Thus the number of linearly independent solutions of the problem (6.21) in the class $L_q(\Gamma, \rho^{1-q})$ over the field of real numbers and the number of linearly independent solutions of the problem (6.2) in the class $E^{\pm}_{q,o}(\Gamma, \rho^{1-q})$ are the same. □

Theorem 6.1. *If $Q(t) \neq 0$ then the index of the problem* (6.1) *in the class $E^{\pm}_{p,o}(\Gamma, \rho)$ is equal to the index of the equation* (6.15) *of the class $L_p(\Gamma, \rho)$ (under the condition, that $1 + \alpha_k \neq p\,\mu_j^{(k)}$, where $\mu_j^{(k)} = \frac{\arg \lambda_j^{(k)}}{2\pi}$, $0 \leqslant \arg \lambda_j^{(k)} < 2\pi$, $\lambda_j^{(k)}$ are the roots of the equations: $\det[H(a_k) - \lambda I] = 0$ or $\det[H^{-1}(b_k) - \lambda I] = 0$ for odd and even k correspondingly; $H(t) = [a(t) + b(t)]^{-1}[a(t) - b(t)]$, the necessary and sufficient solvability conditions for the problem* (6.1) *in the class $E^{\pm}_{p,o}(\Gamma, \rho)$ have the form* (6.11).

Remark 6.1. If $A_+(t) = A_-(t) = A(t)$, then $Q(t) \equiv 0$. In this case instead of the representation (6.13) we shall use the following representation

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{A^{-1}(t)\mu(t)}{t - z}\, dt + F(z), \quad \text{where} \quad \mu = \mu^+ + \mu^-.$$

The equation (6.15) will have the form:

$$\frac{1}{\pi i} \int\limits_{\Gamma} \frac{K(t_0, t)\mu(t)}{t - t_0}\, dt = g(t_0),$$

where

$$K(t_0, t) = \frac{i}{2}\Big[A(t_0)A^{-1}(t) + \overline{A(t_0)}\,\overline{A^{-1}(t_0)} h(t_0, t)\Big].$$

Analogously we obtain the following result

Theorem 6.2. *If $A_+(t) = A_-(t)$ then the index of the problem* (6.1) *coincides with the index of the operator $\int\limits_{\Gamma} \frac{\mu(t)}{t - t_0}\, dr$ of the class $L_p(\Gamma, \rho)$ under the condition that $2(1 + \alpha_k) \neq p$. In this case the necessary and sufficient solvability conditions have the form* (6.11).

Remark 6.2. If $Q(t) = 0$ in some points of Γ then introduce a new desired vector by the formula $\Phi(z) = \Lambda(z)\varphi(z)$, where $\Lambda = \operatorname{diag}[e^{\omega_1(z)}, \dots, e^{\omega_k(z)}]$,

$$\omega_k(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{h_k(t)}{t - z}\, dt, \quad h_k(t) \in H(\Gamma), \quad h_k(a_j) = h_k(b_j) = 0.$$

It is evident,that $\varphi(z) \in E^{\pm}_{p,0}(\Gamma, \rho)$. The matrices $A_{\pm}(t)$ are replaced by the matrices $A_{\pm}(t)\Lambda^{\pm}(t)$. Under the fulfillment of some conditions one may select the functions $h_k(t)$ such that the function $Q(t) \neq 0$ on Γ.

7. Riemann–Hilbert–Poincaré Type Problems on a Cut Plane

In the theory of differential value problems for analytic functions, i.e. boundary value problems containing boundary values of the derivatives of desired functions, very important role plays the integral representation of the analytic functions constructed by I. Vekua [78]. Using this representation we can investigate the Riemann–Hilbert–Poincaré boundary value problem and some of its generalizations. We perform our investigation in two steps: first we consider the Riemann–Hilbert–Poincaré problem on a cut plane and then a general differential boundary value problem.

7.1. Riemann–Hilbert–Poincaré problem on a cut plane. Let S be a complex plane cut along simple arc $c_1c_2 \equiv \Gamma$ of the class C^2_{α}. First consider the following problem: find a vector $\Phi(z) = (\Phi_1, \dots, \Phi_n)$ holomorphic in S and satisfying the boundary condition

$$\operatorname{Re}\big[a_{\pm}(t)\Phi'_{\pm}(t) + b_{\pm}\Phi_{\pm}(t)\big] = f_{\pm}(t), \quad t \in \Gamma, \tag{7.1}$$

where assume that the given matrices $a_\pm(t), b_\pm(t) \in H(\Gamma)$, the given real vectors $f_\pm(t) \in H^*(\Gamma)$. (Note, that the notation $A \in K$, where A is a matrix and K is some class of functions, means that every element $A_{\alpha\beta}$ of A belongs to K.)

The unknown vector $\Phi(z)$ and its derivative $\Phi'(z)$ are continuously extendable from both sides on Γ, $\Phi'(z)$ is supposed to satisfy the following estimate

$$|\Phi'_j(z)| \leqslant const|z - c_k|^{-\alpha}, \ \ 0 \leqslant \alpha < 1 \ \ (j = 1, 2, \dots, n) \tag{7.2}$$

in a neighbourhood of each point c_k.

Let

$$z = \omega(\xi) \tag{7.3}$$

be conformal mapping of the domain S onto the unit circle $D(|\xi| < 1)$ with the boundary γ, let the points c_k turn into the points d_k of the circumference γ, one side of Γ onto the part of circumference γ_1 and another side onto γ_2. Using results of S.Warshavski [80] about the properties of the function (7.3) and its inverse function $\xi = \eta(z)$, we get with respect to a new function

$$\Psi(\xi) \equiv \Phi[\omega(\xi)] \tag{7.4}$$

the Riemann–Hilbert–Poincaré boundary value problem with the boundary condition

$$\text{Re}\,\big[A(\sigma)\Psi'(\sigma) + B(\sigma)\Psi(\sigma)\big] = F(\sigma), \ \ \sigma \in \gamma, \tag{7.5}$$

where $A(\sigma), B(\sigma), F(\sigma) \in H * (\gamma)$ and they are expressed with the help of given matrices $a_\pm(t)$, $b_\pm(t)$ and the vectors $f_\pm(t)$.

Applying the integral representation of I. Vekua [78] (see also [53, 55]) we seek solution $\Psi(\xi)$ in the following form

$$\Psi(\xi) = \int\limits_\gamma \mu(\sigma) \ln\Big(1 - \frac{\xi}{\sigma}\Big)\, ds + \int\limits_\gamma \mu(\sigma)\, ds + iC, \tag{7.6}$$

where $C = (C_1, \dots, C_n)$ is a real constant vector, $\mu(\sigma)$ is a real vector from class $H^*(\gamma)$, under $\ln(1 - \frac{\xi}{\sigma})$ we mean the branch of this function which is equal to zero at point $\xi = 0$, $\mu(\sigma)$ and C are uniquely defined by $\Phi(z)$.

Inserting the representation (7.6) in the boundary condition (7.5) we get for vector μ the following singular integral equation

$$E(\sigma_0)\mu(\sigma_0) + \int\limits_\gamma H(\sigma_0, \sigma)\mu(\sigma)\, ds = F(\sigma_0) - k(\sigma_0)C, \tag{7.7}$$

where

$$k(\sigma) = \text{Re}\,[iB(\sigma)], \quad E(\sigma) = \text{Re}\,\big[A(\sigma)\overline{\sigma}\big],$$
$$H(\sigma_0, \sigma) = \frac{I_m[A(\sigma_0)\overline{\sigma}]}{\Pi(\sigma - \sigma_0)} + \frac{H_0(\sigma_0, \sigma)}{|\sigma - \sigma_0|^\delta}, \quad H_0(\sigma_0, \sigma) \in H(\gamma \times \gamma), \ \ 0 \leqslant \delta < 1. \tag{7.8}$$

From here we have

$$S = A(\sigma_0)\overline{\sigma}_0, \quad D = \overline{A(\sigma_0)}\sigma_0, \quad G = S^{-1}D = \sigma_0^2[A(\sigma_0)]^{-1}\overline{A(\sigma_0)}.$$

For the equation (7.7) to be Fredholm the following condition has to be fulfilled

$$\inf|\det A(t)| > 0 \quad \big(\det a_+(t) \neq 0, \ \ \det a_-(t) \neq 0\big). \tag{7.9}$$

If we denote by $g(\sigma)$ matrix $[A(\sigma)]^{-1}\overline{A(\sigma)}$ and by λ_{kj} the roots of equation $\det[g^{-1}(d_k + 0)g(d_k - 0) - \lambda I] = 0$, then the index of equation (7.7) in the class h_0 is equal to $2n + \varkappa$, where

$$\varkappa = \frac{1}{2\pi}\left[\arg \frac{\det g(\sigma)}{\prod\limits_{k=1}^{2} \det \overset{k}{\underset{0}{\mathrm{X}}}(\sigma)}\right]_\Gamma \tag{7.10}$$

(see [52]), where

$$\det \overset{k}{\underset{0}{X}}(\sigma) = (\sigma - \xi_0)^{q_k}, \quad q_k = \sum_{j=1}^{n} \rho_{k_j}, \quad \rho_{k_j} = \frac{1}{2\pi i} \ln \lambda_{k_j}, \quad -1 < \operatorname{Re} \rho_{k_j} \leqslant 0. \tag{7.11}$$

The necessary and sufficient solvability conditions for the equation (7.7) in the class h_0 have the following form

$$\int_\gamma [F(\sigma) - k(\sigma)] c \overset{j}{\nu}(\sigma)\, ds = 0, \quad j = 1, 2, \dots, l', \tag{7.12}$$

where $\overset{j}{\nu}(\sigma)$ is a complete system of almost bounded linearly independent solutions of the adjoint homogeneous equation.

In order to calculate the index of the problem (7.1) we may assume that $B \equiv 0$. Then $k(\sigma) \equiv 0$ and therefore the index will be

$$n + l - l' = n + (2n + \kappa) = 3n + \kappa, \tag{7.13}$$

where κ is being calculated by the formula (7.10). Thus we get the following result.

Theorem 7.1. *If the condition* (7.9) *is fulfilled then the problem* (7.1) *is normally solvable in the class* $H^*(\Gamma)$. *In this case the necessary and sufficient solvability conditions in the class* h_0 *are* (7.12) *and the index in this class is calculated by* (7.13).

Problems of such kind were the subject of investigations of B. Khvedelidze [38], N. Vekua [79] and others (see, e.g., [2, 3]). The considered problem in more general case will be solved below but not in the same way. We manage reduction to singular integral equation without using conformal mapping.

Before passing to the general case let us consider the case when the boundary condition contains derivatives up to second order. In this case the problem is formulated as follows (for the sake of visuality we only consider the scalar case):

1°. Let S denote the plane of the complex variable $z = x + iy$, cut along simple are $a_1 a_2 \equiv \Gamma$ of the class c^2_α.

Consider the following boundary value problem.

Find the function $\varphi(z)$, holomorphic in S satisfying the boundary condition

$$\operatorname{Re} \sum_{k=0}^{2} \big[a^k_\pm(t)\, \varphi^{(k)}_\pm(t) \big] = f_\pm(t). \tag{7.14}$$

Assume the given functions $a^k_\pm(t)$ $(k = 0, 1, 2)$ belong to the class $H(\Gamma)$, the given real functions $f_+(t), f_-(t) \in H^*(\Gamma)$ with the points of discontinuity a_1 and a_2.

Let the desired function $\varphi(z)$ satisfies the following properties: $\varphi''(z)$, $\varphi'(z)$, $\varphi^{(0)} = \varphi(z)$ are continuously extendable from the left and from the right or Γ, except the points a_1, a_2, the boundary values $\varphi'_\pm(t)$ will have the form

$$\varphi''_\pm(t) = \lambda_\pm(t) \cdot \big[|t - a_1|\, |t - a_2| \big]^{-\delta}, \quad 0 \leqslant \delta < 2, \quad \lambda_+(t), \lambda_-(t) \in H(\Gamma). \tag{7.15}$$

First we seek the solution of the problem (7.14) which may have a pole of the first order at infinity.

Mapping conformally S onto the unit circ D $(|\zeta| < 1)$ with the boundary γ, the points c_k turn into the points $d_k \in \gamma$, one side of the arc Γ turns in to the part of circumference γ_1 and the another side in to the part $\gamma_2 \in \gamma$.

According to B. Warshavwski [80], the function $Z(\zeta) = \omega(\zeta)$ and the inverse function $\zeta = \eta(z)$ have the following properties: the function

$$\eta_k(z) = (z - a_k)^{-1/2} [\eta(z) - \eta(a_k)] \tag{7.16}$$

is continuous in the neighbourhood of the points a_k, $\eta_k(a_k) \neq 0$, and

$$\eta'_k(z) = (z - a_k)^{-1/2} \eta^0_k(z), \tag{7.17}$$

where $\eta_k^0(z)$ is continuous function in the neighbourhood of the points a_k and also $\eta_k^0(a_k) \neq 0$, the function

$$\omega_k(\zeta) = (\zeta - c_k)^{-2}[\omega(\zeta) - \omega(c_k)] \tag{7.18}$$

is continuous function in the neighbourhood of the points c_k, $\omega_k(c_k) \neq 0$ and

$$\omega'(\zeta) = (\zeta - c_k)\omega_k^0(\zeta), \tag{7.19}$$

here $\omega_k^0(\zeta)$ is continuous function in the neighbourhood of the points c_k and also $\omega_k^0(c_k) \neq 0$.

Let $\omega(\zeta) = \frac{A}{\zeta} + \omega_*(\zeta)$, where $\omega_*(\zeta)$ is holomorphic function in D ($\omega(\zeta)$ will have a simple pole in some point of D which corresponds to the point $z = \infty$ while the mapping $z = \omega(\zeta)$).

Denote

$$\varphi[\omega(\zeta)] = \psi(\zeta). \tag{7.20}$$

$\psi(\zeta)$ is holomorphic function in D except possibly at the point $\zeta = 0$, where it may have a pole of the first order. Taking into account the formulas:

$$\frac{dy}{dz} = \frac{\psi'(\zeta)}{\omega'(\zeta)}, \quad \frac{d^2y}{dz^2} = \frac{\psi''(\zeta)}{[\omega'(\sigma)]^2} - \frac{\omega''(\zeta)}{[\omega'(\zeta)]^3}\psi'(\zeta) \tag{7.21}$$

with respect to the new function $\psi(\zeta)$, we get the following boundary value problem

$$\mathrm{Re} \sum_{k=0}^{2} \left[b_k(\sigma)\psi^{(k)}(\sigma)\right] = F(\sigma), \quad \sigma \in \gamma. \tag{7.22}$$

From the formulas (7.21), (7.18), (7.19) we have

$$\psi''(\sigma) = \frac{\lambda_1(\sigma)}{|\sigma - c_1|^\rho |\sigma - c_2|^\rho}, \quad \text{where } \lambda_1(\sigma) \in H(\gamma), \quad 0 \leqslant \rho < 2, \tag{7.23}$$

and on the conversely, if the formula (7.23) holds then

$$\left(\frac{d^2Y(t)}{dz^2}\right)_\pm = \frac{\lambda_2^\pm(t)}{|t - a_1|^\nu |t - a_2|^\nu}, \quad \text{where } \lambda_2^\pm(t) \in H(\Gamma), \quad 0 \leqslant \nu < 2, \tag{7.24}$$

we may rewrite the boundary condition (7.22) in the following way:

$$\mathrm{Re} \sum_{k=0}^{2} \left[b_k(\sigma)h(\sigma)\psi^{(k)}(\sigma)\right] = F(\sigma)h(\sigma), \tag{7.25}$$

where $h(\sigma) = |\sigma - c_1|^3 |\sigma - c_2|^3$.

It is easy to see that $b_0(\sigma)h(\sigma) \in H(\gamma)$, $b_1(\sigma)h(\sigma)$ and $b_2(\sigma)h(\sigma) \in H_0(\gamma)$.

Consider the problem (7.25) first in case, when $a_\pm^1(t) = a_\pm^0(t) = 0$ then the boundary condition (7.25) takes the form:

$$\mathrm{Re} \left\{ \frac{A_2^0(\sigma)h(\sigma)}{[\omega'(\sigma)]^3} \left[\omega'(\sigma)\psi''(\sigma) - \omega''(\sigma)\psi'(\sigma)\right] \right\} = G(\sigma), \tag{7.26}$$

where $A_2^0(\sigma) = a_\pm^2[\omega(\sigma)]$, $\sigma \in \gamma_1$ and $\sigma \in \gamma_2$, $G(\sigma) = F(\sigma)h(\sigma)$.

Denote

$$\omega'(\zeta)\psi''(\zeta) - \omega''(\zeta)\psi'(\zeta) = \frac{\Omega(\zeta)}{\zeta^5}. \tag{7.27}$$

Then $\Omega(\zeta)$ is holomorphic function in D the boundary values of which on γ is the function of the class $H^*(\gamma)$. With respect to the function $\Omega(\zeta)$ we obtain the Riemann–Hilbert problem:

$$\mathrm{Re}\left[A(\sigma)\Omega(\sigma)\right] = G(\sigma), \tag{7.28}$$

where

$$A(\sigma) = \frac{A_2^0(\sigma)h(\sigma)}{\sigma^5[\omega'(\sigma)]^3} \in H_0(\gamma) \ \ (\text{if } \ a_\pm^2(t) \neq 0, \ \text{ then } \ A(\sigma) \neq 0).$$

Note that, if $f_\pm(t)=\frac{f^0_\pm(t)}{|t-a_1|^\varepsilon|t-a_2|^\varepsilon}$, then

$$G(\sigma)=\begin{cases} f^0_+[\omega(\sigma)]\,|\sigma-c_1|^{3-2\varepsilon}|\sigma-c_2|^{3-2\varepsilon}h^0_+(\sigma), & \sigma\in\gamma_1,\\ f^0_-[\omega(\sigma)]\,|\sigma-c_1|^{3-2\varepsilon}|\sigma-c_2|^{3-2\varepsilon}h^0_-(\sigma), & \sigma\in\gamma_2.\end{cases}$$

Assume that the points c_1, c_2 are non-singular. If we solve the problem (7.28), we get (in case when index κ of the class h_0 is non-negative)

$$\Omega(\zeta)=\frac{\chi(\zeta)}{2\pi i}\bigg\{\int\limits_\gamma\frac{G(\sigma)}{A(\sigma)\chi^+(\sigma)(\sigma-\zeta)}\,d\sigma+\zeta^\kappa\int\limits_\gamma\frac{\sigma^{-\kappa}G(\sigma)}{A(\sigma)\chi^+(\sigma)(\sigma-\zeta)}\,d\sigma\bigg\}-$$
$$-\frac{\zeta^\kappa\chi^+(\sigma)}{2\pi i}\int\limits_\gamma\frac{\sigma^{-\kappa}G(\sigma)}{A(\sigma)\chi^+(\sigma)}\cdot\frac{d\sigma}{\sigma}+\chi(\zeta)\sum_{k=0}^{\kappa}c_k\zeta^{\kappa-k}, \tag{7.29}$$

where c_k are arbitrary complex constants connected by the condition $c_k=\overline{c}_{\kappa-k}$, $\chi(z)$ is canonical function of the class h_0.

When $\kappa\leqslant-1$ the solution has the analogous form in case, when $\kappa\leqslant-2$ the solution's existence condition appears:

$$\operatorname{Re}\int\limits_\gamma g(\sigma)\nu_k(\sigma)\,d\sigma=0\ \ (k=0,\dots,\kappa-1),$$

where $\nu_k(\sigma)$ are definite linearly independent functions (over the real numbers field).

From (7.27) we have

$$\psi(\zeta)=D_1+D_2\omega(\zeta)+\omega(\zeta)\nu_1(\zeta)-\nu_2(\zeta), \tag{7.30}$$

where $\nu_1(\zeta)$ is primitive of the holomorphic function $\frac{\Omega(\zeta)}{\zeta^5[\omega'(\zeta)]^2}$, $\nu(\zeta)$ is primitive of the holomorphic function $\frac{\omega(\zeta)\Omega(\zeta)}{\zeta^5[\omega'(\zeta)]^2}$, D_1, D_2 are arbitrary complex constants.

2°. Consider now the problem (7.25) in general case.

It is easy to verify, that the boundary values of the function $\Phi''(\zeta)$ holomorphic in D, where

$$\Phi(\zeta)=\zeta(\zeta-c_1)(\zeta-c_2)\Psi(\zeta) \tag{7.31}$$

will be the functions of the class $H^*(\gamma)$ Therefore, it is possible to represent the function $\phi(\zeta)$ by I. Vekua formula:

$$\Phi(\zeta)=\int\limits_\gamma\mu(\sigma)\Big(1-\frac{\zeta}{\sigma}\Big)\ln\Big(1-\frac{\zeta}{\sigma}\Big)\,ds+\int\limits_\gamma\mu(\sigma)\,ds+iC, \tag{7.32}$$

where $\mu(\sigma)$ is real function of the class $H^*(\gamma)$, c is real constant $\mu(\sigma)$ and C are defined uniquely.

From (7.31) we have

$$\Psi(\zeta)=\frac{\Phi(\zeta)}{\zeta(\zeta-c_1)(\zeta-c_2)}\,. \tag{7.33}$$

It is necessary from here, that $\Phi(c_1)=\Phi(c_2)=0$ and therefore

$$\int\limits_\gamma\mu(\sigma)\Big(1-\frac{c_k}{\sigma}\Big)\ln\Big(1-\frac{c_k}{\sigma}\Big)\,ds+\int\limits_\gamma\mu(\sigma)\,ds+iC=0\ \ (k=1,2). \tag{7.34a}$$

In order for the boundary values $\Psi''(\sigma)$ to have the form (7.15) it is necessary, that $\Phi'(c_1)=\Phi(c_2)=0$, i.e.

$$\int\limits_\gamma\frac{\mu(\sigma)}{\sigma}\Big[\ln\Big(1-\frac{c_k}{\sigma}\Big)+1\Big]\,ds=0\ \ (k=1,2). \tag{7.34b}$$

Hence we obtain, that the desired function

$$\Psi(\sigma)=\frac{1}{\zeta(\zeta-c_1)(\zeta-c_2)}\Big[\int\limits_\gamma\mu(\sigma)\Big(1-\frac{\zeta}{\sigma}\Big)\ln\Big(1-\frac{\zeta}{\sigma}\Big)\,ds+\int\limits_\gamma\mu(\sigma)\,ds+iC\Big], \tag{7.35}$$

where $\mu(\sigma)$ is real function of the class $H^*(\gamma)$, C is real constant, $\mu(\sigma)$ and C are defined uniquely, in addition $\mu(\sigma)$ and C will satisfy the conditions (7.34a) and (7.34b).

If we require, that $\varphi(z)$ be holomorphic at $z=\infty_+$ then should be:

$$\int_\gamma \mu(\sigma)\,ds = C = 0. \tag{7.36}$$

Due to N. Muskhelishvili [4] it follows the formula (7.32), that $\mu(\sigma)$ satisfies Fredholm equation, which in case when γ is unit circle is the equation with degenerate (singular) kernel and it has the form:

$$\mu(\sigma_0) - \frac{1}{2\pi}\int_\gamma \Big(\frac{\sigma_0}{\sigma}+\frac{\sigma}{\sigma_0}+1\Big)\mu(\sigma)\,ds = \frac{1}{\pi}\,\mathrm{Re}\,\big(\sigma^2\Phi''(\sigma_0)\big). \tag{7.37}$$

The solution of this equation will have the form

$$\mu(\sigma_0) = g(\sigma_0) + \sum_{k=1}^{3} d_k g_k(\sigma_0), \tag{7.38}$$

$g_1(\sigma)=1$, $g_2(\sigma)=\sigma+\overline{\sigma}$, $g_3(\sigma)=i\sigma-i\overline{\sigma}$, d_k are arbitrary real constants, $g(\sigma)=\frac{1}{\pi}\,\mathrm{Re}\,(\sigma^2\Phi/(\sigma))$.

Denote

$$\mathcal{L}\mu = \int_\gamma \mu(\sigma)\Big(1-\frac{\zeta}{\sigma}\Big)\ln\Big(1-\frac{\zeta}{\sigma}\Big)\,ds, \tag{7.39}$$

then the formula (7.33) will take the form:

$$\psi(\zeta) = \frac{\mathcal{L}\mu}{\zeta(\zeta-c_1)(\zeta-c_2)}\,. \tag{7.40}$$

Substitute this expression in the boundary condition (7.16). With respect to the function μ we get the singular equation, the coefficients of its principal part are equal to zero in the points c_1 and c_2.

We know how to solve it in case $a^1_\pm = a^0_\pm = 0$ actually. We can construct the solution in this case according lay the arguments mentioned above.

$$\int_\gamma \frac{\mu(\sigma)(1-\frac{\zeta}{\sigma})\ln(1-\frac{\zeta}{\sigma})}{\zeta(\zeta-c_1)(\zeta-c_2)}\,ds =$$

$$= \mathcal{D}_1 + \mathcal{D}_2 w(\zeta) + \omega(\zeta)\int_{\zeta_0}^{\zeta}\frac{\Omega(\tau)}{\tau^5[\omega'(\tau)]^2}\,d\tau - \int_{\zeta_0}^{\zeta}\frac{\omega(\tau)\Omega(\tau)}{\tau^5[\omega'(\tau)]^2}\,d\tau,\quad 0\neq\zeta_0\in\mathcal{D}, \tag{7.41}$$

$\Omega(\zeta)$ is the solution of Riemann–Hilbert boundary value problem (7.28). It is easy to check, that in order for the right-hand side of the formula (7.41) to be single-valued it is necessary and sufficient the fulfillment of the following conditions:

$$\Omega(0)=0,\quad \Omega'(0)=0. \tag{7.42}$$

First from these conditions in case of $\kappa>0$, has the form:

$$\frac{1}{2\pi i}\int_\gamma \frac{G(\sigma)}{\sigma A(\sigma)\chi^+(\sigma)}\,d\sigma + C_0 = 0,$$

the analogous form has the second one also.

Therefore we may rewrite the formula (7.41) in the following form:

$$\frac{\mathcal{L}\mu}{\zeta(\zeta-c_1)(\zeta-c_2)} = \mathcal{D}_1 + \omega(\zeta)\int_{\zeta_0}^{\zeta}\frac{\Omega(\tau)}{\tau^5[\omega'(\tau)]^2}\,d\tau - \int_{\zeta_0}^{\zeta}\frac{\omega(\tau)\Omega(\tau)}{\tau^5[\omega'(\tau)]^2}\,d\tau. \tag{7.43}$$

It is evident that $\mathcal{D}_2 = 0$.

It follows from the last formula, that

$$\mu(\sigma) = KG + \sum_{k=1}^{3} d_k g_k(\sigma), \tag{7.44}$$

where K is linear operator.

Everything till have was related to the case when $a^1_\pm = a^0_\pm = 0$. Consider now general case. In general case we get the equation of the following form:

$$\operatorname{Re}\left\{B_2(\sigma_0)\left[\pi\sigma_0^{-2}\mu(\sigma_0) + \int_\gamma \frac{\mu(\sigma)}{\sigma(\sigma-\sigma_0)}\,d\sigma\right] + N\mu\right\} = H(\sigma_0), \tag{7.45}$$

where

$$B_2(\sigma) = \frac{b_2(\sigma)h(\sigma)h_1(\sigma)}{\sigma(\sigma-c_1)(\sigma-c_2)},$$
$$H(\sigma) = G(\sigma)h_1(\sigma), \quad h_1(\sigma) = |\sigma-c_1|^2|\sigma-c_2|^2,$$

$N\mu$ is Fredholm type operator, transforming the functions of the class H^* into the functions with derivatives from the class H^*. The equation (7.45) will be regulated by the Carleman–Vekua method. We get the equation the solvability condition of which has the following form

$$\int_\gamma \Big(KG + \sum_{k=1}^{3} d_k g_k\Big)\rho_j(t)\,ds = 0 \ \ (j = 1,\dots,M), \tag{7.46}$$

where $\{\rho_j\}$ is a system of linearly independent solutions of conjugate homogeneous equation. Thus we come to the conclusion, that the considered problem is Noetherian in case, when $a^2_\pm(t) \neq 0$.

Remark 7.1. The steps of reduction work when the boundary condition contains the derivatives of arbitrary high order. It is however, more complicated technically. Analogously one may consider the above problems in case of several cuts by applying conformal mapping on an appropriate multiply connected domain.

Глава 3

GENERALIZATIONS AND APPLICATIONS

8. General Differential Boundary Value Problem on a Cut Plane

Assume D denotes a plane of the complex variable $z = x + iy$, cut along Lyapunov-smooth arcs $a_k b_k$ $(k = 1,\dots,p)$. Denote $\Gamma = a_k b_k$ and $\Gamma = \bigcup_{k=1}^{p} \Gamma_k$. Let us consider the following boundary value problem: Find an analytic vector

$$\phi(z) = (\phi_1(z),\dots,\phi_n(z))$$

satisfying the boundary condition:

$$\operatorname{Re}\sum_{k=0}^{m}\left[a_\pm^{(k)}(t)\phi_\pm^{(k)}(t)\right] = f_\pm(t), \ \ t \in \Gamma, \tag{8.1}$$

where $a_+^{(k)}(t)$, $a_-^{(k)}(t)$ $(k = 0,\dots,m)$ are given quadratic matrices of n order on Γ, $f_+(t), = (f_+^1,\dots,f_+^n)$, $f_-(t) = (f_-^1,\dots,f_-^n)$ are given real vectors on Γ, $a_+^{(k)}(t)$, $a_-^{(k)}(t)$ are Hölder-continuous matrices, $f_+(t)$, $f_-(t)$ are satisfying the following condition.

$$f(t)\cdot\Pi(t) \in H^*(\Gamma), \quad \Pi(t) = \prod_{k=1}^{p}(t-a_k)^{m-1}(t-b_k)^{m-1}.$$