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ON SOME CONSTRUCTIVE METHODS FOR THE MATRIX RIEMANN–HILBERT BOUNDARY-VALUE PROBLEM

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ABSTRACT. In this paper we consider the relations between the Riemann–Hilbert monodromy problem and the matrix Riemann–Hilbert boundary-value problem with piecewise continuous coefficient and construct the so-called canonical matrix for the boundary-value problem for a piecewise continuous matrix-function. The formula for the calculation of the index is also obtained.

CONTENTS

1. Partial Indices of Matrix Function and Riemann–Hilbert Monodromy Problem

A simple contour on the Riemann sphere together with a piecewise-constant matrix on it defines the monodromy

$$\pi_1(\texttt{punctured sphere}) \to GL_n(\mathbb{C}). \tag{1.1}$$

This monodromy in turn generates a flat rank-n complex vector bundle over the punctured sphere. This bundle may be naturally continued to the entire sphere. As we know from the Birkhoff–Grothendieck theorem, any holomorphic vector bundle over the sphere is equivalent to the sum of the powers of Hopf line bundles. Those powers are also known as the *partial indices of the factorization problem* (see [4, 7, 8, 13, 15, 16, 25].

One general condition for the solvability of the Riemann–Hilbert problem is the existence of a trivial fibration among all holomorphic fibrations with logarithmic connection on the Riemann sphere having prescribed monodromy ρ and collection of singular points $s_1, \dots, a_n$. The existence of a stable pair (F, Λ) among the aforementioned family of fibrations, where $F \to \mathbb{CP}^1$ is a holomorphic fibration and r is a logarithmic connection with prescribed singular points, is a sufficient condition for solvability of the Riemann–Hilbert problem for Fuchsian systems [4] (see also [11, 12]). In [33] the possibility of the existence of a finite algorithm for checking the existence of a stable pair with a given monodromy representation is investigated. In particular, it is proved that for the representation

$$\rho : \pi_1(\mathbb{CP}^1 \setminus \{s_1, \dots, s_n\}, z_0) \to GL(n, \mathbb{C})$$

with generators $G_1, \dots, G_n$, where to each eigenvalue of the matrix G_i there corresponds exactly one Jordan block for $i = 1, \dots, n$, there exists an algorithm that clarifies in a finite number of steps whether construction of a stable pair with the given monodromy is possible. From this result, in

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particular, follows the existence of a finite algorithm for checking the solvability of the Riemann–Hilbert problem. In [31], the algorithm for determining the splitting types of vector bundles associated to the representation ρ is presented.

To determine partial indices or at least to find some information about them is a difficult problem. For rank $n = 2$ and number of punctures $m = 4$, and in the case $n = m = 3$, the detailed classification of the problem is given in [29] (see also [30]), and the partial indices are found (but the answer is not constructive).

In [30] the authors consider piecewise constant 2×2 matrix functions with at most three jumps on the one-point compactification of $\mathbb{R}$. For such matrices, they establish criteria for factorizability and Φ-factorizability (in the Wiener–Hopf sense), formulas for the partial indices, and criteria for the stability of the partial indices.

In [29] is constructed the factorization in $L_{p,\rho}(\mathbb{R}^1)$ of a piecewise-constant matrix function $G \in L_\infty$ in the form $G = G_+ \Lambda G_-$, where

$$\Lambda = \operatorname{diag}(\zeta^{\kappa_1}, \dots, \zeta^{\kappa_n}), \quad \kappa_j \in \mathbb{Z}, \quad \zeta = \frac{t-i}{t+i},$$

$$(t \pm i)^{-1} G_\pm \in H^\pm_{p,\rho} = P_\pm L_{p,\rho}, \quad (t \pm i)^{-1} G_\pm^{-1} \in H_{p,\rho^{-1}}, \quad q = \frac{p}{p-1},$$

$$\rho(t) = (1+t^2)^{\frac{\nu}{2}} \prod_{k=0}^{N-1} |t - t_k|^{\nu_k},$$

$$-\frac{1}{p} < \nu_k < 1 - \frac{1}{p}, \quad k = 0, 1, \dots, N-1, \quad \nu_N = 1 - \frac{2}{p} - \nu - \nu_0 - \dots - \nu_{N-1}.$$

A factorization in $L_{p,\rho}$ with the additional property that the operator $G_- \Lambda^{-1} P_- G_+^{-1}$ is invertible in $L_{p,\rho}$ is called a Φ-factorization. The main result is as follows.

Theorem 1.1 (see [9]). *For G to be Φ-factorizable in $L_{p,\rho}$ it is necessary and sufficient that*

(i) $\det G(t) \neq 0$ *for* $t \overline{\in} \{t_j\}_{j=0}^N$, $\det G(t_j \pm 0) \neq 0$ *for* $j = 0, 1, \dots, N$, *and*

(ii) $\dfrac{1}{p} + \nu_j - \gamma_{jr} \overline{\in} \mathbb{Z}$, $k = 1, \dots, n$, $j = 0, \dots, N$, *where* $2\pi\gamma_{jk}$ *denotes the argument of the kth eigenvalue of the matrix* $G(t_j - 0)^{-1} G(t_j + 0)$.

If (i) *and* (ii) *are satisfied, then*

$$\kappa = \kappa_1 + \kappa_2 + \dots + \kappa_n$$
$$= \frac{1}{2\pi} \sum \big(\arg \det G \big)\big|_{\Gamma_j} + \sum_{k=0}^{n} \sum_{j=0}^{N} \left(\Gamma_{jk} + \left[\frac{1}{p} + \nu_j - \gamma_{jk} \right] \right).$$

Here the first sum is taken over the following intervals: $(-\infty, t_0)$, (t_0, t_1), $\dots$, $(t_{N-1} + \infty)$.

The paper [27] deals with the Riemann boundary-value problem

$$\varphi^+(t) = G(t)\varphi^-(t) + g(t), \quad t \in C, \tag{1.2}$$

where C is a finite set of simple, closed, non-intersecting Lyapunov curves bounding the interior domain D^+, the domain complementary to $D^+ + C$ with respect to the extended plane being denoted by D^-. The problem requires determination of n-vectors $\varphi^+(t)$ and $\varphi^-(t)$ analytic in D^+ and D^-, respectively, whose limiting values on C satisfy (1.2), where $g(t)$ is a specified n-vector and $G(t)$ is a given n-square matrix. The cases $n = 1$ and $n > 1$ are considered separately.

The case $n = 1$ was first solved by Gakhov assuming $G(t)$ and $g(t)$ Hölder-continuous on C. This was extended by Khvedelidze who assumed $g(t) \in L_p(C)$, $p > 1$ (see [10, 25]). Further weakening of the requirements on G and g was made by Mikhlin, Gohberg, Ivanov, Danilyuk, Widom, et al. (see [27, 28]). Using methods of functional analysis, they show that the basic Noether theorems associated with

(1.2) continue to hold for $g \in L_p(C)$ and with $G(t)$ bounded, measurable, and such that for each t_0 on C there is a neighborhood of t_0 and a $\delta > 0$ for which $G(t)$ is contained in a sector with vertex at the origin and of opening $\dfrac{2\pi - \delta}{max(p, \frac{p}{p-1})}$. For the case $n > 1$, one can assume that the components of the n-vector g belong to $L_2(C)$ and the n-square matrix $G(t)$ has the following properties: (i) its elements are bounded and measurable, (ii) it may be written as the product $G_1(t)G_2(t)G_3(t)$ of matrices wherein $G_1(t), G_3(t)$ are continuous and nonsingular and $G_2(t)$ is such that

$$\operatorname{Re} G_2(t) = \frac{1}{2}(G(t) + G^*(t)) > \nu > 0 \tag{1.3}$$

(ν is independent of t and $\geq$ refers to a comparison of Hermitian matrices). The number $\frac{1}{2\pi} \arg \Delta(G_1 G_3)_C$ is invariant with respect to the representation (1.3) and defines the index of G on C.

Let $\mathrm{L}^p(\Gamma)$ be the space of Lebesgue measurable functions satisfying the condition on that norm

$$\|f\|_{\mathrm{L}^p(\Gamma)} = \left(\int\limits_{\Gamma} |f(\tau)|^p |d\tau| \right)^{\frac{1}{p}} < \infty,$$

so that $\mathrm{L}^p(\Gamma)$ is a Banach space.

Consider the singular integral operator

$$(S_\Gamma f)(t) = \frac{1}{\pi i} \int\limits_{\Gamma} \frac{f(\tau)}{\tau - t} dt, \quad t \in \Gamma.$$

This operator is bounded on $\mathrm{L}^p(\Gamma)$ and $S_\Gamma^2 = \mathbf{1}$. Let us introduce the following projectors:

$$P_\Gamma = \frac{\mathbf{1} + S_\Gamma}{2}, \quad Q_\Gamma = \frac{\mathbf{1} - S_\Gamma}{2}.$$

All $f \in \mathrm{L}^p_+(\Gamma)$ can be identified with functions $\hat{f}$ holomorphic in U_+. Thus, $\hat{f}$ is an analytic continuation of f to U_+. Here $\mathrm{L}^p_+(\Gamma)$ denotes the space of those holomorphic functions which are boundary values of functions from $\mathrm{L}^p(\Gamma)$; similarly let $\mathrm{L}^p_-(\Gamma)$ denote the space of those holomorphic functions on U_- whose extension to Γ gives an element of $\mathrm{L}^p(\Gamma)$.

Let $\mathrm{L}^\infty(\Gamma)$ be the Banach space of Lebesgue measurable and essentially bounded functions.

Definition 1.1. The factorization of a matrix-function $G \in \mathrm{L}^\infty(U)^{n\times n}$ in the space $\mathrm{L}^p(\Gamma)$ is its representation in the form

$$G(t) = G_+(t)\Lambda(t)G_-(t), \quad t \in \Gamma, \tag{1.4}$$

where

$$\Lambda(t) = \operatorname{diag}(t^{k_1}, \dots, t^{k_n}), \quad k_i \in \mathbb{Z}, \quad i = 1, \dots, n,$$
$$G_+ \in L_+(\Gamma)^{n\times n}, \quad G_+^{-1} \in \mathrm{L}^q_+(\Gamma)^{n\times n}, \quad G_- \in \mathrm{L}^q_-(\Gamma)^{n\times n}, \quad G_-^{-1} \in \mathrm{L}^p_-(\Gamma)^{n\times n},$$
$$\frac{1}{p} + \frac{1}{q} = 1.$$

We say that G admits the *canonical* factorization in $\mathrm{L}^p(\Gamma)$ if $k_1 = \cdots = k_n = 0$.

This definition implies that the operator $G_-^{-1} Q_\Gamma G_+^{-1}$ is defined on the everywhere dense subspace of the space $\mathrm{L}^p(\Gamma)^n$ consisting of those rational vector-functions that are allowed to have poles on Γ, and maps this subspace onto $\mathrm{L}^1(\Gamma)^n$. If this operator is bounded in the L^p norm, then it can be extended to the whole $\mathrm{L}^p(\Gamma)^n$, and the obtained operator is still bounded, in which case the representation (1.4) from definition (1.1) will be called the Φ-*factorization* of $G(t)$.

It is known that a matrix-function $G \in \mathrm{L}^\infty(\Gamma)^{n\times n}$ is Φ-factorizable in the space $\mathrm{L}^p(\Gamma)$ if and only if the operator $P_\Gamma + GQ_\Gamma$ is a Fredholm operator on the space $\mathrm{L}^p(\Gamma)^n$ (see [9]).

Let us consider the particular case, namely, on the subspace $\mathrm{PC}(\Gamma)^{n\times n}$ of piecewise continuous matrix-functions. For elements of this subspace, there exist the one-sided limits $G(t+0)$ and $G(t-0)$ for each $t \in \Gamma$. For such matrix-functions, the necessary and sufficient condition for the existence of Φ-factorization is given by the following theorem.

Theorem 1.2 (see [9]). *A matrix-function $G \in \mathrm{PC}(\Gamma)^{n\times n}$ is Φ-factorizable in the space $\mathrm{L}_p(\Gamma)$ if and only if*

(a) *the matrices $G(t+0)$ and $G(t-0)$ are invertible for each $t \in \Gamma$;*
(b) *for each $j = 1, \dots, n$ and $t \in \Gamma$ one has*

$$\frac{1}{2\pi}\arg\lambda_j(t) + \frac{1}{p} \notin \mathbb{Z}.$$

Here $\lambda_1(t), \dots, \lambda_n(t)$ are eigenvalues of the matrix-function $G(t-0)G(t+0)^{-1}$.

If a matrix-function G is Φ-factorizable, then $\xi_j(\tau) = \frac{1}{2\pi}\arg\lambda_j(\tau)$ is a single-valued function taking values in the interval $\left(\frac{1}{p} - 1, \frac{1}{p}\right)$.

Suppose that G has m singular points $s_1, \dots, s_m \in \Gamma$; then

$$\kappa = \sum_{k=1}^{m}\left[\frac{1}{2\pi}\arg\det G(t)\right]_{t=a_k+0}^{a_{k+1}-0} + \sum_{k=1}^{m}\sum_{j=1}^{n}\xi_j(s_k). \tag{1.5}$$

It can be seen from (1.5) that κ depends on $\mathrm{L}_p(\Gamma)$. If the $\lambda_j(\tau)$ are positive real numbers, then $\xi_j(\tau) = 0$, and consequently κ does not depend on the space $\mathrm{L}_p(\Gamma)$.

Suppose that $G \in \mathrm{PC}(\Gamma)^{n\times n}$ is, moreover, a piecewise constant matrix function with singular points $s_1, \dots, s_m \in \Gamma$, occurring in this order on Γ. Suppose that G is factorizable in the space $\mathrm{L}_p(\Gamma)$. Let us denote $M_k = G(s_k - 0)G(s_k + 0)^{-1}$, $k = 1, \dots, m$. Thus G is constant on the arc (s_k, s_{k+1}), and clearly $M_1M_2\cdots M_k = 1$. Suppose that the matrices are similar to the matrices $\exp(-2\pi i E_k)$ and eigenvalues of E_k belonging to the interval $\left(\frac{1}{p} - 1, \frac{1}{p}\right)$, where the matrices E_k are determined uniquely up to similarity since the length of that interval is 1. The numbers $\xi_1(s_k), \dots, \xi_n(s_k)$ are equal to real parts of eigenvalues of E_k. This implies that for the index κ one has the formula $\kappa = \sum\limits_{k=1}^{m} \mathrm{tr}E_k$. Thus the matrices $E_1, \dots, E_k$ depend on the space $\mathrm{L}_p(\Gamma)$. They also depend on the choice of a logarithm of eigenvalues of the matrices M_j. Thus $G \in \mathrm{PC}(\Gamma)^{n\times n}$ produces two m-tuples $(M_1, \dots, M_m)$ and $(E_1, \dots, E_m)$ of matrices.

Let

$$\frac{df}{dz} = \Omega(z)f(z) \tag{1.6}$$

be a system of differential equations with regular singularities, having $s_1, \dots, s_m$ as singular points, and ∞ as an apparent singular point. It is known that such a system has n linearly independent solutions in a neighborhood of a regular point.

Let us denote such a fundamental system of solutions by $F(\tilde{z})$. It is possible to characterize $F(\tilde{z})$ by its behavior near the singular points $s_1, \dots, s_m$, using the monodromy matrices $M_1, \dots, M_m$, which are determined by the matrices $E_1, \dots, E_m$, and by the behavior at ∞ which is characterized by the partial indices $k_1, \dots, k_m$. Therefore, it is said that the system (1.6) has the standard form with respect to the matrices $(M_1, \dots, M_m)$ and $(E_1, \dots, E_m)$ satisfying the condition $M_1 \cdots M_m = 1$ such that M_k are similar to $\exp(-2\pi i E_k)$, $k = 1, \dots, m$ and E_j are not resonant, with singular points $s_1, \dots, s_m$ and partial indices $k_1 \geq \cdots \geq k_n$, if

(i) $s_1, \dots, s_m$ are the only singular points of (1.6), with ∞ as an apparent singular point;
(ii) the monodromy group of (1.6) is conjugate to the subgroup of $\mathrm{GL}_n(\mathbb{C})$ generated by the matrices $M_1, \dots, M_m$;

(iii) in a neighborhood U_j of the point s_j the solution has the form

$$F(\tilde{z}) = Z_j(z)(\tilde{z} - s_j)^{E_j} C,$$

where $Z_j(z)$ is an analytic and invertible matrix-function on $U_j \cup \{s_j\}$ and C is a nondegenerate matrix;

(iv) the solution of the system in a neighborhood U_∞ of ∞ has the form

$$F(z) = \operatorname{diag}(z^{k_1}, \dots, z^{k_n}) Z_\infty(z) C, \quad z \in U_\infty,$$

with $Z_\infty(z)$ holomorphic and invertible on U_∞.

Theorem 1.3 (see [9])**.** *Suppose that $G \in \mathrm{PC}(\Gamma)^{n\times n}$ is a piecewise constant function with jump points $s_1, \dots, s_m$. Suppose that G has a Φ-factorization in the space $\mathrm{L}_p(\Gamma)$, $1 < p < \infty$, and $(M_1, \dots, M_m)$ and $(E_1, \dots, E_m)$ are matrices associated to G on $\mathrm{L}_p(\Gamma)$. Suppose that there exists a system of differential equations in standard form* (1.6) *with singular points $s_1, \dots, s_m$ and partial indices $\kappa_1, \dots, \kappa_m$. Let $F_1(z)$, $F_2(z)$ be a fundamental system of its solutions in U_+ and $U_- \setminus \{\infty\}$. Then there exist nondegenerate $(n \times n)$-matrices C_1 and C_2 such that*

$$G(t) = G_+(t) \Lambda(t) G_-(t)$$

is a Φ-factorization of G in $\mathrm{L}_p(\Gamma)$, where $\Lambda(t) = \operatorname{diag}(t^{k_1}, \dots, t^{k_n})$,

$$G_+(z) = C_1^{-1} F_1^{-1}(z), \ z \in U_+, \quad G_-(z) = \Lambda^{-1}(z) F_2(z) C_2, \ z \in U_- \setminus \{\infty\}.$$

This result can be used to obtain the solvability condition of the Riemann–Hilbert problem.

Let Γ the closed simple contour, $s_1, \dots, s_m \in \Gamma$ and $M_1, \dots, M_m \in GL_n(\mathbb{C})$. We call that the piecewise constant matrix function $G(t)$ induced from collections $s = \{s_1, \dots, s_m\}$, $M = \{M_1, \dots, M_m\}$ if it is constructed in following manner:

$$G(t) = M_j \cdot \dots \cdot M_1, \quad \text{if } t \in [s_j, s_{j+1}),$$

where M_j are monodromy matrices corresponding to going around small loop singular points s_j.

Theorem 1.4. *Let*

$$\rho : \pi_1(\mathbb{CP}^1 \setminus \{s_1, \dots, s_m\}) \to GL_n(\mathbb{C}) \tag{1.7}$$

be the representation such that $(\rho(\gamma_1) = M_1, \ \dots, \ \rho(\gamma_m) = M_m)$ and $(E_1, \dots, E_m)$ is admissible.

Then for the representation (1.7) *the Riemann–Hilbert monodromy problem is solvable if $G(t)$ admits a canonical factorization in $L^\alpha(\Gamma)$, for some $\alpha > 1$ sufficiently close to 1.*

Proof. It is known that for the given monodromy matrices $M_1, \dots, M_m$, and singular points $s_1, \dots, s_m$ there exists the system of differential equations of the type

$$df = \omega f, \tag{1.8}$$

such that $s_1, \dots, s_m$ are the poles of first order for (1.8), ∞ is an apparent singular point, the matrices $M_1, \dots, M_m$ are monodromy matrices of (1.8), and the solution of the (1.8) in the neighborhood of the singular point s_j has the form

$$\Phi_j(\tilde{z}) = U_j(z)(\tilde{z} - s_j)^{E_j} C,$$

where the matrix function $U_j(z)$ is invertible and analytic in the neighborhood of s_j and C is a nondegenerate matrix. In the neighborhood of ∞ the solution has the form

$$\Phi_\infty(\tilde{z}) = \operatorname{diag}(k_1, \dots, k_n) U_\infty(z) C, \tag{1.9}$$

where $U_\infty(z)$ is analytic and invertible at ∞ [9]. By Theorem 1.1, the piecewise constant matrix function $G(t)$ admits a Φ-factorization; therefore, $\xi_j(\tau) = \frac{1}{2\pi} \arg \lambda_j(\tau)$ is a single-valued function taking values in the interval $\left(\frac{1}{p}-1, \frac{1}{p}\right)$. From the factorization condition $G(t) = G_+(t)\Lambda(t)G_-(t)$ and by Theorem 1.2, we have

$$G_+(z) = C_1^{-1}F_1^{-1}(z),\ z \in U_+, \quad G_-(z) = \Lambda^{-1}(z)F_2(z)C_2,\ z \in U_- \setminus \{\infty\}.$$

By the assumption, $G(t)$ admits a canonical factorization, i.e., $k_1 = \cdots = k_n = 0$. From this it follows that ∞ is a regular point of system (1.8). $\square$

Below is given the solution of the Riemann–Hilbert problem for the regular systems. In this case the problem is reduced to the linear conjugation problem for the Hölder class functions [26, 32].

Suppose that Γ is the same as above and $G : \Gamma \to GL(n, \mathbb{C})$ has a discontinuity of the first kind at the point $s_1 \in \Gamma$. Introduce the notation

$$G(s_1+0) = \lim_{t\to s_1+0} G(t), \quad G(s_1-0) = \lim_{t\to s_1-0} G(t)$$

and put

$$M = G^{-1}(s_1+0)G(s_1-0), \quad E = \frac{1}{2\pi i}\ln M,$$

so that if λ_i are eigenvalues of M, then $\mu_i = \frac{1}{2\pi i}\ln\lambda_i$ satisfies the condition $0 \le \operatorname{Re}\mu_i < 1$.

Consider the functions

$$\omega^+(z) = (z-s_1)^E \overset{def}{=} e^{E\ln(z-s_1)},\ \omega^-(z) = \left(\frac{z-s_1}{z-z_0}\right)^E \overset{def}{=} e^{E\ln\left(\frac{z-s_1}{z-z_0}\right)},$$

where z_0 is some fixed point in U^+. It is known that $\omega^+(z)$ is a single-valued matrix-function on $\mathbb{C}\setminus l_1$, where l_1 is a curve with endpoints s_1 and ∞, and $\omega^-(z)$ is a single-valued matrix-function on $\mathbb{C}\setminus l_2$, where l_1 is a curve with endpoints z_0 and s_1.

Introduce new vector-functions $f_1^+(z)$ and $f_1^-(z)$:

$$f_1^+(z) = (z-s_1)^E G^{-1}(s_1+0)f^+(z), \quad f_1^-(z) = \left(\frac{z-s_1}{z-z_0}\right)^{-E} f^-(z).$$

They are holomorphic respectively on $U^\pm$ and satisfy the transmission condition

$$f_1^+(z) = (z-s_1)^{-E}G^{-1}(s_1+0)G(t)\left(\frac{z-s_1}{z-z_0}\right)^E f_1^-(z).$$

Denoting

$$G_1(z) = (z-s_1)^{-E}G^{-1}(s_1+0)G(t)\left(\frac{z-s_1}{z-z_0}\right)^{-E},$$

one can prove that $G_1(t)$ is continuous at the point s_1 [32]. Granted this, we can deal with the general case.

Let $s_1, \ldots, s_m \in \Gamma$ be points of discontinuity, and let there exist finite limits

$$G(s_j+0) = \lim_{t\to s_1+0} G(t), \quad G(s_j-0) = \lim_{t\to s_1-0} G(t).$$

The curve Γ is assumed to be a union of smooth nonintersecting arcs $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ with fixed orientations. The ends of arcs Γ_j, $j = 1, 2, \ldots, m$, are s_j and s_{j+1}.

Suppose that $M_j = G^{-1}(s_j+0)G(s_j-0)$ and $E_j = \frac{1}{2\pi i}\ln M_j$ so that if λ_j^i are eigenvalues of G^j; then $\mu_i^j = \frac{1}{2\pi i}\ln\lambda_j^i$. Denote $\rho_i^j = \operatorname{Re}\mu_i^j$ and normalize the logarithm demanding that $0 \le \rho_i^j < 1$.

Consider the matrix-functions

$$\Omega_j^+(z) = A_jG(s_j+0)(z-s_j)^{E_j}, \quad \Omega_j^-(z) = B_j\left(\frac{z-s_j}{z-z_0}\right)^{E_j},$$

where A_j and B_j are constant matrices:

$$A_1 = E, \quad A_j = \left[\prod_{k=1}^{j-1} \Omega_k^+(s_j) \right]^{-1}, \quad B_1 = E, B_j = \left[\prod_{k=1}^{j-1} \Omega_k^-(s_j) \right]^{-1}, \quad j = 2, 3, \dots .$$

The functions $\Omega_j^{\pm}(z)$ are holomorphic, respectively, in $U^{\pm}$. Introduce the new vector-functions

$$f^+(z) = \prod_{j=1}^{m} \Omega_j^+(z) f_1^+(z), \quad f^-(z) = \prod_{j=1}^{m} \Omega_j^-(z) f_1^-(z).$$

One can use the transmission condition (2) to obtain

$$f_1^+(t) = \left[\left(\prod_{j=1}^{m} \Omega_j^+(t) \right)^{-1} G(t) \prod_{j=1}^{m} \Omega_j^-(t) \right] f_1^-(t).$$

Proposition 1.1. *The matrix-function*

$$G_1(t) = \left(\prod_{j=1}^{m} \Omega_j^+(t) \right)^{-1} G(t) \prod_{j=1}^{m} \Omega_j^-(t)$$

is continuous at points $s_1, \dots, s_m$.

According to [32], there exists a system of canonical solutions $\chi_0(z)$ to our linear conjugation problem that satisfy the following conditions:

1. $\det \chi(z) \neq 0$ on $\mathbb{C}$, with the possible exception of points $s_1, s_2, \dots, s_m$.
2. There exists a diagonal matrix-function d_K such that $\lim\limits_{z \to \infty} \chi(z) d_K(z)$ is invertible at ∞.
3. If s_j is some singular point, then

$$\lim_{z \to s_j} (z - s_j)^{\varepsilon} \chi(z) = 0,$$

for some real number $\varepsilon > 0$.

Let $\omega = d\chi \cdot \chi^{-1}$ be a holomorphic 1-form on $\mathbb{CP}^1 \setminus \{s_1, \dots, s_m\}$. The corresponding system of differential equations $df = \omega f$ is a regular system with singular points $s_1, \dots, s_m$ and given monodromy, which gives a *solution of the Riemann–Hilbert problem in the class of regular systems.*

Let $a : \Gamma \to \mathrm{GL}_n(\mathbb{C})$ be a continuous and invertible matrix function on the contour Γ. In this section we will consider methods for calculation of partial indices of such matrix functions [1–3]. A recursive procedure allowing one to construct the factorizations (in particular, to compute the partial indices) was described by Litvinchuk and Spitkovsky [17]. An alternative approach, making use of the moments of a^{-1} with respect to Γ and yielding finite formulas for the factors, was proposed by Adukov (see [1–3]).

The formulas for the indices in [2] are obtained in the case where $a(t)$ is a meromorphic matrix function with no poles and zeros on Γ. The formulas are given in terms of ranks of the matrices T_l.

Definition 1.2. The power moment of the matrix function $a(t)$ with respect to the contour Γ is called the following matrix:

$$c_j = \frac{1}{2\pi i} \int_{\Gamma} t^{-j-1} a^{-1}(t) dt, \quad j \in \mathbb{Z}.$$

Let $k = \operatorname{ind} a(t)$ and consider the family of block Töplitz matrices:

$$T_l = \begin{pmatrix} c_l & c_{l-1} & \cdots & c_{-2k} \\ c_{l+1} & c_l & \cdots & c_{-2k+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_0 & c_{-1} & \cdots & c_{-2k-l} \end{pmatrix}, \qquad -2k \le l \le 0.$$

The following theorem holds.

Theorem 1.5 (see [1–3])**.** *The left and right partial indices of the matrix function $a(t)$ are calculated by the formulas*

$$k_j^{\mathrm{r}} = \operatorname{card}\{l \mid n + r_{-l-1} - r_{-l} \le j-1,\ l = 2k, 2k-1, \ldots 0\} - 1,$$
$$k_j^{\mathrm{l}} = 2k + 1 - \operatorname{card}\{l \mid r_{-l-1} - r_{-l} \le j-1,\ l = 2k, 2k-1, \ldots, 0\},$$

where $j = 1, \ldots, n$, r_l is the rank of the Töpliz matrix T_{-l}, and it is assumed that $r_{-2l-1} = 0$ and $l = 2k, 2k-1, \ldots, 0$.

Suppose that the matrix function $a(t)$ admits analytic continuation to U_+ and has in that domain p poles $z_1, \ldots, z_p$ of multiplicities $\kappa_1, \ldots, \kappa_p$. Then the matrix function

$$\tilde{a}(t) = \prod_{j=1}^{p} (z_{-} z_j)^{r_j} a(t)$$

is analytic and

$$\operatorname{ind}_\Gamma \det \tilde{a}(t) = \operatorname{ind}_\Gamma \det a(t) + (\kappa_1 + \kappa_2 + \cdots + \kappa_p)n.$$

For such matrix function, the left and right partial indices k_j^{l}, k_j^{r}, $j = 1, \ldots, n$, can be expressed by the formulas

$$k_j^{\mathrm{r}} = \operatorname{card}\{l \mid n + r_{-l-1} - r_{-l} \le j-1,\ l = 2k, 2k-1, \ldots 0\} - \kappa - 1, \tag{1.10}$$
$$k_j^{\mathrm{l}} = 2(\kappa + n\kappa)k - \kappa + 1 - \operatorname{card}\{l \mid r_{-l-1} - r_{-l} \le j-1,\ l = 2(\kappa + n\kappa), \ldots, 0\}, \tag{1.11}$$

where $\kappa = \kappa_1 + \cdots + \kappa_p$ is the total multiplicity of poles of $a(t)$.

Let $\kappa > 0$.

(i) Suppose that $\kappa > 2k$; then all right partial indices of the meromorphic function $a(t)$ are negative.

(ii) Suppose that $n \le 2\kappa$; then in order for the inequalities $k_j^{\mathrm{r}} \le 0$ to hold for all j, it is necessary and sufficient that

$$r_{\kappa-2k} \le r_{\kappa-2k+1} + 1.$$

If among the right partial indices occur both negative and positive numbers, then let us introduce the following numbers:

$$\alpha = \sum_{k_j^{\mathrm{r}} < 0} |k_j^{\mathrm{r}}|, \quad \beta = \sum_{k_j^{\mathrm{r}} > 0} |k_j^{\mathrm{r}}|.$$

As we will see later, the numbers α and β are dimensions of the kernel and cokernel of a certain Fredholm operator. Suppose that among the partial indices of the meromorphic function $a(t)$, there are positive as well as negative numbers. Then the possibility of calculating these numbers using rank of the matrix T_{-k} is seen from the following:

$$\alpha = (\kappa + 1)n - r_{\kappa-2k}, \quad \beta = k + n - r_{\kappa-2k}.$$

In order that among the partial indices of the meromorphic matrix function there are some that are nonnegative, it is necessary and sufficient that the following conditions hold:

(i) $\operatorname{ind}_\gamma \det a(t) \ge 0$,

(ii) $r_{\kappa-2k} = (\kappa + 1)n$.

In order that the meromorphic matrix function $a(t)$ has canonical factorization, it is necessary and sufficient that

(i) $\operatorname{ind}_\gamma \det a(t) = 0$,

(ii) $r_{-(2n-1)\kappa} = (\kappa+1)n$.

Suppose that

$$k = \operatorname{ind}_\gamma \det a(t) + \kappa n = 0;$$

then right partial indices of the meromorphic matrix function $a(t)$ are equal to $-\kappa$ and hence $a(t)$ is stable.

Let $k \neq 0$. We can find the numbers q and r, $0 \le r < q$ from the relation

$$\operatorname{ind}_\Gamma \det a(t) = nq + r.$$

Then for the stability of partial indices it is necessary and sufficient that

$$r_{\kappa+q-2k} = (\kappa+q+1)n,$$
$$r_{\kappa+q-2k+1} = (\kappa+q+1)n + r.$$

Theorem 1.6. *For the left partial indices, the following estimate is valid*:

$$-N \le k_j \le 2\operatorname{ind}_\Gamma \det a(t) + N(2n-1) + 1,$$

where N is the number of poles counted with multiplicities and n is the dimension of $a(t)$.

Proof. Indeed,

$$k_j = 2\operatorname{ind}_\Gamma \det a(t) + N(2n-1) + 1 - \operatorname{card}\{k \mid r_{-k-1} - r_{-k} \le j-1,\ k = 2\kappa, 2\kappa-1, \dots, 0\},$$

which gives

$$\max_{1\le j\le n} \operatorname{card}\{k \mid r_{-k-1} - r_{-k} \le j-1,\ k = 2\kappa, \dots, 0\}$$
$$= \max \operatorname{card}\{k \mid r_{-k-1} - r_{-k} \le 0,\ k = 2\kappa, \dots, 0\} = 2\kappa + 1$$

and

$$\min_{1\le j\le n} \operatorname{card}\{k \mid r_{-k-1} - r_{-k} \le j-1,\ k = 2\kappa, \dots, 0\}$$
$$= \min \operatorname{card}\{k \mid r_{-k-1} - r_{-k} \le 0,\ k = 2\kappa, \dots, 0\} = 0,$$

which proves the above inequalities (see also [11, 16]). □

2. Riemann–Hilbert Boundary-Value Problem

By the problem of linear conjugation, we mean the following problem:

Let Γ be a simple, closed, piecewise-smooth curve Γ, $a(t)$ and $b(t)$ be given $(n \times n)$-matrices on Γ, $a(t)$ be a piecewise-continuous matrix, $\inf|\det a(t)| > 0$, $b(t) \in L_p(\Gamma, \rho)$, $p > 1$, and the weight function ρ have the form

$$\rho(t) = \prod_{k=1}^{r} |t - t_k|^{\nu_k}, \quad t_k \in \Gamma, \quad -1 < \nu_k < p - 1. \tag{2.1}$$

The set $\{t_k\}$ contains all discontinuity points of the matrix $a(t)$; it may contain also other points of Γ. Find an $(n \times l)$-matrix $\Phi(z) \in E_p^{\pm}(\Gamma, \rho)$ satisfying the boundary condition

$$\Phi^+(t) = a(t)\Phi^-(t) + b(t) \tag{2.2}$$

almost everywhere on Γ.

Let c be some point of discontinuity of the matrix $a(t)$; denote by $\lambda_1, \dots, \lambda_n$ the roots of the equation

$$\det[a^{-1}(c+0)a(c-0) - \lambda I] = 0.$$

Consider the following numbers:

$$\tau_k = \frac{1}{2\pi i} \ln \lambda_k;$$

these numbers are defined to within the integer summands. We say that the point c is *singular* if $\operatorname{Re} \tau_k$ are integers; otherwise c is called nonsingular (see [25]).

The quadratic matrix $\chi(z)$ of order n is called the normal matrix of the boundary-value problem (2.2) (or for the matrix $a(t)$) if it satisfies the following conditions:

$$\chi(z) \in E_q^{\pm}(\Gamma, \rho), \quad \chi^{-1}(z) \in E_p^{\pm}(\Gamma, \rho^{1-q}), \quad q = \frac{p}{p-1}, \quad \chi^+(t) = a(t)\chi^-(t)$$

almost everywhere on Γ.

The normal matrix $\chi(z)$ is said to be *canonical* if it has normal form at infinity, i.e., $\lim\limits_{z\to\infty} (z^{-\sigma} det\chi(z))$ (σ is the sum of the column orders of $\chi(z)$) is finite and nonzero. In connection with the fact that it is possible to consider the different classes $E_p^{\pm}(\Gamma, \rho)$, we shall speak of the canonical (normal) matrices of the classes $E_p^{\pm}(\Gamma, \rho)$.

We shall say that the matrix $a(t)$ is *factorizable* in $E_p^{\pm}(\Gamma, \rho)$ if for $a(t)$, there exists the canonical matrix of the same class $E_p^{\pm}(\Gamma, \rho)$, and in this case we shall write $a(t) \in \mathfrak{F}_p(\Gamma, \rho)$.

It is easy to prove the following proposition. If $\chi_1(z)$ and $\chi_2(z)$ are normal matrices (in particular canonical) of the problem (2.2) of one and the same class, then $\chi_1(z) = \chi_2(z)P(z)$, where $P(z)$ is a polynomial matrix with constant and nonzero determinant.

Consequently the determinants of all normal (canonical) matrices of the given class of the boundary-value problem (2.2) have the same orders at infinity.

Definition 2.1. We define the index (or the total index) of the problem (2.2) of the class $E_p^{\pm}(\Gamma, \rho)$ (or the index of class $E_p^{\pm}(\Gamma, \rho)$ of the matrix $a(t)$) as the order at infinity of the determinant of the normal (canonical) matrix of the given class $E_p^{\pm}(\Gamma, \rho)$ taken with the opposite sign.

Having the normal matrix $\chi(z)$ of some class, we may obtain the canonical matrix by multiplying $\chi(z)$ from the right on the corresponding polynomial matrix with the constant nonzero determinant.

Let $\chi(z)$ be a canonical matrix (of the given class) for the matrix $a(t)$. Denote by $-\varkappa_1, \dots, -\varkappa_n$ the orders of the columns of $\chi(z)$ at infinity. The integers $\varkappa_1, \dots, \varkappa_n$ are called the partial indices of the matrix $a(t)$ or of the boundary-value problem (1.2) (of the given class). The sum of the partial indices $\varkappa_1 + \varkappa_2 + \cdots + \varkappa_n$ is equal to the index of $a(t)$ (or of the problem (2.2) of the given class).

Note that if $\chi(z)$ is a canonical matrix of $E_p^{\pm}(\Gamma, \rho)$ of the matrix $a(t)$, then the matrix $[\chi'(z)]^{-1}$ will be a canonical matrix of the class $E_p^{\pm}(\Gamma, \rho^{1-q})$ of the matrix $[a'(t)]^{-1}$.

It is easy to prove the following lemmas (see [21]).

Lemma 2.1. *Let $\chi(z)$ be a normal (canonical) matrix of the class $E_p^{\pm}(\Gamma, \rho)$ of the problem* (1.2). *If* (1.2) *is solvable for the given matrix $b(t) \in L_p(\Gamma, \rho)$, then all solutions of the problem* (1.2) *of the class $E_p^{\pm}(\Gamma, \rho)$ are given by the following formula*:

$$\Phi(z) = \frac{\chi(z)}{2\pi i} \int\limits_{\Gamma} \frac{[\chi^+(t)]^{-1} b(t)dt}{t-z} + \chi(z)P(z),$$

where $P(z)$ is an arbitrary polynomial $(n \times l)$-matrix. In particular, the solutions of the homogeneous problem $(b(t) \equiv 0)$ have the form $\chi(z)P(z)$.

Lemma 2.2. *Let $\chi(z)$ be a normal (canonical) matrix of the class $E_p^{\pm}(\Gamma, \rho)$ of the problem* (1.2), *and let the angular boundary values of the matrix of the form $\Phi(z) = f(z)\varphi(z)g(z)$ (here $\varphi(z) \in E_p^{\pm}(\Gamma, \rho)$, $f(z)$ and $g(z)$ are the piecewise meromorphic matrices that are continuously extendable from both sides everywhere on Γ) satisfy the boundary-value problem* (2.2) *for the given $b(t) \in L_p(\Gamma, \rho)$. Then the boundary-value problem* (2.2) *has the solution of the class $E_p^{\pm}(\Gamma, \rho)$.*

We can prove now the following propositions.

Proposition 2.1. *If the boundary-value problem* (2.2) *is solvable for an arbitrary* $b(t) \in L_p(\Gamma, \rho)$ *and there exists the normal (canonical) matrix* χ *of the class* $E_p^{\pm}(\Gamma, \rho)$*, then the expressions*

$$L_1 b \equiv \chi^+(t) \int\limits_{\Gamma} \frac{[\chi^+(\tau)]^{-1} b(\tau)}{\tau - t} d\tau,$$

$$L_2 b \equiv \chi^-(t) \int\limits_{\Gamma} \frac{[\chi^-(\tau)]^{-1} b(\tau)}{\tau - t} d\tau.$$

are the linear bounded operators in the space $L_p(\Gamma, \rho)$*.*

Proof. Indeed, let $b_m(t) \to b(t)$ and $L_1 b_m \to g$ with respect to the norm of the space $L_p(\Gamma, \rho)$. From $b_m(t) \to b(t)$ it follows that $L_1 b_m \to L_1 b$ with respect to the measure (see [21]); therefore, $g = L_1 b$ and the operator L_1 is a closed operator; since $L_p(\Gamma, \rho)$ is a Banach space, $L_1 b$ will be the bounded operator. □

Proposition 2.2. *The partial indices* $\varkappa_1, \ldots, \varkappa_n$ *of the problem* (1.2) *of the class* $E_p^{\pm}(\Gamma, \rho)$ *do not depend on the choice of a canonical matrix.*

Proof. See [21, 25]. Let $\chi(z)$ be a canonical matrix of the class $E_p^{\pm}(\Gamma, \rho)$, and D^+ and D^- be finite and infinite domains bounded by Γ. We have

$$\chi(z) = \chi_0(z)\Lambda(z), \quad z \in D^-,$$
$$\Lambda(z) = \operatorname{diag}\left[(z-c)^{-\varkappa_1}, \ldots, (z-c)^{-\varkappa_n}\right], \quad c \in D^+, \quad \det \chi_0(\infty) \neq 0.$$

Rewrite the boundary condition of the homogeneous problem (2.2) in the following form:

$$[\chi^+(t)]^{-1}\Phi^+(t) = \Lambda^{-1}(t)[\chi_0^-(t)]^{-1}\Phi^-(t),$$

from which it follows that

$$\begin{gathered} {}^{-1}\Phi(z) = P(z), \quad z \in D^+, \quad [\chi_0(z)]^{-1}\Phi(z) = \xi(z)P(z), \\ \xi(z) = \operatorname{diag}[(z-c)^{-\varkappa_1}, \ldots, (z - c^{-\varkappa_n})], \quad P(z) = (p_1, \ldots, p_n), \end{gathered} \tag{2.3}$$

$P_j(z)$ is an arbitrary polynomial of order j; $P_j(z) = 0$ when $j < 0$.

Denoting by λ the number of linear independent solutions of the homogeneous problem 2.2 of the class $E_{p,0}^{\pm}(\Gamma, \rho)$, from the equalities (2.3) we obtain $\lambda = \sum\limits_{\varkappa_k \geq 0} \varkappa_k$. It is evident that the number μ of linear independent solutions of the conjugate homogeneous problem $\Phi^+(t) = [a'(t)]^{-1}\Phi^-(t)$ of the class $E_{q,0}^{\pm}(\Gamma, \rho^{1-q})$ is equal to $\mu = -\sum\limits_{\varkappa_k \leq 0} \varkappa_k$.

Obviously, λ and μ are the invariants of the problem.

Let $\chi_1(z)$ and $\chi_2(z)$ be the canonical matrices of the problem (1.2) of the class $E_p^{\pm}(\Gamma, \rho)$. Denote by $-\varkappa_k^{(i)}$, $i = 1, 2, k = 1, 2, \ldots, n$, the orders of the columns of $\chi_i(z)$ at infinity. Let

$$\varkappa_1^{(i)} \geq \varkappa_2^{(i)} \geq \cdots \geq \varkappa_n^{(n)}, \quad \varkappa_1^{(1)} \geq \varkappa_1^{(2)}.$$

Consider the matrix $a_0(t) = (t-c)^{-\varkappa_1^{(2)}} a(t)$, and for this matrix as a canonical matrix we may take the matrix

$$\chi_i^0(z) = \begin{cases} \chi_i(z), & z \in D^+, \\ (z-c)^{\varkappa_1^{(2)}} \chi_i(z), & z \in D^-, \end{cases} \qquad i = 1, 2.$$

Note that the orders of columns of the matrix $\chi_1^0(z)$ at infinity are equal to $-\varkappa_k^{(1)} + \varkappa_1^{(2)}$; we get $\varkappa_1^{(1)} - \varkappa_1^{(2)} \leq 0$, $\varkappa_1^{(1)} = \varkappa_1^{(2)}$.

If we continue the argument, then we have that $\varkappa_k^{(1)} = \varkappa_k^{(2)}$, $k = 2, \ldots, n$. □

Boundary-value problem of linear conjugation with continuous coefficient. Consider the following boundary-value problem:

$$\Phi^+(t) = a(t)\Phi^-(t) + b(t), \quad t \in \Gamma, \tag{2.4}$$

where $a(t)$ and $b(t)$ are given $(n \times n)$-matrices on Γ, $b(t) \in L_p(\Gamma)$, $p > 1$, $a(t) \in C(\Gamma)$, and $\det a(t) \neq 0$.

For an arbitrary $\varepsilon > 0$, there exists the rational matrix $r(z)$ satisfying the conditions; $r(z)$ has no poles on Γ, $detr(t) \neq 0$ when $t \in \Gamma$, and

$$\|a(t)r^{-1}(t) - I\|_{C(\Gamma)} \leq \varepsilon, \quad \|a^{-1}(t)r(t) - I\|_{C(\Gamma)} \leq \varepsilon, \tag{2.5}$$

where I is a unit matrix.

Let us consider the sequence of matrices

$$\varphi_m(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{a_0(t)\varphi^-_{m-1}(t)}{t-z}dt + \frac{1}{2\pi i}\int\limits_\Gamma \frac{b(t)dt}{t-z}, \tag{2.6}$$

where

$$a_0 = ar^{-1} - I, \quad m = 1, 2, \ldots, \quad \varphi^-_0(t) = 0.$$

It is evident that $\varphi^-_m(t) \in L_p(\Gamma)$. Using the Sokhotsky–Plemelj formulas, from (2.6) we obtain

$$\varphi^-_{m+1}(t) - \varphi^-_m(t) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{a_0(\tau)[\varphi^-_m(\tau) - \varphi^-_{m-1}(\tau)] - a_0(t)[\varphi^-_m(t) - \varphi^-_{m-1}(t)]}{\tau - t}d\tau.$$

Hence

$$\|\varphi^-_{m+1} - \varphi^-_m\|_{L_p(\Gamma)} \leq A_p\varepsilon\|\varphi^-_m - \varphi^-_{m-1}\|_{L_p(\Gamma)}. \tag{2.7}$$

From inequality (2.7) it follows that if $A_p\varepsilon < 1$, then the sequence $\varphi^-_m(t)$ converges by the norm of $L_p(\Gamma)$ to some matrix $\varphi^-(t) \in L_p(\Gamma)$. Whence it follows that for every $z \notin \Gamma$ there exists the $\lim \varphi_m(z) = \varphi(z)$ representable by the following formula:

$$\varphi(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{a_0(t)\varphi^-(\tau)}{\tau - t}d\tau + \frac{1}{2\pi i}\int\limits_\Gamma \frac{b(t)dt}{\tau - t}. \tag{2.8}$$

The matrix $\varphi(z)$ defined by the formula (2.8) belongs to the class $E^\pm_{p,0}(\Gamma)$ and satisfies the boundary condition

$$\varphi^+(t) = a(t)r^{-1}(t)\varphi^-(t) + b(t).$$

If we take $b(t)$ equal to $a(t)r^{-1}(t)$, then $\varphi(z)$ will satisfy the following boundary condition:

$$\varphi^+(t) = a(t)r^{-1}(t)[\varphi^-(t) + I]. \tag{2.9}$$

Substituting the matrix $a(t)$ into $a'^{-1}(t)$ and $r(t)$ into $r'^{-1}(t)$ (this is possible by virtue of (2.5)), we obtain that there exists the matrix $\psi(z) \in E^\pm_{p,0}(\Gamma)$ such that $\psi^+(t) = a'^{-1}(t)r'(t)[\psi^-(t) + I]$, or

$$\psi'^+(t) = [\psi'^-(t) + I]\tau(t)a^{-1}(t). \tag{2.10}$$

It follows from (2.9) and (2.10) that

$$\psi'^+(t)\varphi^+(t) = [\psi'^-(t) + I][\varphi^-(t) + I]. \tag{2.11}$$

Let $p \geq 2$. The matrix defined by the formula

$$\chi(z) = \begin{cases} \psi'(z)\varphi(z), & z \in D^+, \\ (\psi'(z) + I)(\varphi(z) + I), & z \in D^-, \end{cases}$$

belongs to the class $E^\pm_1(\Gamma)$, and from (2.11) we have $\chi(z) \equiv I$, i.e.,

$$[\varphi(z)]^{-1} = \psi'(z), \quad z \in D^+, \quad [\varphi(z) + I]^{-1} = \psi'(z) + I, \quad z \in D^-.$$

Consider now the matrix

$$\chi(z) = \begin{cases} \varphi(z)R(z), & z \in D^+, \\ r^{-1}(z)[\varphi(z) + I]R(z), & z \in D^-, \end{cases}$$

where $R(z)$ is a rational matrix chosen in the following way: it liquidates the zeros of $detr^{-1}(t)$ in the domain D^- and the poles of $r^-(z)$ in the same domain and gives to $\chi(z)$ the normal form at infinity; there exists such a matrix [25]. It is easy to see that $\chi(z) \in E_p^{\pm}(\Gamma)$, $\chi^{-1}(z) \in E_p^{\pm}(\Gamma)$; therefore for an arbitrary continuous nonsingular matrix $a(t)$ there exists a canonical matrix of the class $E_p^{\pm}(\Gamma)$ for an arbitrary $p \geq 2$.

Let $\chi_1(z)$ and $\chi_2(z)$ be canonical matrices of the classes $E_{p_1}^{\pm}(\Gamma)$ and $E_{p_2}^{\pm}(\Gamma)$, respectively, $2 \leq p_1 < p_2$.

We obtain

$$\chi_1(z) = \chi_2(z)P_1(z), \quad \left[\chi_1'(z)\right]^{-1} = [\chi_2'(z)]^{-1}P_2(z),$$

where $P_1(z)$ and $P_2(z)$ are some polynomial matrices. From the last equalities it follows that $\chi_1(z) \in E_{p_2}^{\pm}(\Gamma)$ and $[\chi_1(z)]^{-1} \in E_{p_2}^{\pm}(\Gamma)$.

Consequently the canonical matrix of an arbitrary class $E_p^{\pm}(\Gamma)$ $(p \geq 2)$ has the property

$$\chi(z) \in E_\infty^{\pm}(\Gamma), \quad \chi^{-1}(z) \in E_\infty^{\pm}(\Gamma).$$

It is evident that these matrices are the canonical matrices also for $1 < p < 2$. So it follows from these arguments that the boundary-value problem (2.4) is solvable for an arbitrary $b(t) \in L_p(\Gamma, \rho)$ in the class $E_p^{\pm}(\Gamma)$, and all solutions of this class are given by the following formula:

$$\Phi(z) = \frac{\chi(z)}{2\pi i} \int_\Gamma \frac{[\chi^+(\tau)]^{-1} b(\tau) d\tau}{\tau - t} + \chi(z)P(z),$$

where $P(z)$ is an arbitrary polynomial $(n \times n)$-matrix.

Let now the matrix $a(t)$ be Hölder-continuous. Then the canonical matrix $\chi(z)$ is continuously extendable for all points of the curve Γ from both sides, and the matrices $\chi^+(t)$ and $\chi^-(t)$ are Hölder-continuous, $\det \chi^{\pm}(t) \neq 0$.

Let us construct again the sequence $\varphi_m(\Gamma)$ by the formula (2.6); however, we take the rational matrix $\chi(z)$ such that the inequalities (2.5) will be fulfilled by the norm of the space $H_\beta(\Gamma)$, $0 < \beta < \alpha$ $(a(t) \in H_\alpha(\Gamma))$. Then the sequence $\varphi_m^-(t)$ converges by the norm of $H_\beta(\Gamma)$, $\varphi^-(t) \in H_\beta(\Gamma)$ and the matrix $\varphi(z)$ defined by the formula (2.8) will be Hölder-continuous in closures $\bar{D}^+$ and $\bar{D}^-$. This proves the above proposition.

3. Boundary-Value Problem with Piecewise Continuous Coefficient

In this section we give a review of an important approximation method of G. Manjavidze for piecewise continuous matrix-function by rational matrices. We follows Manjavidze's papers [18–21] (see also [22–24]).

Let Γ be a simple, smooth, closed contour in the complex plane with interior D^+ and exterior D^-, on which are defined matrices $G(t) = (G_{ik}(t))$ and $F(t) = (F_{ik}(t))$ whose elements are Hölder continuous on Γ with index μ; in addition, it is assumed that $\det G(t) \neq 0$ on Γ. Consider the problem of finding a sectionally-holomorphic matrix $\Phi(z)$ with elements of finite degree at infinity whose boundary values $\Phi^+(t)$ and $\Phi^-(t)$, along the inner and outer edges of Γ, respectively, satisfy

$$\Phi^+(t) = G(t)\Phi^-(t) + F(t). \tag{3.1}$$

The solution of this problem in terms of Cauchy integrals is given by Mushelishvili in his well-known book [25]. In [20] the author determines this solution approximately by the following scheme. Define the sectionally-holomorphic matrix $\varphi(z)$ to be $\Phi(z)$ for z in D^+ and $R(z)\Phi(z)$ for z in D^-, where

$R = (R_{ik})$ is a matrix whose elements are rational functions such that on Γ one has $|G_{ik} - R_{ik}|_\nu < \varepsilon$, $i, k = 1, 2, \dots, n$, where ε is a small positive number and the norm is defined by

$$\|f\|_\nu = \max_{t\in\Gamma} |f(t)| + \sup \frac{|f(t_1) - f(t_2)|}{|t_1 - t_2|^\nu}, \quad 0 < \nu < 1, \quad \nu < \mu. \tag{3.2}$$

The boundary relation (3.1) can then be rewritten as

$$\varphi^+(t) - \varphi^-(t) = g(t)\varphi^-(t) + F(t), \tag{3.3}$$

where $g = (G - R)R^{-1}$ (ε is taken sufficiently small so that $\det R(t) \neq 0$ on Γ). It is then shown, for the sequence of sectionally-holomorphic matrices $\varphi_m(z)$ defined recursively by

$$\varphi_m(z) = \frac{1}{2\pi i} \int_\Gamma \frac{g(t)\varphi_{m-1}(t)dt}{t - z} + \frac{1}{2\pi i} \int_\Gamma \frac{F(t)dt}{t - z}, \tag{3.4}$$

that the sequences $\varphi_m^+(t)$ and $\varphi_m^-(t)$ converge in the sense of the norm (3.2) whenever ε is small enough, and hence $\varphi_m(z) \to \varphi(z)$. As a corollary to the method, there arises the possibility to construct an approximate solution of the homogeneous problem $\alpha^+(t) = G(t)\alpha^-(t)$ with poles possible only in D^-, whence by known methods the canonical solution may be obtained. From the results above, it follows also to allow $G(t)$ and $F(t)$ to have at most a finite number of jump points on Γ.

Scalar case. First let us consider the homogeneous problem for $n = 1$

$$\Phi^+(t) = a(t)\Phi^-(t), \quad a(t) \in C_0(\Gamma; c_1, \dots, c_m). \tag{3.5}$$

Now we make the substitution (see [26, 32]):

$$\Phi(z) = \prod_{k=1}^m \chi_k^1(z)\varphi(z), \quad z \in D^+, \quad \Phi(z) = \prod_{k=1}^m \chi_k(z)\varphi(z), \quad z \in D^-, \tag{3.6}$$

where

$$\chi_k^1(z) = (z - c_k)^{\tau_k}, \quad \chi_k(z) = \left(\frac{z - c_k}{z - z_0}\right)^{\tau_k}, \quad z_0 \in D^+,$$
$$\tau_k = \frac{1}{2\pi i} \ln \lambda_k, \quad \lambda_k = \frac{a(c_k - 0)}{a(c_k + 0)}, \quad -1 < \operatorname{Re} \tau_k \leq 0,$$

and where χ_k^1 and χ_k are the univalent branches of the elementary multi-valued functions defined as follows: $\chi_k^1(z)$ is the univalent branch in the plane cut along the line e_k that connects the point c_k with the point $z = \infty$ and lies in the domain D^-, and $\chi_k(z)$ is the univalent branch in the plane cut along the line ℓ_k^1 that connects the point z_0 with the point c_k and lies in the domain D^+, $\chi_k(\infty) = 1$. With respect to the function $\varphi(z)$, we obtain the boundary condition $\varphi^+(t) = g(t)\varphi^-(t)$, where

$$g(t) = a(t) \left[\prod_{k=1}^r \chi_k^{1+}(t)\right]^{-1} \prod_{k=1}^r \chi_k^-(t) = a(t) \prod_{k=1}^r (t - z_0)^{-\tau_k};$$

$g(t)$ is a continuous function, $g(t) \neq 0$.

In the previous section we proved that, for the continuous function $g(t)$, there exists the canonical function $A(z) \in E_\infty^\pm(\Gamma)$, $A^{-1}(z) \in E_\infty^\pm(\Gamma)$.

Consider the function

$$\chi_0(z) = \begin{cases} A(z) \prod_{k=1}^r X_k^1(z), & z \in D^+, \\ A(z) \prod_{k=1}^r X_k(z), & z \in D^-. \end{cases}$$

It is evident that $\chi_0^{-1}(z) \in E_\infty^\pm(\Gamma)$ and $\chi_0(z) \in E_\varepsilon^\pm(\Gamma)$ for some $\varepsilon > 1$.

Let $\Phi(z)$ be some solution of the problem (3.5) of the class $E_\delta^\pm(\Gamma)$, $\delta > 1$. Consider the function $\Phi_1(z) = \Phi(z)/\chi_0(z)$. Obviously $\Phi_1(z) \in E_{\delta_1}^\pm(\Gamma)$, $\delta_1 > 1$, and $\Phi_1^+(t) = \Phi_1^-(t)$, $t \in \Gamma$.

Consequently, $\Phi_1(z)$ is a polynomial $P(z)$ and $\Phi(z) = \chi_0(z)P(z)$.

Assume that there exists the canonical function of the problem (3.5) of the class $E_p^\pm(\Gamma,\rho)$, $\rho(t) = \prod\limits_{K=1}^{r} |t-a_k|^{\nu_k}$, $-1 < \nu_k < p-1$. Then it will have the form $\chi(z) = \chi_0(z)Q(z)$, where $Q(z)$ is some polynomial; in addition,

$$\chi_0(z)Q(z) \in E_p^\pm(\Gamma,\rho), \quad [\chi_0(z)Q(z)]^{-1} \in E_q^\pm(\Gamma,\rho^{1-q}). \tag{3.7}$$

One can see from (3.7) that the polynomial $Q(z)$ may have zeros only in the points c_k and

$$\begin{gathered} A^+(t)\prod_{k=1}^{r}(t-c_k)^{\tau_k}Q(t) \in L_p(\Gamma,\rho), \\ \left[A^+(t)\prod_{k=1}^{r}(t-c_k)^{\tau_k}Q(t)\right]^{-1} \in L_q(\Gamma,\rho^{1-q}). \end{gathered} \tag{3.8}$$

Denote by m_k ($m_k \geq 0$) the order of zero of the polynomial $Q(z)$ at the point c_k. The relations

$$\begin{gathered} |A^+(t)|^p|Q_s(t)|^p\prod_{k=1}^{r}|t-c_k|^{m_kp+\nu_k}|(t-c_k)^{\tau_k}|^p \in L_1(\Gamma), \\ |A^+(t)|^q|Q_s(t)|^q\prod_{k=1}^{r}|t-c_k|^{-m_kq+\nu_k(1-q)}|(t-c_k)^{-\tau_k}|^q \in L_1(\Gamma), \\ Q_k(z) = (z-c_k)^{-m_k}Q(z). \end{gathered}$$

hold.

From these relations, it follows that

$$\begin{gathered} \prod_{k=1}^{r}|t-c_k|^{m_kp+\nu_k}|(t-c_k)^{\tau_k}|^p \in L_{1-\lambda}(\Gamma), \\ \prod_{k=1}^{r}|t-c_k|^{-m_kq+\nu_k(1-q)}|(t-c_k)^{-\tau_k}|^q \in L_{1-\lambda}(\Gamma), \end{gathered} \tag{3.9}$$

where λ is an arbitrary small positive number.

Denoting by $\tau_k = \alpha_k + i\beta_k$, from (3.9) we obtain

$$\begin{gathered} \prod_{k=1}^{r}|t-c_k|^{(\alpha_k+m_k)p+\nu_k} \in L_{1-\lambda}(\Gamma), \\ \prod_{k=1}^{r}|t-c_k|^{-(\alpha_k+m_k)q+\nu_k(1-q)} \in L_{1-\lambda}(\Gamma), \end{gathered}$$

from which it follows that

$$(\alpha_k+m_k)p+\nu_k > -1, \quad -(\alpha_k+m_k)q+\nu_k(1-q) > 1,$$

or

$$-\alpha_k - \frac{1}{p} - \frac{\nu_k}{p} < m_k < -\alpha_k + \frac{1}{q} - \frac{\nu_k}{p}.$$

Denote $|\alpha_k| = \mu_k$ and call this number the parameter of the function $a(t)$ at the point c_k. The parameter μ_k may also be defined by the following relations:

$$\mu_k = \mathrm{Re}\,\frac{1}{2\pi i}\frac{a(c_k+0)}{a(c_k-0)}, \quad 0 \le \arg\frac{a(c_k+0)}{a(c_k-0)} < 2\pi.$$

Introduce the notation

$$\mu_k - \frac{1+u_k}{p} = \varepsilon_k.$$

Evidently,

$$-\frac{1+\nu_k}{p} < \varepsilon_k < \nu_k$$

and therefore $-1 < \varepsilon_k < 1$. So we have

$$\varepsilon_k < m_k < 1 + \varepsilon_k.$$

If $\varepsilon_k = 0$, then the inequality is unrealizable; if $\varepsilon_k > 0$, then $m_k = 1$; if $\varepsilon_k < 0$, then $m_k = 0$. Hence we get the following result.

Theorem 3.1. *If $\mu_k p = 1 + \nu_k$ for some k, then a canonical function of the corresponding class does not exist.*

If $\mu_k p \neq 1+\nu_k$, $k = 1, 2, \ldots, r$, then the canonical function of the class $E_p^{\pm}(\Gamma, \rho)$ exists and is given by the formula

$$\chi(z) = \chi_0(z)Q(z),$$

where

$$Q(z) = \prod_{k=1}^{r}(z-c_k)^{m_k}, \quad m_k = \begin{cases} 1, & \textit{if}\ \ \mu_k - \dfrac{1+\nu_k}{p} > 0, \\ 0, & \textit{if}\ \ \mu_k - \dfrac{1+\nu_k}{p} < 0. \end{cases}$$

The index of the class $E_p^{\pm}(\Gamma, \rho)$ of the function $a(t)$ (or the problem (3.5)) is given by the formula $\varkappa = \mathrm{ind}\, g(t) - \sum\limits_{k=1}^{r} m_k$ or by the formula

$$\varkappa = \frac{1}{2\pi}\left\{\arg\frac{a(t)}{\prod\limits_{k=1}^{r}(t-z_0)^{s_k}}\right\}_\Gamma, \tag{3.10}$$

where $s_k = \dfrac{1}{2\pi i}\ln\lambda_k$ and

$$-1 < \mathrm{Re}\, s_k \le 0 \quad \text{if} \quad \mu_k < \frac{1+\nu_k}{p} \quad (\text{i.e., } s_k = \tau_k),$$
$$0 \le \mathrm{Re}\, s_k < 1 \quad \text{if} \quad \mu_k > \frac{1+\nu_k}{p} \quad (\text{i.e., } s_k = \tau_{k+1}).$$

Note that the condition $\mu_{kp} \neq 1+\nu_k$ is trivially fulfilled if the point c_k is singular, because in this case $\mu_k = \alpha_k = 0$; $s_k = \tau_k$, $\mathrm{Re}\, s_k = 0$.

Remark 3.1. If $\chi_i(z)$, $i = 1, 2$, are the canonical functions of the classes $E_{p_i}^{\pm}(\Gamma, \rho_i)$ (ρ_i are the functions of the form (1.1), Sec. 1), then

$$\chi_2(z) = \chi_1(z)\prod_{k=1}^{r}(z-c_k)^{m_k},$$

where $m = +1, -1$ or 0.

In particular, if $\chi_1(z)$ and $\chi_2(z)$ are canonical functions of the classes $E^{\pm}_{1+\varepsilon}(r)$ and $E^{\pm}_p(r)$ (ε is a sufficiently small positive number and p is a sufficiently large number), then in the last equality $m_k = 0$ for singular points and $m_k = 1$ for nonsingular points. Between the indices of the these classes, the relation $\varkappa_p = \varkappa_{1+\varepsilon} - \tau_0$ holds, where τ_0 is the number of nonsingular points (see [25, p. 78]).

Consider now the nonhomogeneous problem

$$\phi^+(t) = a(t)\phi^- + b(t), \quad b(t) \in L_p(\Gamma, \rho), \tag{3.11}$$

and make the substitution (3.6).

Instead of $-1 < \operatorname{Re}\tau_k \leq 0$ we suppose

$$\frac{-1+\nu_k}{p} < \operatorname{Re}\tau_k < 1 - \frac{1+\nu_k}{p}. \tag{3.12}$$

Since $\operatorname{Re}\tau_k$ is defined to within an integer, the inequalities

$$\frac{1+\nu_k}{p} \leq \operatorname{Re}\tau_k < 1 - \frac{1+\nu_k}{p}$$

are always fulfilled.

Since the equality

$$\frac{-1+\nu_k}{p} = \operatorname{Re}\tau_k$$

is eliminated, inequalities (3.11) are satisfied.

We obtain the nonhomogeneous problem

$$\phi^+(t) = g(t)\phi^-(t) + f(t), \quad f(t) = b(t)\left(\prod_{k=1}^{r} \chi^1_k(t)\right)^{-1}. \tag{3.13}$$

It is obvious that

$$f(t) \in L_p(\Gamma, \rho_1), \quad \rho_1(t) = \prod_{k=1}^{r} |(t-c_k)|^{\nu^1_k}, \quad \nu^1_k = \alpha_k p + \nu_k, \quad \alpha_k = \operatorname{Re}\tau_k.$$

It is easy to see that $-1 < \nu^1_k < p-1$ as this inequality coincides with (3.11).

We shall construct the solution of (3.12) in the class $E^{\pm}_p(\Gamma, \rho_1)$. Take the rational function $R(z)$ such that $\|g(t) - R(t)\|_{C(\Gamma)} \leq \varepsilon$, where ε is a sufficiently positive number, and consider the following sequence:

$$\psi_{m+1}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{g_0(t)\psi^-_m(t)}{t-z}dt + \frac{1}{2\pi i}\int_\Gamma \frac{f(t)dt}{t-z}, \quad \psi^-_0(t) = 0, \quad g_0 = gR^{-1} - I. \tag{3.14}$$

It is evident that $\psi_m(z) \in E^{\pm}_{\rho,0}(r, \rho_1)$. From (3.14) we have

$$\psi^-_{m+1}(t_0) - \psi^-_m(t_0) = -\frac{1}{2}g_0(t)\big[\psi^-_m(t_0) - \psi^-_{m-1}(t_0)\big] + \frac{1}{2\pi i}\int_\Gamma \frac{g_0(t)\big[\psi^-_m(t) - \psi^-_{m-1}(t)\big]}{t-z}dt.$$

Consequently the sequence $\psi^-_m(t)$ converges by the norm of the space $L_p(\Gamma, \rho)$ to some function $h(t) \in L_p(\Gamma, \rho_1)$.

From (3.14) we also have

$$h(t_o) = -\frac{1}{2}\big[g_0(t_0)h(t_0) + f(t_0)\big] + \frac{1}{2\pi i}\int_\Gamma \frac{g_0(t)h(t) + f(t)}{t-t_0}dt.$$

Hence $h(t_0)$ is a boundary value of some analytic function on Γ in the domain D^- vanishing at infinity. Finally, we obtain

$$\psi(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{g_0(t)\psi^-(t)}{t-z}dt + \frac{1}{2\pi i}\int\limits_\Gamma \frac{f(t)}{t-z}dt$$

(here $\psi^-(t)$ denotes $h(t)$).

From the last equality we get

$$\psi^+(t) - \psi^-(t) = g_0(t)\psi^-(t) + f(t)$$

or

$$\psi^+(t) = gR^{-1}\psi^-(t) + f(t). \tag{3.15}$$

Comparing (3.12) and (3.14) we can see that the function $\phi(z) = \psi(z)$, if $z \in D^+$, and $\phi(z) = R^{-1}(z)\psi(z)$, if $z \in D^-$ is a solution of the problem (3.13). As far as the problem (3.12) has a canonical function of the class $E_p^{\pm}(\Gamma,\rho)$, it is solvable in this class for any $f(t) \in L_p(\Gamma,\rho)$, and the initial problem (3.11) is solvable in $E_p^{\pm}(\Gamma,\rho)$ for an arbitrary function $b(t) \in L_p(\Gamma,\rho)$. Whence, by virtue of Lemma 1.3, the expressions

$$\chi^+(t_0)\int\limits_r \frac{[\chi^+(t)]^{-1}b(t)}{t-t_0}dt, \quad \chi^-(t_0)\int\limits_r \frac{[\chi^+(t)]^{-1}b(t)}{t-t_0}dt$$

are the linear bounded operators in $L_p(\Gamma,\rho)$.

Case of triangular matrix. Consider now the following boundary-value problem:

$$\phi^+(t) = a(t)\phi^-(t) + b(t), \quad t \in \Gamma, \tag{3.16}$$

where $a(t)$ is a triangular piecewise-continuous nonsingular matrix $a = (a_{ik})$, $a_{ik} = 0$, where $i < k$, $b \in L_p(\Gamma,\rho)$. Denote by $c_1,\dots,c_r$ all discontinuity points of the functions $a_{ii}(t)$, $i = 1,\dots,n$. By μ_{ik} denote the parameters of the functions $a_{ii}(t)$ at the points c_k, $k = 1,\dots,r$. It is evident that $\mu_{ik} = 0$ if the function $a_{ii}(t)$ is continuous at the point c_k. Let us assume that the inequalities

$$\frac{1+\nu_k}{p} \neq \mu_{ik}, \quad k = 1,\dots,r, \quad i = 1,\dots,n, \tag{3.17}$$

are valid and show that in this case there exists a canonical matrix of the problem (3.16) of the corresponding class. Obviously, if inequalities (3.17) are fulfilled, then every function $a_{kk}(t)$ is a canonical function of the class $E_p^{\pm}(\Gamma,\rho)$. Denote it by $\chi_k(z)$.

Consider the triangular matrix $\chi(z) = (\chi_{ik})$, $i,k = 1,\dots,n$; $\chi_{ik} = 0$ when $i < k$, $\chi_{ik}(z) = \chi_k(z)$, and the remaining elements are defined by the formulas

$$\chi_{s1}(z) = \frac{\chi_s(z)}{2\pi i}\int\limits_\Gamma \frac{\sum\limits_{i=1}^{s-1} a_{si}(t)\chi_{i1}^-(t)dt}{\chi_s^+(t)(t-z)}, \qquad s = 2,\dots,n,$$

$$\chi_{s2}(z) = \frac{\chi_s(z)}{2\pi i}\int\limits_\Gamma \frac{\sum\limits_{i=1}^{s-1} a_{si}(t)\chi_{i2}^-(t)dt}{\chi_s^+(t)(t-z)}, \qquad s = 3,\dots,n,$$

$$\chi_{n,n-1}(z) = \frac{\chi_s(z)}{2\pi i}\int\limits_\Gamma \frac{\sum\limits_{i=1}^{s-1} a_{n,n-1}(t)\chi_{n-1,n-1}^-(t)dt}{\chi_n^+(t)(t-z)}.$$

It can be easily seen that the matrix constructed in this manner belongs to the class $E_p^{\pm}(\Gamma,\rho)$ and satisfies the relation

$$\chi^+(t) = a(t)\chi^-(t).$$

Moreover,

$$\det \chi(z) = \prod_{k=1}^{n} \chi_k(z).$$

Construct now the same matrix $\chi_*(z)$ for the matrix $[a'(t)]^{-1}$ as above, where

$$\chi_*(z) \in E_q(\Gamma,\rho^{1-q}), \quad \det \chi_*(z) = \prod_{k=1}^{n} [\chi_k(z)]^{-1}, \quad \chi_*^+(t) = [a'(t)]^{-1}\chi_*^-(t).$$

Consider the matrix $\chi_*'(t)\chi(z) = \chi_0(z)$. We have

$$\chi_0(z) \in E_1^{\pm}(\Gamma), \quad \chi_*'^+(t) = \chi_*'^-((t)[a(t)]^{-1}, \quad \chi_*'^+(t)\chi^+(t) = \chi_+'^-(t)\chi^-(t).$$

Whence $\chi_0(z) = P(z)$, where $P(z)$ is some polynomial matrix. But $\det P(z) = 1$ and $P^{-1}(z)$ is also a polynomial matrix. Thus,

$$P^{-1}(z)\chi_*'(z)\chi(z) = I$$

and $\chi(z)$ has a inverse matrix equal to

$$P^{-1}\chi_*'(z) \in E_q^{\pm}(\Gamma,\rho^{1-q}).$$

Thus, we have proved that $\chi(z)$ is a normal matrix for $a(t)$ of the class $E_p^{\pm}(\Gamma,\rho)$.

It is easy to see that the boundary-value problem (3.16) is solvable for an arbitrary vector $b(t) \in L_p(\Gamma,\rho)$, and therefore the operators

$$\chi^+(t_0)\int_\Gamma \frac{[\chi^+(t)]^{-1}b(t)dt}{t-t_0}, \quad \chi^-(t_0)\int_\Gamma \frac{[\chi^-(t)]^{-1}b(t)dt}{t-t_0}$$

are linear bounded operators in $L_p(\Gamma,\rho)$.

The index of the problem (3.16) of the class $E_p^{\pm}(\Gamma,\rho)$ is equal to the sum of the indices of the boundary-value problems $\varphi_k^+(t) = a_{kk}(t)\varphi_k^-(t)$, i.e., $\kappa = \sum_{k=1}^{n}\kappa_k$, and κ_k is calculated by the formula (3.10):

$$\kappa_k = \frac{1}{2\pi}\left\{\arg \frac{a_{kk}(t)}{\prod_{j=1}^{r}(t-z_0)^{s_{kj}}}\right\}_\Gamma,$$

where

$$s_{kj} = \frac{1}{2\pi i}\ln \lambda_{kj}, \quad \lambda_{kj} = \frac{a_{kk}(c_j-0)}{a_{kk}(c_j+0)},$$

$$-1 < \operatorname{Re} s_{kj} \le 0 \quad \text{if} \quad \mu_{kj} < \frac{1+\nu_j}{p}, \qquad 0 \le \operatorname{Re} s_{kj} < 1 \quad \text{if} \quad \mu_{kj} > \frac{1+\nu_j}{p}.$$

General case. Consider now the following problem:

$$\Phi^+(t) = a(t)\Phi^-(t) + b(t), \quad b(t) \in L_p(\Gamma, \rho), \tag{3.18}$$

where $a(t)$ is an arbitrary piecewise-continuous matrix, and $\inf|\det a(t)| > 0$. Let us represent the matrix $a(t)$ in the form $a(t) = a_1(t)\Lambda(t)a_2(t)$, where $a_1(t)$ and $a_2(t)$ are continuous nonsingular matrices and $\Lambda(t)$ is a piecewise-continuous nonsingular triangular matrix. This is possible by virtue of the lemma proved in [21].

Take the rational matrices $R_1(z)$ and $R_2(z)$ such that $\|a_k(t) - R_k(t)\| \le \varepsilon$, $k = 1, 2$, where ε is a sufficiently small positive number. Rewrite the boundary condition (3.18) in the following form:

$$\Phi^+ = R_1(t)\Lambda(t)R_2(t)\Phi^-(t) + \big[a(t) - R_1(t)\Lambda(t)R_2(t)\big]\Phi^-(t) + b(t).$$

Introduce the following notation:

$$R_1^{-1}(z)\Phi(z) = \varphi(z), \quad z \in D^+, \qquad R_2(z)\Phi(z) = \varphi(z), \quad z \in D^-, \qquad R_1^{-1}(t)b(t) = B(t);$$

we have

$$\varphi^+(t) = \Lambda(t)\varphi^-(t) + \big[R_1^{-1}(t)a_1(t)\Lambda(t)a_2(t)R_2^{-1}(t) - \Lambda(t)\big]\varphi^-(t) + B(t).$$

It is evident that

$$a_0(t) = R_1^{-1}(t)a(t)R_2^{-1}(t) - \Lambda(t)$$

is a piecewise-continuous matrix and

$$\sup_{t\in\Gamma}|a_0(t)| < C_1\varepsilon,$$

where C_1 is constant.

Consider now a sequence of matrices

$$\varphi_{m+1}(z) = \frac{\chi(z)}{2\pi i}\int_\Gamma \frac{[\chi^+(t)]^{-1}a_0(t)\varphi_m^-(t)}{t-z}dt + \frac{\chi(z)}{2\pi i}\int_\Gamma \frac{[\chi^+(t)]^{-1}B(t)}{t-z}dt, \tag{3.19}$$

where $\varphi_0^-(t) = 0$ and $\chi(z)$ is a canonical matrix of the class $E_p^\pm(\Gamma, \rho)$ of the matrix $\Lambda(t)$. It is evident that $\varphi_m(z) \in E_p^\pm(\Gamma, \rho)$, $m \ge 1$. From (3.19) we have

$$\varphi_{m+1}^-(t) - \varphi_m^-(t_0) = -\frac{1}{2}a^{-1}(t)a_0(t)\big[\varphi_m^-(t_0) - \varphi_{m-1}^-(t_0)\big] + \frac{\chi(t_0)}{2\pi i}\int_\Gamma \frac{[\chi^+(t)]^{-1}a_0(t)\big[\varphi_m^-(t) - \varphi_{m-1}^-(t)\big]}{t-z}dt.$$

Whence

$$\|\varphi_{m+1}^- - \varphi_m^{-1}\|_{L_p(\Gamma,\rho)} \le C_2\varepsilon,$$

where C_2 is a constant. Therefore, if $C_1C_2\varepsilon < 1$, then the sequence φ_m^- converges in the space $L_p(\Gamma, \rho)$. It follows from (3.19) that φ_m^+ also converges in the space $L_p(\Gamma, \rho)$. The limit matrix $\varphi(z) \in E_{p,0}^\pm(\Gamma, \rho)$ and satisfies the following boundary condition:

$$\varphi^+(t) = R_1^{-1}(t)a(t)R_2^{-1}(t)\varphi^-(t) + R_1^{-1}(t)b(t).$$

Consequently the matrix

$$\Phi(z) = \begin{cases} R_1(z)\varphi(z), & z \in D^+, \\ R_2^{-1}(z)\varphi(z), & z \in D^-, \end{cases} \tag{3.20}$$

will be the solution of the boundary-value problem, which may have poles in some points of the domains D^+ and D^-.

Now we consider the adjoint boundary-value problem, i.e., the problem

$$\Psi^+(t) = [a'(t)]^{-1}\Psi^-(t) + g(t), \quad g \in L_q(\Gamma, \rho^{1-q}). \tag{3.21}$$

Substituting in the previous arguments the matrices R_1 and R_2, respectively, by the matrices $R_1^{'-1}$ and $R_2^{'-1}$, we construct the solution in the form

$$\Psi(z) = \begin{cases} [R_1'(z)]^{-1}\psi(z), & z \in D^+, \\ R_1'(z)\psi(z), & z \in D^-. \end{cases}$$

Take now $b = aR_2^{-1}\chi^-$ and $g = a^{'-1}R_2^1(\chi'^-)^{-1}$. We obtain

$$\Phi^+(t) = a(t)\big[\Phi^-(t) + R_2^{-1}(t)\chi^-(t)\big],$$
$$\Psi^+(t) = [a^{'}(t)]^{-1}\big[\Psi^-(t) + R_2'(t)(\chi^{1-}(t))^{-1}\big].$$

It follows from these equalities that

$$\Psi^{'+}(t)\Phi^+(t) = \big[\Psi^{'-}(t) + (\chi^-(t))^{-1}\big]\big[\Phi^-(t) + \chi^-(t)\big].$$

Consider the matrix

$$Q(z) = \begin{cases} \psi'(z)\varphi(z), & z \in D^+, \\ \big[\psi'(z) + \chi^{-1}(z)\big]\left[\varphi(z) + \chi(z)\right], & z \in D^-. \end{cases}$$

It is evident that $Q(z) \in E_1^{\pm}(\Gamma)$, $Q(\infty) = I$. Therefore, $Q(z) \equiv I$ and

$$[\varphi(z)]^{-1} = \psi^{'}(z), \quad z \in D^+,$$
$$[\varphi(z) + \chi(z)]^{-1} = \psi^{'}(z) + \chi^{-1}(z), \quad z \in D^-.$$

Consequently, the matrix

$$\omega(z) = \begin{cases} \varphi(z), & z \in D^+, \\ \varphi(z) + \chi(z), & z \in D^-, \end{cases}$$

has the following properties:

$$\omega(z) \in E_p^{\pm}(\Gamma, \rho), \quad \omega^{-1}(z) \in E_q^{\pm}(\Gamma, \rho^{1-q}),$$

and the matrix

$$\Phi(z) = \begin{cases} R_1(z)\omega(z), & z \in D^+, \\ R_2^{-1}(z)\omega(z), & z \in D^-, \end{cases} \tag{3.22}$$

is suitable for the preparation of the canonical matrix.

Now we shall show this. First cite the following auxiliary propositions.

Lemma 3.1. *Let $\varphi_1(z)$ be a quadratic matrix of order n and have the following form*:

$$\varphi_1(z) = P(z)\varphi(z)[\varphi(c)]^{-1}P^{-1}(z), \quad c \in \Gamma,$$

where $P(z)$ is a diagonal matrix, $P_{kk}(z) = 1$, $k = 1, \ldots, s$, $P_{kk}(z) = z - c$, $k = s+1, \ldots, n$ (or all $P_{kk}(z) = z - c$), $\varphi \in E_p^{\pm}(\Gamma, \rho)$, and $\varphi^{-1} \in E_q(\Gamma, \rho^{1-q})$. Then $\varphi_1(z) \in E_p^{\pm}(\Gamma, \rho)$ and $\varphi_1^{-1}(z) \in E_q^{\pm}(\Gamma, \rho^{1-q})$.

From the equalities

$$\varphi_1(z) = P(z)[\varphi(z)[\varphi(c)]^{-1} - I]P^{-1}(z) + I, \quad \varphi_1^{-1}(z) = P^{-1}(z)[\varphi(c)\varphi^{-1}(z) - I]P(z) + I$$

it follows immediately that the lemma is valid.

Lemma 3.2. *Let $\Phi(z)$ be a matrix defined by the formula*

$$\Phi(z) = \begin{cases} r_1(z)\varphi(z), & z \in D^+, \\ r_2(z)\varphi(z), & z \in D^- \end{cases}$$

(here $r_k(z)$, $k=1,2$, are rational matrices whose poles are not situated on Γ, $\det r_k(t)\neq 0$, $t\in\Gamma$, $\varphi(z)\in E_p^{\pm}(\Gamma,\rho)$, $\varphi^{-1}(z)\in E_q(\Gamma,\rho^{1-q})$). If $\Phi(z)$ satisfies the condition

$$\Phi^+(t)=a(t)\Phi^-(t),\quad t\in\Gamma, \tag{3.23}$$

where $a(t)$ is a given piecewise-continuous matrix on Γ, then there exists a rational matrix $R(z)$ such that $\Phi(z)R(z)$ is a canonical matrix for the matrix $a(t)$ of the class $E_p^{\pm}(\Gamma,\rho)$. The index of the matrix $a(t)$ of the class $E_p^{\pm}(\Gamma,\rho)$ is equal to

$$\varkappa=\frac{1}{2\pi i}\left[\arg\frac{\det r_1(t)}{\det r_2(t)}\right]_\Gamma - s, \tag{3.24}$$

where s is the order of $\det\varphi(z)$ *at infinity.*

Proof. Let us represent the matrices $r_k(z)$, $k=1,2$, in the following form:

$$r_k(z)=P_k^{(1)}(z)Q_k(z)P_k^{(2)}(z)/\lambda_k(z),$$

where $\lambda_k(z)$ are polynomials, $P_k^{(1)}(z)$ and $P_k^{(2)}(z)$ are polynomial matrices with nonzero, constant determinants, $Q_k(z)$ is a diagonal polynomial matrix, and the polynomial $Q_k^{s+1,s+1}$ is divisible by the polynomial $Q_k^{s,s}$.

Represent the polynomial λ_k and the matrix Q_k in the following form:

$$\lambda_k(z)=\lambda_k^{(1)}(z)\lambda_k^{(2)}(z),\quad Q_k(z)=Q_k^{(1)}(z)Q_k^{(2)}(z),$$

where the polynomials $\lambda_k^1(z)$ (respectively, $\lambda_k^2(z)$) may have poles only in the domain D^+ (respectively, D^-), and the elements of the main diagonal of the matrix $Q_k^{(1)}(z)$ (respectively, $Q_k^{(2)}(z)$) may have zeros only in the domain D^+ (respectively, D^-).

Write the matrix $\Phi(z)$ in the following form:

$$\Phi(z)=\begin{cases}\dfrac{P_1^{(1)}(z)}{\lambda_1^{(1)}(z)}q_1(z)\Psi(z), & z\in D^+,\\[2mm] \dfrac{P_2^{1}(z)}{\lambda_2^{(2)}(z)}q_2(z)\Psi(z), & z\in D^-,\end{cases}$$

where the following notation is introduced:

$$\Psi(z)=\begin{cases}\dfrac{Q_1^{(2)}(z)P_1^{(2)}(z)}{\lambda_1^{(2)}(z)}\varphi(z), & z\in D^+,\\[2mm] \dfrac{Q_2^{(1)}(z)P_2^{(2)}(z)}{\lambda_2^{(1)}(z)}\varphi(z), & z\in D^-,\end{cases}$$

$$q_k(z)=Q_k^{(k)}(z)=\operatorname{diag}(q_k^1,\dots,q_k^n),\quad k=1,2.$$

It is evident that

$$\Psi(z)\in E_p^{\pm}(\Gamma,\rho),\quad \Psi^{-1}(z)\in E_q^{\pm}(\Gamma,\rho^{1-q}).$$

Consider the matrix

$$\Phi_1(z)=\frac{\lambda_1^{(1)}(z)\lambda_2^{(2)}(z)}{q_1'(z)q_2'(z)},$$

$$\Phi(z)=\begin{cases}P_1(z)[q_1'(z)]^{-1}q_1(z)\Psi(z), & z\in D^+,\\ P_2(z)[q_2'(z)]^{-1}q_2(z)\Psi(z), & z\in D^-.\end{cases}$$

$$P_1=\lambda_2^2P_1^{(1)}/q_2^1(z),\quad P_2=\lambda_1^1P_2^{(1)}/q_1^1(z).$$

It is clear that $\Phi(z)$ satisfies the boundary condition (3.23).

Denote by c a zero of the polynomial $q_1^2(z)/q_1^1(z)$ (if such exist) and consider the matrix

$$\Phi_2(z) = \Phi_1(z)[\Psi(c)]^{-1}M^{-1}(z) = \begin{cases} P_1(z)(q_1'(z))\Psi(z)[\Psi(c)]^{-1}M^{-1}(z), & z \in D^+, \\ P_2(q_2'(z))^{-1}q_2(z)\Psi(z)[\Psi(c)]^{-1}M^{-1}(z), & z \in D^-, \end{cases}$$

where $M(z) = \mathrm{diag}[1, z-c, \dots, z-c]$. It is evident that $\Phi(z)$ also satisfies the boundary condition (3.23). If we continue this process, then we will get the solution of the homogeneous problem (3.23), the determinant of which is not equal to zero in the domains D^+, D^-. Consequently we obtain the normal matrix of the class $E_p^{\pm}(\Gamma, \rho)$. Bringing this matrix to the normal form at infinity (for this we shall multiple it by the corresponding polynomial matrix from the right) we get the canonical matrix. □

Tracing the construction of the normal matrix, we see that formula (3.24) is valid. If we apply this formula to the matrix (3.20), we will obtain the index for the problem (3.19) of the class $E_p^{\pm}(\Gamma, \rho)$ (if the corresponding conditions are fulfilled) as

$$\varkappa = \frac{1}{2\pi}\{\arg\det[a_1(t)a_2(t)]\}_\Gamma + \varkappa_\Lambda,$$

where $\varkappa_\Lambda$ is the index of the matrix $\Lambda(t)$ of the class $E_p^{\pm}(\Gamma, \rho)$. Thus we have the following theorem.

Theorem 3.2. *Let $a(t)$ be a piecewise-continuous nonsingular matrix with points of discontinuity t_k, $k = 1, \dots, r$, and let λ_{kj}, $k = 1, \dots, r$, $j = 1, \dots, n$, be the roots of the equation*

$$\det[a^{-1}(t_{k-0})a(t_{k+0}) - \lambda I] = 0.$$

Denote $\mu_{kj} = \arg\lambda_{kj}/2\pi$, $0 \le \arg\lambda_{kj} < 2\pi$.

If the inequalities

$$\frac{1+\nu_k}{p} \neq \mu_{kj} \tag{3.25}$$

are fulfilled, then there exists the canonical matrix of the problem (3.19) of the class $E_p^{\pm}(\Gamma, \rho)$, and the index of the matrix $a(t)$ is calculated by the formula

$$\varkappa = \frac{1}{2\pi}\left[\arg\frac{\det a(t)}{\prod\limits_{k=1}^{r}(t-z_0)^{\sigma_k}}\right]_\Gamma, \tag{3.26}$$

where $\sigma_k = \sum\limits_{j=1}^{r}\rho_{kj}$,

$$\begin{aligned} 1 < \mathrm{Re}\,\rho_{kj} \le 0 \quad &\text{if} \quad \mu_{kj} < \frac{1+\nu_k}{p}, \\ 0 \le \mathrm{Re}\,\rho_{kj} < 1 \quad &\text{if} \quad \mu_{kj} > \frac{1+\nu_k}{p}, \end{aligned} \qquad \rho_{kj} = -\frac{1}{2\pi}\ln\lambda_{kj}.$$

Formula (3.26) is analogous to the formula mentioned in [32, p. 18]. Consider now the nonhomogeneous problem. Denote by $\chi(z)$ the canonical matrix of the class $E_p^{\pm}(\Gamma, \rho)$. By virtue of Lemmas 1.1 and 1.2, the problem (3.18) is solvable in the class $E_p^{\pm}(\Gamma, \rho)$, and solutions of this class are given by the formula

$$\Phi(z) = \frac{\chi(z)}{2\pi i}\int \frac{[\chi^+(t)]^{-1}b(t)dt}{t-z} + \chi(z)P(z), \tag{3.27}$$

where $P(z)$ is an arbitrary polynomial vector.

We look now for the solutions of (3.19) vanishing at infinity. Without loss of generality, it is possible to assume that the partial indices $\varkappa_1, \varkappa_2, \dots, \varkappa_n$ are situated in the decreasing order: $\varkappa_1 \ge \varkappa_2 \ge \dots \ge \varkappa_n$. For this purpose, it is enough to change the position of the columns, i.e., to multiply

$\chi(z)$ from the right by a constant nonsingular matrix. Let $\varkappa_1 \geq \cdots \geq \varkappa_m \geq 0 > \varkappa_{m+1} \geq \cdots \geq \varkappa_n$, $\lambda = \varkappa_1 + \varkappa_2 + \cdots + \varkappa_n$, and $\mu = -(\varkappa_{m+1} + \cdots + \varkappa_n)$.

Introduce the following notation:

$$[\chi^+(t)]^{-1}b(t) = (b_1, \dots, b_n), \quad P(z) = (P_1, \dots, P_n);$$

denote also the columns of the canonical matrix by $\chi^1(z), \dots, \chi^n(z)$. It is possible to write the formula (3.16) in the form

$$\Phi(z) = \sum_{k=1}^{n} \chi^k(z) \left[\frac{1}{2\pi i} \int_\Gamma \frac{h_k(t)dt}{t-z} + P_k(z) \right]. \tag{3.28}$$

Expanding the Cauchy-type integral (3.28) in the neighborhood of the point $z = \infty$,

$$\int_\Gamma \frac{h_k(t)dt}{t-z} = -\sum_{s=0}^{\infty} \frac{1}{z^{s+1}} \int_\Gamma t^s h_k(t)dt,$$

we obtain that for the existence of the desired solution it is necessary and sufficient that the free term $b(t)$ satisfy the $\mu = -\sum\limits_{k=m+1}^{n} \varkappa_k$ conditions

$$\int_\Gamma t^s h_k(t)dt = 0, \quad s = 0, 1, \dots, -\varkappa_{k-1}, \quad k = m+1, \dots, n, \tag{3.29}$$

and when these conditions are fulfilled, the general solution of the desired form is given by the formula (3.27), in which we assume that

$$P_k(z) = P_{\varkappa_{k-1}}(z),$$

where $P_\alpha(z)$ denotes an arbitrary polynomial of order α; $P_\alpha(z) \equiv 0$ if $\alpha < 0$. The union of the conditions (3.28) can be written in the form of one relation:

$$\int_\Gamma q(t)h(t)dt = 0 \quad \text{or} \quad \int_\Gamma q(t)[\chi^+(t)]^{-1}h(t)dt = 0, \tag{3.30}$$

where $q(t)$ is defined by the formula

$$q(t) = (q_{-\varkappa_1-1}, \dots, q_{-\varkappa_n-1});$$

q_α are arbitrary polynomials of order α ($q_\alpha = 0$ in case $\alpha < 0$). The condition (3.30) can be rewritten in the form

$$\int_\Gamma h'(t)[\chi'^+(t)]^{-1}q'(t)dt = 0. \tag{3.31}$$

Note that the expression $[\chi'^+(t)]^{-1}q'(t)$ in (3.30) is a boundary value of the general solution from the domain D^+ of the adjoint homogeneous problem

$$\Psi^+(t) = [a'(t)]^{-1}\Psi^-(t) \tag{3.32}$$

of the class $E^\pm_{q,0}(\Gamma, \rho^{1-q})$. Therefore we get the following theorem.

Theorem 3.3. *If the conditions* (3.22) *are fulfilled, then for the problem* (3.19) *to be solvable in the class* $E^\pm_{p,0}(\Gamma, \rho)$ *it is necessary and sufficient that the conditions*

$$\int_\Gamma h(t)\Psi^+(t)dt = 0$$

be fulfilled, where $\Psi(z)$ *is an arbitrary solution of the adjoint homogeneous problem* (3.30) *of the class* $E^\pm_{q,0}(\Gamma, \rho^{1-q})$.

Let $l(l')$ be a number of linear independent solutions of the homogeneous problem (3.18) (of the homogeneous problem (3.30)) of the class $E_p^{\pm}(\Gamma,\rho)$ (of the class $E_q^{\pm}(\Gamma,\rho^{1-q})$. Then $l-l'=\varkappa$, where $\varkappa$ is the index of the matrix $a(t)$ of the class $E_p^{\pm}(\Gamma,\rho)$.

Remark 3.2. If $\chi(z)$ is a canonical matrix of the problem (3.18) of the class $E_p^{\pm}(\Gamma,\rho)$, then $\chi(z)$ is a canonical matrix of the same problem of the class $E_{p+\varepsilon}(\Gamma,\rho_\eta)$, $\rho_\eta=\Pi|t-t_k|^{\nu_k+\eta_k}$ if ε and η_k are sufficiently small numbers.

Remark 3.3. For the boundary-value problem (3.18) the following proposition is valid: if $a(t),b(t)\in H(\Gamma)$, then the solution of this problem of an arbitrary class $E_p^{\pm}(\Gamma,\rho)$ is Hölder-continuous in the closures $\overline{D}^+$ and $\overline{D}^-$ (except perhaps the point $z=\infty$ if the solution have the pole there). If $a(t),b(t)\in H_0(\Gamma)$), then the solution of the problem of an arbitrary class consists of piecewise-holomorphic vectors; they are continuously extendable on all points of Γ, except perhaps the points of discontinuity of $a(t)$ and $b(t)$.

4. Stability of Partial Indices

The partial indices of a continuous matrix are unstable in general. The necessary and sufficient stability condition is the following condition:

$$\varkappa_1-\varkappa_n\le 1,$$

where $\varkappa_1$ (respectively, $\varkappa_n$) is the greatest (respectively, smallest) among the partial indices (see [5, 6, 14, 21]).

Consider the problem of stability of the partial indices of a piecewise-continuous matrix. Let the matrix $a(t)\in C_0(\Gamma,t_1,\dots,t_2)$, $\inf|\det a(t)|>0$.

Let the matrix-function $g(t)$ of the class $C_0(\Gamma,t_1,\dots,t_r)$ satisfies the following conditions:

(a) $g(c\pm 0)=a(c\pm 0)$, c is an arbitrary singular point of the matrix a,

(b) $\sup|a(t)-g(t)|\le\varepsilon$; for small ε we shall say that $g(t)$ is close to $a(t)$.

It is evident that if the Noetherian conditions (3.22) for the matrix $a(t)$ are fulfilled, then these conditions are fulfilled also for the matrix $g(t)$, and we may speak of the partial indices of $g(t)$.

Let $\chi(z)$ be a canonical matrix of the class $E_p^{\pm}(\Gamma,\rho)$, and let the vector Φ be some solution of the class $E_{p,0}^{\pm}(\Gamma,\rho)$ of the boundary-value problem

$$\Phi^+(t)=g(t)\Phi^-(t),\quad t\in\Gamma. \tag{4.1}$$

Rewrite (4.1) in the form

$$\begin{gathered}[\chi^+(t)]^{-1}\Phi^+(t)=[\chi^-(t)]^{-1}\Phi^-(t)+F(t),\\ F(t)=[\chi^+(t)][g(t)-a(t)]\Phi(t).\end{gathered} \tag{4.2}$$

If the partial indices of the matrix $a(t)$ are nonpositive, then it follows from (4.2) that

$$[\chi(z)]^{-1}\Phi(z)=\frac{1}{2\pi i}\int\limits_\Gamma\frac{F(t)dt}{t-z},$$

$$\Phi^-(t_0)=-\frac12 a^{-1}(t_0)[g(t_0)-a(t_0)]\Phi^-(t_0)+\frac{\chi^-(t_0)}{2\pi i}\int\limits_\Gamma\frac{[\chi^+(t)]^{-1}[g(t)-a(t)]\Phi^-(t)dt}{t-t_0}.$$

It follows from the last equality that

$$\|\Phi^-\|_{L_p(\Gamma,\rho)}\le B\sup|g(t)-a(t)|\|\Phi^-(t)\|_{L_p(\Gamma,\rho)}, \tag{4.3}$$

where B is constant.

If $\sup|g(t)-a(t)|$ is sufficiently small, then from inequality (4.3) it follows that $\Phi^-(t)\equiv 0$ and $\Phi(z)\equiv 0$. Therefore, if the matrix has nonpositive partial indices, then the boundary-value problem

(4.1) has nontrivial solutions of the class $E^{\pm}_{p,0}(\Gamma,\rho)$ when the matrices $a(t)$ and $g(t)$ are close, and hence such matrices $g(t)$ have also nonpositive indices.

Let now the matrix $a(t)$ have arbitrary partial indices

$$\varkappa_1 \geq \cdots \geq \varkappa_n,$$

and $g(t)$ be a matrix close to $a(t)$ with the partial indices

$$\eta_1 \geq \cdots \geq \eta_k.$$

It is clear that the matrix

$$a_1(t)(t-b)^{-\varkappa_1}a(t)[g_1(t)=(t-b)^{-\varkappa_1}g(t)],$$

where b is a fixed point inside of Γ, has the numbers $\varkappa_k-\varkappa_1 \leq O(\eta_k-\eta_1)$ as the partial indices. Hence, when the matrices $a(t)$ and $g(t)$ are sufficiently close, then the partial indices of the matrix $g_1(t)$ will be nonpositive and therefore $\eta_1 \leq \varkappa_1$.

Passing from the matrices a and g to the matrices $(a')^{-1}$ and $(g')^{-1}$ and to the classes $E^{\pm}_p(\Gamma,\rho)$ and $E^{\pm}_q(\Gamma,\rho^{1-q})$, we get $\eta_n \geq \varkappa_k$,

$$\varkappa_1 \geq \eta_1 \geq \cdots \geq \eta_n \geq \varkappa_n. \tag{4.4}$$

The relations (4.4) imply that if the partial indices of the matrix $a(t)$ satisfy the condition $\varkappa_1-\varkappa_n \leq 1$, then for all sufficiently close matrices

$$\eta_k=\varkappa_k, \quad k=1,\ldots,n.$$

Due to [5] it follows that if $\varkappa_1-\varkappa_n \geq 2$, then the partial indices are unstable.

Let

$$\varkappa_1=\cdots=\varkappa_s>\varkappa_{s+1}\geq\cdots\geq\varkappa_n$$

be the partial indices of the matrix $a(t)$ of the class $E^{\pm}_p(\Gamma,\rho)$.

Consider the case where the matrix $a(t)$ has only one point of discontinuity $c\in\Gamma$; this restriction is not essential and is made because of the simplicity of the formulas.

Construct the sequence of matrices $a_m(t)\in H^1_0(\Gamma,C)$, $a_m(c\pm 0)=a(c\pm 0)$ convergent to the matrix $a(t)$:

$$\sup_t |a_m(t)-a(t)| \to 0, \quad m\to 0.$$

Consider two possible cases:

(a) the partial indices of $a_m(t)$ coincide with the partial indices starting from some m_0;
(b) when case (a) is not possible.

In case (b), the partial indices are unstable. Therefore, we consider case (a).

As is known, the partial indices of the matrix $a(t)$ of the class $E^{\pm}_p(\Gamma,\rho)$ $(\rho=|t-c|^{\nu})$ coincide with the partial indices of the Hölder-continuous matrix

$$A_m(t)=Y_+^{-1}a_mY_-(t),$$

where

$$\begin{aligned}
&Y_+(z)=AU[u_1]\chi_1(z), && z\in D^+,\\
&Y_-(z)=BU[u]\chi(z), && z\in D^-,\\
&\chi_1(z)=\operatorname{diag}[(z-c)^{\rho_1},\ldots,(z-c)^{\rho_n}], && \chi=\chi_1\chi_0^{-1},\\
&\chi_0(z)=\operatorname{diag}[(z-z_0)^{\rho_1},\ldots,(z-z_0)^{\rho_n}], && z_0\in D^+,\\
&-\frac{1+\nu}{p}<\operatorname{Re}\rho_n<1-\frac{1+\nu}{p}, && \rho_k=\frac{1}{2\pi i}\ln\lambda_k,
\end{aligned}$$

A and B are constant nonsingular matrices.

The λ_k are the roots of the equation $\det(a^{-1}(c+0)a(c-0)-\lambda I)=0$,

$$u_1 = \frac{1}{2\pi i}\ln(z-c), \quad u_2 = \frac{1}{2\pi i}\ln\frac{z-c}{z-z_0},$$

and $u(\xi)$ is a definite polynomial matrix of ξ. These matrices are defined in [32, Sec. 18].

Represent the matrix A_m in the form (see [32, Sec. 7])

$$A_m = \chi_m^+ \Lambda \chi^-,$$

where $\chi_m^{\pm}(t)$ are Hölder-continuous matrices and

$$\Lambda(t) = \mathrm{diag}[t^{\varkappa_1}, t^{\varkappa_2}, \dots, t^{\varkappa_{n-1}}]$$

(we suppose that $O \in D^+$).

Consider the matrix

$$A_m^{\varepsilon} = A_m(t) + \varepsilon(t-c)q(t),$$

$$q(t) = \begin{pmatrix} 0 & t^{\varkappa_2} & \dots & 0 \\ t^{\varkappa_2 - 2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}. \tag{4.5}$$

Let $\tilde{\varkappa}_1 \geq \tilde{\varkappa}_2 \geq \dots \geq \tilde{\varkappa}_n$ be the partial indices of A_m^{ε}. It is not difficult to check that for sufficiently small ε for the matrix $A_m^{\varepsilon}(t)$ we will have

$$\tilde{\varkappa}_s = \varkappa_s - 1.$$

It follows from (4.5) that

$$a_m^{\varepsilon} = Y_+ A_m^{\varepsilon} Y_-^{-1} = a_m + \varepsilon(t-c)Y_+ q Y_-^{-1},$$

and hence

$$a_m^{\varepsilon}(c \pm 0) = a_m(c \pm 0),$$

$$\sup|a_m^{\varepsilon} - a_m| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

The sequence $a_m^{\varepsilon_m}(t)$ ($\varepsilon_m \to 0$) converges to the matrix $a(t)$ with respect to the above mentioned norm; therefore, the condition $\varkappa_1 - \varkappa_n \leq 1$ is not only sufficient but also necessary for the partial indices to be stable.

REFERENCES

1. V. M. Adukov, “On the factorization of analytic matrix functions,” *Teor. Mat. Fiz.*, **118**, No. 3, 255–263 (1999).
2. V. M. Adukov, “Fractional and Wiener–Hopf factorizations,” *Linear Algebra Appl.*, **340**, 199–213 (2002).
3. V. M. Adukov, “Wiener–Hopf factorization of piecewise meromorphic matrix functions,” *Mat. Sb.*, **200**, No. 8, 3–24 (2009).
4. D. V. Anosov and A. A. Bolibruch, *The Riemann–Hilbert problem*, Aspects of Mathematics, Vieweg, Braunschweig, Wiesbaden (1994).
5. B. Bojarski, “Stability of the Hilbert problem for a holomorphic vector,” *Bull. Georgian Acad. Sci.*, **21**, 391–398 (1958).
6. B. Bojarski, “Connection between Complex and global analysis: analytical and geometrical aspects of the Riemann–Hilbert transition problem,” in: *Complex Analysis: Methods and Applications*, Berlin, A.V. (1983).
7. A. A. Bolibruch, “The Riemann–Hilbert problem,” *Russ. Math. Surv.*, **45** , No. 2, 1–47 (1990).
8. A. A. Bolibruch, “Construction of a Fuchsian equation from a monodromy representation,” *Mat. Zametki*, **48**, No. 5, 1090–1099 (1990).

9. T. Ehrhardt and I. Spitkovsky, “Factorization of piecewise constant matrix functions and systems of linear differential equations,” *Algebra Anal.*, **13**, 56–123 (2001).
10. F. D. Gakhov, *Boundary-Value Problems* [in Russian], Fizmatgiz, Moscow (1958).
11. G. Giorgadze, “Regular systems on Riemann surfaces,” *J. Math. Sci. (N.Y.)*, **118**, No. 5, 5347–5399 (2003).
12. G. Giorgadze, “G-Systems and holomorphic principal bundles on Riemann surfaces,” *J. Dynam. Control Syst.*, **8**, No. 2, 245–291 (2002).
13. G. Giorgadze and G. Khimshiashvili, “The Riemann–Hilbert problem in loop spaces,” *Dokl. Math.*, **73**, No. 2, 258–260 (2006).
14. I. Gohberg and M. G. Krein, “Systems of integral equations on semiaxis with kernel depending on the arguments difference,” *Usp. Mat. Nauk*, **13**, 3–72 (1958).
15. G. N. Khimshiashvili, “Lie groups and transmission problems on Riemann surface,” *Contemp. Math.*, **131**, 164–178 (1992).
16. G. Khimshiashvili, “Geometric aspects of Riemann–Hilbert problems,” *Mem. Differ. Equ. Math. Phys.*, **27** (2002).
17. G. S. Litvinchuk and I. M. Spitkovsky, “Factorization of measurable matrix functions,” in: *Operator Theory: Advances and Applications*, **25**, Birkhäuser, Basel (1987).
18. G. F. Manjavidze, “On approximate solution of boundary-value problems of the theory of functions of a complex variable,” *Soobshch. Akad. Nauk Gruz. SSR,* **14**, 577–582 (1953).
19. G. F. Manjavidze, “Approximate solution of boundary-value problems of the theory of analytic functions,” in: *Studies in Modern Problems of Complex Analysis* [in Russian], Moscow (1960), pp. 365–370.
20. G. F. Manjavidze, “A boundary-value problem of linear conjugacy with a piecewise continuous matrix-valued coefficient,” in: *Continuum Mechanics and Related Problems of Analysis* (1969), pp. 297–304.
21. G. Manjavidze, *Boundary-value problems with conjugation for generalized analytic functions*, preprint I. Vekua Inst. Appl. Math., Tbilisi State Univ. (1990).
22. G. F. Manjavidze and B. V. Khvedelidze, “A problem for the linear conjugate and singular integral equations with Cauchy kernel with continuous coefficients,” *Proc. A. Razmadze Math. Inst.*, **28**, 85–105 (1962).
23. G. Manjavidze and N. Manjavidze, “Boundary-value problems for analytic and generalized analytic functions,” *J. Math. Sci. (N.Y.)*, **160**, No. 6, 745–821 (2009).
24. G. Manjavidze, W. Tutschke, and H. L. Vasudeva, “A complex method for solving boundary value problems with boundary data piecewise belonging to fractional order spaces,” *Mem. Differ. Equ. Math. Phys.*, **33**, 103–120 (2004).
25. N. I. Muskhelishvili, *Singular Integral Equations*, Noordhoff, Groningen (1953).
26. J. Plemelj, *Problems in the sense of Riemann and Klein*, Intersience Publ., New York–London–Sidney (1964).
27. I. B. Simonenko, “The Riemann boundary-value problem for n pairs of functions with measurable coefficients and its application to the study of singular integrals in L_p spaces with weights,” *Izv. Akad. Nauk SSSR, Ser. Mat.*, **28**, 277–306 (1964).
28. I. B. Simonenko, “The Riemann boundary-value problem with a measurable coefficient,” *Dokl. Akad. Nauk SSSR*, **135**, 538–541 (1960).
29. I. M. Spitkovskii and A. M. Tashbaev, “Factorization of certain piecewise constant matrix functions and its applications,” *Math. Nachr.*, **151**, 241–261 (1991).
30. I. M. Spitkovskii and A. M. Tashbaev, “Factorization of piecewise constant matrix functions with 3 points of discontinuity in the classes $L_{p,\rho}$ and some of its applications,”*Dokl. Akad. Nauk SSSR*, **307**, No. 2, 291–296 (1989).
31. A. A. Ryabov, “Splitting types of vector bundles constructed from the monodromy of a given Fuchsian system,” *Zh. Vychisl. Mat. Mat. Fiz.*, **44**, No. 4, 640–648 (2004).

32. N. P. Vekua, *Systems of Singular Integral Equations and Certain Boundary-Value Problems* [in Russian], Nauka, Moscow (1970).
33. I. V. Vyugin, "On constructive conditions for the solvability of the Riemann–Hilbert problem," *Mat. Zametki*, **77**, No. 5, 643–655 (2005).

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