Mem. Differential Equations Math. Phys. 10(1997), 125–128

A. Lomtatidze and S. Mukhigulashvili

ON A TWO-POINT BOUNDARY VALUE PROBLEM FOR SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS, I

(Reported on April 29 and May 6, 1996)

In the present note, we consider the question of solvability of the boundary value problem

$$u''(t) = F(u)(t), \tag{1}$$

$$u(a) = 0, \quad u(b) = 0, \tag{2}$$

where the continuous operator $F : C'([a, b]) \to L([a, b])$ satisfies the Carathéodory conditions.

Before we proceed to formulate the basic results, let us introduce the following notation:

$R =]-\infty, +\infty[\,$, $R_+ = [0, +\infty[\,$;

$C([a, b])$ is the space of continuous functions $f : [a, b] \to R$ with the norm $\|f\|_C = \max\{|f(t)| : a \le t \le b\}$;

$C'([a, b])$ is the space of continuously differentiable functions $f : [a, b] \to R$ with the norm $\|f\|_{C'} = \|f\|_C + \|f'\|_C$; $C'_0([a, b]) = \Big\{f \in C'([a, b]) : \ f(a) = 0, \ \ f(b) = 0\Big\}$;

$\widetilde{C}'([a, b])$ is the set of absolutely continuous, with its first derivative, functions $f : [a, b] \to R$;

$L([a, b])$ is the space of summable on $[a, b]$ functions $f :]a, b[\to R$ with the norm $\|f\|_L = \int_a^b |f(s)|\, ds$.

$M(A, B)$ is the set of measurable functions $F : A \to B$;

$K_0([a, b])$ is the set of operators $p : C'([a, b]) \to M([a, b], R)$;

$\mathcal{L}([a, b])$ is the set of linear continuous operators $l : C([a, b]) \to L([a, b])$ such that for any $r > 0$ there exists $g_r \in L([a, b])$ satisfying

$$|l(u)(t)| \le g_r(t) \quad \text{for} \quad a < t < b, \quad \|u\|_C \le r;$$

$K([a, b])$ is the set of continuous operators $F : C'([a, b]) \to L([a, b])$ such that for any $r > 0$ there exists $g_r \in L([a, b])$ satisfying

$$|F(u)(t)| \le g_r(t) \quad \text{for} \quad a < t < b, \quad \|u\|_{C'} \le r;$$

$K_1([a, b] \times R, R_+)$ is the set of functions $q :]a, b[\times R \to R_+$ satisfying the Carathéodory condition;

$\sigma : L([a, b]) \to L([a, b])$ is an operator defined by

$$\sigma(p)(t) = \exp\Bigg[\int\limits_{\frac{a+b}{2}}^{t} p(s)\, ds\Bigg].$$

1991 *Mathematics Subject Classification.* 34K10.

Key words and phrases. Functional differential equation, boundary value problem.

$\sigma_\tau : L([a,b]) \to L([a,b])$ is an operator defined by

$$\sigma_\tau(p)(t) = \frac{1}{\sigma(p)(t)} \left| \int_\tau^t \sigma(p)(s)\,ds \right|,$$

$[p(t)]_+ = \frac{1}{2}\left(|p(t)| + p(t)\right)$, $[p(t)]_- = \frac{1}{2}\left(|p(t)| - p(t)\right)$.

An operator $l \in \mathcal{L}([a,b])$ is said to be positive (negative) if for any nonnegative function $u \in C([a,b])$ the function $l(u)$ is nonnegative (nonpositive).

In what follows, we assume $F \in K([a,b])$. Under solution of the equation (1) it is understood a function $u \in \widetilde{C}'([a,b])$ which almost everywhere satisfies it.

Definition. Let $l \in \mathcal{L}([a,b])$. We say that a vector function $(p, g_1, g_2) :]a,b[\to R^3$ belongs to the set $V(]a,b[\,;l)$ if p, g_1, $g_2 \in L([a,b])$ and for any function $g \in M([a,b],R)$ satisfying

$$g_1(t) \le g(t) \le g_2(t) \quad \text{for} \quad a < t < b,$$

there exists a positive function $w \in \widetilde{C}'([a,b])$ such that

$$w''(t) \le p(t)w(t) + g(t)w'(t) + l(w)(t) \quad \text{for} \quad a < t < b.$$

Remark. Let $l \in \mathcal{L}([a,b])$ be a negative operator and $p(t) + l(1)(t) \ge 0$ for $a < t < b$. Then for any g_1, $g_2 \in L([a,b])$ satisfying $g_1(t) \le g_2(t)$ for $a < t < b$, we have $(p, g_1, g_2) \in V(]a,b[\,;l)$.

Theorem 1. *Let on the set $C_0'([a,b])$ the inequalities*

$$\Big[F(v)(t) - p_1(t)v(t) - p_2(v)(t)v'(t) - l(v)(t)\Big] \operatorname{sgn} v(t) \ge -q\Big(t, \|v\|_{C'}\Big),$$
$$g_1(t) \le p_2(v)(t) \le g_2(t) \tag{3}$$

be fulfilled, where $l \in \mathcal{L}([a,b])$ is a negative operator, $p_2 \in K_0([a,b])$, $q \in K_1([a,b] \times R, R_+)$ is nondecreasing in the second argument and

$$\lim_{x \to +\infty} \frac{1}{x} \int_a^b q(s,x)\,ds = 0. \tag{4}$$

Let, moreover,

$$(p_1, g_1, g_2) \in V\Big(]a,b[\,,l\Big).$$

Then the problem (1), (2) *has at least one solution.*

Mention two corollaries of Theorem 1 for the equation

$$u''(t) = h(t)u(\tau(t)) + G(u)(t), \tag{5}$$

where $G \in K([a,b])$, $\tau \in M([a,b],[a,b])$, and $h \in L([a,b])$ is a nonpositive function.

Corollary 1. *Let on the set $C_0'([a,b])$ the inequality*

$$G(v)(t) \operatorname{sgn} v(t) \ge -q\Big(t, \|v\|_{C'}\Big) \tag{6}$$

be fulfilled, where $q \in K_1([a,b]\times R, R_+)$ *is nondecreasing in the second argument and satisfies* (4). *Moreover, let*

$$\big(b-\tau(t)\big)\int\limits_a^{\tau(t)}(s-a)|h(s)|\,ds+$$

$$+\big(\tau(t)-a\big)\int\limits_{\tau(t)}^{b}(b-s)|h(s)|\,ds<b-a \quad \textit{for} \quad a<t<b.$$

Then the problem (5), (2) *has at least one solution.*

Corollary 2. *Let on the set* $C_0'([a,b])$ *the inequality* (6) *be fulfilled, where* $q \in K_1([a,b]\times R, R_+)$ *is nondecreasing in the second argument and satisfies* (4). *Let, moreover, there exist* $c \in [a,b]$ *such that*

$$\int\limits_a^c \sigma_a(p)(s)|h(s)|\,ds<1, \qquad \int\limits_c^b \sigma_b(p)(s)|h(s)|\,ds<1,$$

$$\big(t-\tau(t)\big)\sigma(p)(t)\int\limits_t^c \frac{|h(s)|}{\sigma(p)(s)}\,d\le 1 \quad \textit{for} \quad a<t<b,$$

where $p(t)=h(t)(\tau(t)-t)$ *for* $a<t<b$. *Then the problem* (5), (2) *has at least one solution.*

Finally, we give a corollary of Theorem 1 for the equation

$$u''(t)=p_1(t)u(t)+p_2(u)(t)u'(t)+h(t)u(\tau(t))+G(u)(t), \tag{7}$$

where p_2, $G\in K([a,b])$, $\tau\in M([a,b],[a,b])$, p_1, $h\in L([a,b])$ and h is positive.

Corollary 3. *Let on the set* $C_0'([a,b])$ *the inequalities* (3) *and* (6) *be fulfilled, where* g_1, $g_2 \in L([a,b])$, $q\in K_1([a,b]\times R,R_+)$ *is nondecreasing in the second argument and satisfies* (4). *Let, moreover, there exist* $\lambda_i\in[0,1[$, $\alpha_{ij}\in[0,+\infty[$, $i,j=1,2$, *and* $c\in[a,b]$ *such that*

$$\int\limits_0^{+\infty}\frac{ds}{\alpha_{11}+\alpha_{12}s+s^2}>\frac{(c-a)^{1-\lambda_1}}{1-\lambda_1}\,, \qquad \int\limits_0^{+\infty}\frac{ds}{\alpha_{21}+\alpha_{22}s+s^2}>\frac{(b-c)^{1-\lambda_2}}{1-\lambda_2}$$

and

$$(t-a)^{2\lambda_1}\Big[p_1(t)+h(t)\Big]\ge-\alpha_{11}, \quad (t-a)^{\lambda_1}\bigg[g_1(t)+\frac{\lambda_1}{t-a}+\big(\tau(t)-t\big)h(t)\bigg]\ge-\alpha_{12} \quad \textit{for} \ \ a<t<c,$$

$$(b-t)^{2\lambda_2}\Big[p_1(t)+h(t)\Big]\ge-\alpha_{21}, \quad (b-t)^{\lambda_2}\bigg[g_2(t)-\frac{\lambda_2}{b-t}+\big(\tau(t)-t\big)h(t)\bigg]\le\alpha_{22} \quad \textit{for} \ \ c<t<b.$$

Then the problem (7), (2) *has at least one solution.*

References

1. I. T. Kiguradze and B. L. Shekhter, Singular boundary value problems for second-order differential equations. *In: "Current Problems in Mathematics: Newest Results", vol.* 30, *pp.* 105–201, *Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuzn. Inst. Nauchn. i Tekhn. Inform., Moscow*, 1987.

Authors' addresses:

A. Lomtatidze
N. Muskhelishvili Institute of Computational Mathematics
Georgian Academy of Sciences
8, Akuri St., Tbilisi 380093
Georgia

S. Mukhigulashvili
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 390093
Georgia