.1. A linear operator $p : \mathbb{C}([a,b]) \to L([a,b])$ is said to be *nonnegative* if $p(x)(t) \ge 0$ almost everywhere on $[a,b]$ for every nonnegative function $x \in \mathbb{C}([a,b])$.

Definition 2.2. We say that a linear operator $p : \mathbb{C}([a,b]) \to L([a,b])$ is *concentrated on a set* $A \subset [a,b]$ if $p(x)(t) \equiv 0$ almost everywhere on $[a,b]$ for every function $x \in \mathbb{C}([a,b])$ such that $x(t) = 0$ for $t \in A$.

Then the following assertions are valid.

Theorem 2.1$_1$. *Let p be a nonnegative operator. Suppose that there exist numbers $\alpha \in [a,b]$ and $\beta \in [\alpha, b]$ such that p is concentrated on the set $[a,\alpha] \cup [\beta, b]$ and*

$$\beta - \alpha \neq b - a, \tag{2.1$_1$}$$

$$\int_a^b p(1)(s)ds \le 16\frac{b-a}{(b-a)^2 - 4\delta^2}, \tag{2.2}$$

where $\delta = [\beta - \alpha - (b-a)/2]_+$. Then problem (1.1), (1.2) *has exactly one solution.*

Theorem 2.1$_2$. *Let p be a nonnegative operator. Suppose that there exist numbers $\alpha, \beta \in [a,b]$ such that p is concentrated on the set $[\alpha, \beta]$,*

$$\alpha < \beta, \tag{2.1$_2$}$$

and condition (2.2) *is satisfied with $\delta = [(b-a)/2 - (\beta - \alpha)]_+$. Then problem* (1.1), (1.2) *has exactly one solution.*

Remark 2.1. One can readily see that if condition (2.1$_i$) in Theorem 2.1$_i$ ($i = 1, 2$) is not satisfied, then problem (1.1), (1.2) has exactly one solution without the additional condition concerning the smallness of the expression $\int_a^b p(1)(s)ds$.

Remark 2.2. Theorems 2.1$_1$ and 2.1$_2$ generalize the results obtained in [9]. In particular, it was shown there that problem (1.1), (1.2) has exactly one solution provided that p is a nonnegative operator and the condition $\int_a^b p(1)(s)ds \le 16/(b-a)$ (which was shown to be sharp) is satisfied.

Example 2.1. This example shows that condition (2.2) in Theorem 2.1$_1$ is sharp in the sense that it cannot be replaced by the condition

$$\int_a^b p(1)(s)ds \le 16\frac{b-a}{(b-a)^2 - 4\delta^2} + \varepsilon \tag{2.2$_\varepsilon$}$$

however small the constant $\varepsilon \in]0,1[$ is. Let $a = 0$, $b = 1$, $\delta \in]0, 1/4[$, and $\varepsilon_0 \in]0, 1/8[$, and let α, β, μ_i, and ν_i ($i = 1, 2$) be the positive constants given by the relations

$$\begin{aligned} \mu_1 &= \frac{1 - 2\delta - 4\varepsilon_0}{4}, & \mu_2 &= \frac{1 - 2\delta + 4\varepsilon_0}{4}, \\ \nu_1 &= \frac{3 + 2\delta - 4\varepsilon_0}{4}, & \nu_2 &= \frac{3 + 2\delta + 4\varepsilon_0}{4}, \\ \alpha &= \mu_1 + \varepsilon_0, & \beta &= \nu_2 - \varepsilon_0. \end{aligned}$$

Furthermore, let $x \in \tilde{\mathbb{C}}'([\mu_1, \mu_2])$ and $y \in \tilde{\mathbb{C}}'([\nu_1, \nu_2])$ be arbitrary functions such that

$$\begin{gathered} x(\mu_1) = x(\mu_2) = 1, \qquad x'(\mu_1) = \frac{1}{\mu_1}, \qquad x'(\mu_2) = -\frac{1}{\mu_1 + \delta}, \\ x''(t) \le 0 \quad \text{for} \quad \mu_1 \le t \le \mu_2, \end{gathered} \tag{2.3$_1$}$$

$$\begin{gathered} y(\nu_1) = y(\nu_2) = -1, \qquad y'(\nu_1) = -\frac{1}{\mu_1 + \delta}, \qquad y'(\nu_2) = \frac{1}{\mu_1}, \\ y''(t) \ge 0 \quad \text{for} \quad \nu_1 \le t \le \nu_2. \end{gathered} \tag{2.3$_2$}$$