Part III

Boundary value problems

Part III is dedicated to the memory of Prof. G. Manjavidze

The following Part III was originally written in Russian. It has been translated and revised by G. Akhalaia and N. Manjavidze.

Preface of G. Akhalaia and N. Manjavidze

Among the most remarkable and important works of the well-known Georgian mathematician Professor Giorgi Manjavidze (1924-1999) is his monograph "Boundary value problems for analytic and generalized analytic functions", which was first published in 1990 by Tbilisi University Press in Russian. This book presents an original approach to the subject, that has not yet appeared anywhere except Georgia. English version will allow wide mathematical audience to get acquainted with this approach and apply in problems of mechanics and engineering.

That is why we translated this book and we are eager to dedicate this publication to the author's memory. The present addition differs from the original Russian in one added chapter (Chapter 20), which contains the theory of boundary value problems for nonlinear systems of partial differential equations on the plane and is written by the author together with Prof. Wolfgang Tutschke (note that the references are also enlarged).

Finally, with the most sincere feeling of gratitude we would like to thank all of them who have helped to turn our expectation into reality for great support and cooperation. Special thanks to the mastermind of all Prof. W. Tutschke, to Prof. H. Begehr who encouraged us to keep on with the idea and to our Georgian colleagues Prof. G. Giorgadze, Prof. G. Khimshiashvili, Prof. A. Tsiskaridze, Prof. N. Chinchaladze. We are pleased to express our deepest gratitude to Prof. C. C. Yang and to the "Science Press" Publishing House.

G. Akhalaia and N. Manjavidze

Goal

Part III is devoted to boundary value problems for analytic and generalized analytic functions and vectors. A complete solvability theory for boundary value problems of linear conjugation with shift for analytic functions and vectors with piecewise continuous coefficients is developed. Fundamental results on factorization of discontinuous matrix functions are systematically developed from scratch. Connection with the theory of singular integral equations is worked out in great detail. Explicit conditions of normal solvability and index formulae are obtained. The classical theory is extended to the case of generalized analytic functions and vectors. The theory of boundary value problems for nonlinear systems of partial differential equations on the plane is presented.

References to all chapters of Part III are given at the end of Chapter 20.

Introduction to the Chapters 17-19

The Chapters 17-19 of Part III of this book are devoted to boundary value problems of linear conjugation with displacement (or with shift). In these problems the boundary values of the desired functions are conjugated at points which are displaced to each other.

The model problem is to find a function $\Phi(z)$ holomorphic on the complex plane z, cut along some simple closed curves, the boundary values of which $\Phi^+(t)$ and $\Phi^-(t)$ are satisfying the condition

$$\Phi^+[\alpha(t)] = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma \tag{$*$}$$

from the both sides of Γ, where $G(t)$, $g(t)$ are given continuous functions on Γ, $\alpha(t)$ is continuous function mapping Γ onto γ in one-to-one manner.

The first researches concerning the theory of linear conjugation problems with displacement belong to Haseman C. [59] and to Carleman T. [28].

Complete solution of the problems of the form (*) was given by Kveselava D. [81], [82], [83], [84].

Several articles were dedicated to the problems of linear conjugation with displacement in case of vectors by Vekua N; his results and also the results of other authors in this connection are given in the monograph of Vekua N.: “Systems of Singular Integral Equations”, [136]

Later on the problems of linear conjugation with displacement were studied by the various authors. In the monograph [88] of Litvinchuk G., “Boundary Value Problems and Singular Integral Equations with Shift”, are given the articles concerning these problems, published till 1975.

Studies in the theory of boundary value problems of linear conjugation with displacement are continued; some of them published recently are indicated in the references of this book.

The present book is divided into three chapters.

In the chapter I short, “concise” presentation of the theory of problems of linear conjugation for analytic functions and based on it the theory of (one-dimensional) singular integral equations is given.

Chapter II is devoted to the theory of linear conjugation problems with displacement for analytic functions properly; the main attention is paid to the construction of the canonical matrices which are used in the construction of the general solutions of the considered problems.

The third chapter focuses on the studies of linear conjugation problems with displacement for the generalized analytic functions (vectors). In this chapter the differential boundary problems i.e., the problems containing the derivatives of boundary values of the desired vector are also considered.

Basic definitions and notations

We apply the terms and the notations basically from the books [108], [135], [136], [118], [81] and from the paper [23]. Sometimes there will be some changes in the definitions and notations.

0.1. Let S be some set in the plane of the complex variable $z = x + iy$.

Denote by $C(S)$ the class of all bounded continuous functions $f(z)$ defined in S. By $C(S)$ also the Banach space with the norm

$$\|f\|_c = \sup|f(z)|, \quad z \in S \tag{1}$$

is denoted.

0.2. We say that a function $f(z)$ satisfies a $H(\mu)$- condition (i.e. a Hólder condition with exponent μ) if $f(z)$ defined on S satisfies the inequality

$$|f(z_1) - f(z_2)| \leqslant A|z_1 - z_2|^\mu, \quad z_1, z_2 \in S, \tag{2}$$

where A and μ are constants not depending on z_1, z_2 (where $A \geqslant 0$, $0 < \mu \leqslant 1$).

We shall denote by $H_\mu(S)$ the class of the functions satisfying the condition (2) (the constant A is not fixed). $H_\mu(S)$ also denotes the Banach space with the norm

$$\|f\|_{H_\mu} = \|f\|_c + \sup\frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^\mu}, \quad z_1, z_2 \in S$$

where $\|f\|_c$ is defined by the equation (1).

The union of the classes $H_\mu(S)$, $0 < \mu \leqslant 1$, is denoted by $H(S)$. It is evident that the functions of the class $H(S)$ are continuous; therefore sometimes the functions from this class will be referred to Hölder-continuous.

It is easy to generalize the notation of a Hölder-condition to the case of several variables: the function $f(z_1, \cdots, z_m)$ defined in $z_k \in S_k$ $(k = 1, \cdots, m)$ satisfies the $H(\mu)$-condition if

$$|f(z_1', \cdots, z_m') - f(z_1'', \cdots, z_m'')| \leqslant A\sum_{k=1}^{m}|z_k' - z_k''|^\mu,$$

where A, μ are constants, $A \geqslant 0$, $0 < \mu \leqslant 1$.

The class of all functions satisfying this condition is denoted by $H_\mu(S_1 \times \cdots \times S_m)$; denote, finally, by $H(S_1 \times \cdots \times S_m)$ the union of all $H_\mu(S_1 \times \cdots \times S_m)$, $0 < \mu \leqslant 1$.

0.3. If S is a domain, then denote by $C^m(S)[H^m_\mu(S)]$ the class of all functions satisfying the following conditions

$$f \in C(\bar{S})[f \in H(S)], \ \ \frac{\partial^m f}{\partial x^{m-k}\partial y^k} \in C(S)[f \in H(S)], \ \ k = 0, \cdots m.$$

Consider, moreover, the class of functions $f(z) = f(x, y)$ defined and measurable in S and satisfying the condition

$$\int_S |f(z)|^p dxdy < \infty, \ \ p \geqslant 1.$$

The class of all functions satisfying this condition is denoted by $L_p(\bar{S})$; by $L_p(\bar{S})$ we denote also the Banach space with the norm

$$\|f\|_{L_p} = \left(\int_S |f(z)|^p dxdy\right)^{1/p}.$$

Denote by $L_p(S)$ the class of all functions f for which the $p-$th power of the absolute value $|f|$ is summable on every subset of the domain S.

0.4. Let Γ be a simple rectifiable curve $z = z(s)$, where s is the arc length, $0 \leqslant s \leqslant \ell$, and ℓ is the length of Γ.

We say that $\Gamma \in C^m$ if the derivatives of the function $z(s)$ with respect to s up to and including the order m are continuous on the segment $[0, \ell]$ (it is assumed that if Γ is closed, then $z^{(k)}(0) = z^{(k)}(\ell)$, $k = 1, \cdots, m$); if in addition, the derivative $z^{(m)} \in H_\mu([0, \ell])$ then we say that $\Gamma \in H^m_\mu$. Curves of the class C^1 are called the smooth ones.

Curves consisting of a finite number of smooth curves are called the piecewise smooth ones.

We say that the curve belongs to the class K, if the relation

$$\frac{s(t_1, t_2)}{|t_1 - t_2|}$$

is bounded for arbitrary $t_1, t_2 \in \Gamma$. By $s(t_1, t_2)$ is denoted the length of the least arc connecting the points t_1 and t_2.

We write $D \in C^m[D \in H^m_\mu]$ if the boundary of the domain D consists of a finite number of simple closed curves of the class $C^m[H^m_\mu]$.

0.5. Let Γ be a simple curve, c_1, $c_2, \cdots, c_r$ are points of Γ ordered according to the orientation of Γ. Denote by $C_0(\Gamma, c_1, \cdots, c_r)$ the class of functions which are continuous on Γ except perhaps the points c_k where they may have discontinuities of the first kind; we call such functions the piecewise-continuous functions.

We shall say that a function $f(t)$ belongs to the class $H^\mu_0(\Gamma, c_1, \cdots, c_r)$ if $f \in C_0(\Gamma, c_1, \cdots, c_\Gamma)$ and f satisfies the $H(\mu)$-condition on each closed arc $c_k c_{k+1}$

provided the limits $f(c_k+0)$ and $f(c_{k+1}-0)$ are interpreted as the values of f at the points c_k and c_{k+1} where $k=1,\cdots,r$ and $c_{r+1}=c_1$.

Denote by $C_0(\Gamma)$ [resp. $H_0(\Gamma)$] the union of the classes $C_0(\Gamma, c_1,\cdots,c_r)$ [resp. $H_0^{\mu}(\Gamma, c_1,\cdots,c_r)$], $0<\mu\leqslant 1$.

We shall say that $f(t)\in H^*(\Gamma)$ if the function $f(t)$ given on Γ admits the representation

$$f(t)=f_0(t)\prod_{k=1}^{r}|t-c_k|^{-\alpha},\ \ c_k\in\Gamma,\ \ f_0(t)\in H_0(\Gamma),\ \alpha<1.$$

If $\prod_{k=1}^{r}|t-c_k|^{\varepsilon}f(t)\in H(\Gamma)$ for arbitrary small $\varepsilon>0$ then we write $f(t)\in H_{\varepsilon}^*(\Gamma)$.

0.6. Let Γ be a rectifiable curve $t=t(s)$, $0\leqslant s\leqslant \ell$ and $f(t)$ be a function defined on Γ. We shall say that $f(t)$ is measurable [resp. summable] on Γ if the function $f(t(s))$ of the real variable s is measurable [resp. summable] on the segment $[0,\ell]$; if $f(t)$ is summable, we define

$$\int_{\Gamma}f(t)dt=\int_0^{\ell}f(t(s))t'(s)ds.$$

Let $\rho\geqslant 0$, $f(t)$ be measurable functions defined on Γ. We shall say that $f(t)\in L_p(\Gamma,\rho)$ if $\rho(t)|f(t)|^p$ $(p\geqslant 1)$ is a summable function on Γ; we write $L_p(\Gamma)$ instead of $L_p(\Gamma,1)$.

By $L_p(\Gamma,\rho)$ also the Banach space with the norm

$$\|f\|_{L_p(\Gamma,\rho)}=\left(\int_{\Gamma}\rho(t)|f(t)^p|dt\right)^{1/p}$$

is denoted.

The spaces $L_p(\Gamma,\rho)$ and $L_q(\Gamma,\rho^{1-q})$, are called conjugate classes if

$$\frac{1}{p}+\frac{1}{q}=1$$

i.e., $q=p/(p-1)$.

As a rule we assume that the weight function has the form

$$\rho(t)=\prod_{k=1}^{r}|t-t_k|^{\nu_k},\ \ t_k\in\Gamma,\ -1<\nu_k<p-1,\ p>1. \tag{3}$$

It is clear that in this case $L_p(\Gamma,\rho)\subset L_{\lambda}(\Gamma)$ for some $\lambda>1$.

0.7. Let Γ be a union of simple smooth curves in the complex $z-$plane.

Let $\phi(z)$ be a function given and continuous in a neighborhood of Γ except perhaps at the points of Γ themselves. Let t be some point of Γ different from the end points and the points of self-intersection (if there are any). We say that the function $\phi(z)$ is continuously extendable to the point t from the left [resp. from the right] if $\phi(z)$ tends to a well-defined limit $\phi^+(t)$ [resp. $\phi^-(t)$] when z tends to t along any path remaining on the left [on the right respectively] on Γ.

If the mentioned limits exist when z tends to t along any non-tangential path remaining on the left [on the right respectively] on Γ, then we say that $\phi(z)$ has the angular boundary value $\phi^+(t)[\phi^-(t)]$.

Under a piecewise-holomorphic function ϕ we understand a holomorphic function in the plane cut along Γ (except perhaps at the point of infinity) continuously extendable to Γ from both sides everywhere except perhaps the finite set of points c_k; near these points c_k the function $\phi(z)$ is supposed to satisfy the following estimate

$$|\phi(z)| \leqslant \frac{\text{const}}{|z-c_k|^\alpha}, \quad 0 \leqslant \alpha < 1.$$

At the point $z=\infty$ the function may have a pole. An analogous definition can be given for generalized analytic vectors.

0.8. The notation $A \in K$, where A is a matrix and K is some class of functions, means that every element $A_{\alpha\beta}$ of A belongs to K. If K is some linear normed space with the norm $\|\cdot\|_K$, then $\|A\|_K = \max\limits_{\alpha,\beta}\|A_{\alpha\beta}\|_K$.

Sometimes an $(n \times 1)$-matrix A is called a vector, and it is convenient to write it as a row,

$$A = (A_1, \cdots, A_n).$$

0.9. Let D be a simply connected domain in the extended complex plane bounded by a rectifiable Jordan curve Γ.

By definition the class $E_p(D)$, $p>0$, is the set of all analytic functions in D for which

$$\sup \int_{\Gamma_K} |f(z)|^p |dz| < \infty,$$

where D_k are subdomains of D with rectifiable boundaries Γ_K such that

$$\overline{D}_k \subset D, \quad D_k \subset D_{k+1}, \quad \bigcup_k D_k = D,$$

i.e., the D_k form an exhaustion of D.

The class $E_p(D)$ can be defined by the requirement that the curves Γ_K are the images of circles $|\zeta| = r < 1$ under the conformal mapping $z = \omega(\zeta)$ of the unit disk $|\zeta| < 1$ onto the domain D. Then we may define the class $E_p(D)$ with the help of the Hardy classes: $f(z) \in E_p(D)$ (D is a finite domain) if and only if $f(\omega(\zeta))[\omega'(\zeta)]^{1/p} \in H_p$.

0.10. Let Γ be a rectifiable curve, $f(t) \in L_1(\Gamma)$. The expression

$$\phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)dt}{t-z}, \quad z \in \Gamma$$

is called a Cauchy-type integral, and the function $f(t)$ is called its density.

Different properties of a Cauchy type integral are studied in monographs [55], [39], [65], [140], [108], [118], [72] and in survey articles [40], [41], [74].

Let Γ be a simple closed rectifiable curve bounding the finite domain D^+ and the infinite domain D^- (the domain D^+ remains on the left when passing along Γ in the positive direction); the Cauchy type integral has the angular boundary values $\phi^+(t)$ and $\phi^-(t)$ almost everywhere on Γ from D^+ and D^- (from both sides of Γ); these boundary values are given by the formulas

$$\phi^{\pm}(t) = \pm\frac{1}{2}f(t) + \frac{1}{2\pi i}\int_{\Gamma}\frac{f(\tau)d\tau}{\tau - t}, \tag{4}$$

where the integral

$$Sf \equiv \frac{1}{\pi i}\int_{\Gamma}\frac{f(\tau)d\tau}{\tau - t} \tag{5}$$

is to be understood in the sense of Cauchy's principal value.

The formulas (4) are called Sokhotsky-Plemely formulas.

If Γ is a simple closed smooth curve and $f(t) \in H(\Gamma)$, then $\phi(z)$ is continuously extendable on Γ from both sides and the formulas (4) take place everywhere on Γ [108], [118].

Denote by R the class of rectifiable curves for which the singular integral (5) is a linear bounded operator in $L_p(\Gamma)$, $p > 1$. The class R contains the piecewise-smooth curves, the curves of the class K and etc. (see [56], [36], [25], [26], [33], [81], [83]); geometrically R is described in [36].

Sf is a linear bounded operator in the weighted spaces $L_p(\Gamma\rho)$ under some restrictions imposed on Γ and on $\rho(t)$. When $\rho(t)$ has the form (3) and Γ is a piecewise smooth curve (or the curve of the class K), then S is a bounded operator in $L_p(\Gamma, \rho)$. This problem is studied for more general classes of the weighted functions (see [35], [75], [76], [123], [60]).

Sf is a linear bounded operator in the space $H^{\mu}(\Gamma)$, $\mu < 1$, where Γ is a simple closed smooth curve (see [108], [118]).

Denote by $E_p^{\pm}(\Gamma)$, $p \geqslant 1$ ($E_p^{\pm}(\Gamma, \rho)$, ρ is a function (3)) the class of the functions $\phi(z)$ representable in the form

$$\phi(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)dt}{t - z} + P(z), \quad f \in L_p(\Gamma)\ (L_p(\Gamma, \rho)) \tag{6}$$

where $P(z)$ is some polynomial, Γ is a simple closed rectifiable curve.

Denote also by $E_{p,0}^{\pm}(\Gamma)$ ($E_{p,0}^{\pm}(\Gamma, \rho)$) the class of functions of the form (6) with $P(z) = 0$. By $E_{\infty}^{\pm}(\Gamma)$ ($E_{\infty,0}^{\pm}(\Gamma)$) we denote the intersection

$$\bigcap_{p>1} E_p^{\pm}(\Gamma) \quad \left(\bigcap_{p>1} E_{p,0}^{\pm}(\Gamma)\right)$$

For the functions of the class $E_p^{\pm}(\Gamma)$ the following propositions are valid:

a) $E_p^{\pm}(\Gamma) \subset E_r^{\pm}(\Gamma)$, $p > r$,

b) If $\phi(z) \in E_1^{\pm}(\Gamma)$ and $\phi^+(t) = \phi^-(t)$ almost everywhere on Γ, then $\Phi(z)$ is some polynomial,

c) If $\phi(z) \in E_p^{\pm}(\Gamma)$ $p > 1$, $\Gamma \in R$, then $\phi(z) \in E_p(D^+)$, $\phi(z) - P(z) \in E_p(D^-)$.

It is evident that if

$$\phi_1(z) \in E_p(D^+), \;\; \phi_2(z) \in E_p(D^-), \;\; p \geqslant 1,$$

then the function

$$\phi(z) = \begin{cases} \phi_1(z), & z \in D^+, \\ \phi_2(z), & z \in D^-, \end{cases}$$

belongs to $E_p^{\pm}(\Gamma)$.

d) Let $\phi_1(z) \in E_p^{\pm}(\Gamma, \rho)$, $\phi_1(z) \in E_q^{\pm}(\Gamma, \rho^{1-q})$, ρ be the function of the form (3). Then

$$\phi_1(z)\phi_2(z) \in E_1^{\pm}(\Gamma).$$

e) If $\phi(z) \in E_1^{\pm}(\Gamma)$ then $\phi(z) \in E_{1-\varepsilon}(D^+)$, $\phi(z) \in E_{1-\varepsilon}(D^-)$, for arbitrary small positive ε [36], [115].

0.11. Let $\Gamma_k (k = 1, 2)$ be the rectifiable Jordan curve bounding finite and infinite domains D_k^+ and D_k^-.

Let $\varphi^+(z)$ and $\varphi^-(z)$ be a couple of functions representable in the following form

$$\varphi^+(z) = \frac{1}{2\pi i}\int_{\Gamma_2} \frac{f_2(t)dt}{t - z}, \; z \in D_2^+, \;\; \varphi^-(z) = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{f_1(t)dt}{t - z} + P(z), \; z \in D_1^-,$$

where $f_k \in L_p(\Gamma_k, \rho_k)$, $\rho > 1$, ρ_k are the functions of the form (3), $k = 1, 2$, $P(z)$ is some polynomial. The class of such couples of functions we denote by $E_p(\Gamma_1, \Gamma_2, \rho_1, \rho_2)$. The class $E_p^{\pm}(\Gamma_1, \Gamma_2, 1, 1)$ we denote by $E_p^{\pm}(\Gamma_1, \Gamma_2)$. The subclass of the class $E_p^{\pm}(\Gamma_1, \Gamma_2)$ for which $P(z) \equiv 0$ is denoted by $E_{p,0}^{\pm}(\Gamma_1, \Gamma_2)$. By $E_\infty^{\pm}(\Gamma_1, \Gamma_2)$ the intersection

$$\bigcap_{p>1} E_p(\Gamma_1, \Gamma_2)$$

is denoted.

It is easy to verify that the functions of the class $E_p^{\pm}(\Gamma_1, \Gamma_2, \rho_1, \rho_2)$ have the properties analogous of the properties of functions of the class $E_p^{\pm}(\Gamma_1, \rho)$ from 0.10.

0.12 Let X and Y be Banach spaces, and A is a linear bounded operator mapping X into Y.

The operator A is said to be Noetherian if

a) the image of the operator A in Y is closed (the operator A is normally solvable according to Hausdorff);

b) the null spaces $N = \{x \in X,\ Ax = 0\}$ and $N^*\{f \in Y^* : A^* f = 0\}$ are finite dimensional subspaces (A^* is the conjugate operator, X^* and Y^* are the conjugate spaces).

Where ℓ and ℓ^* denote the dimensions of the subspaces N and N^*, respectively, is called the index of the Noether operator A.

Let A_0 and A_1 be Noether operators and $A(\lambda)$ is a family of Noether operators depending countinuously on real parameter $\lambda \in [0, 1]$, $A(0) = A_0$, $A(1) = A_1$. Then the operators A_0 and A_1 are called homotopic and $\mathrm{ind} A_0 = \mathrm{ind} A_1$ (see for example [105], p.27).

Chapter 17

The Problem of Linear Conjugation and Systems of Singular Integral Equations

by Giorgi F. Manjavidze

17.1 Formulation of the problem

Under the problem of linear conjugation we mean the following problem.

Let Γ be a simple closed piecewise-smooth curve $\Gamma, a(t)$ and $b(t)$ are given $(n \times n)$ and $(n \times l)$ matrices respectively on Γ; $a(t)$ is a piecewise $-$ continuous matrix, $\inf|deta(t)| > 0, b(t) \in L_p(\Gamma, \rho), p > 1$, the weight function ρ has the form

$$\rho(t) = \prod_{k=1}^{r} |t - t_k|^{\nu_k}, \quad t_k \in \Gamma, \quad -1 < \nu_k < p - 1. \tag{1.1}$$

The set $\{t_k\}$ contains all discontinuity points of the matrix $a(t)$, it may contain also other points of Γ. Find a $(n \times l)-$ matrix $\Phi(z) \in E_p^{\pm}(\Gamma, \rho)$ satisfying the boundary condition

$$\Phi^+(t) = a(t)\Phi^-(t) + b(t) \tag{1.2}$$

almost everywhere on Γ.

Let c be some point of discontinuity of the matrix $a(t)$; denote by $\lambda_1, \cdots, \lambda_n$ the roots of the equation

$$det[a^{-1}(c+0)a(c-0) - \lambda I] = 0.$$

Consider the following numbers

$$\tau_k = \frac{1}{2\pi i} \ln \lambda_k;$$

these numbers are defined to within the integer summands. We say that the point c is singular if Re τ_k are integers otherwise c is called non $-$ singular (see [138]).

The quadratic matrix $\chi(z)$ of order n is called to be normal matrix of the boundary problem (1.2) (or for the matrix $a(t)$) if it satisfies the following conditions:

$$\chi(z) \in E_q^{\pm}(\Gamma, \rho), \quad \chi^{-1}(z) \in E_p^{\pm}(\Gamma, \rho^{1-q}), \quad q = \frac{p}{p-1},$$
$$\chi^+(t) = a(t)\chi^-(t)$$

almost everywhere on Γ.

We call the normal matrix $\chi(z)$ canonical if it has normal form at infinity i.e. $\lim_{z\to\infty}(z^{-\sigma}det\chi(z))$ (σ is the sum of columns orders of $\chi(z)$) is finite and nonzero. In connection that it is possible to consider the different classes $E_p^{\pm}(\Gamma,\rho)$, we shall speak about the canonical (normal) matrices of the classes $E_p^{\pm}(\Gamma,\rho)$.

We shall say that the matrix $a(t)$ is factorizable in $E_p^{\pm}(\Gamma,\rho)$, if for $a(t)$ there exists the canonical matrix of the same class $E_p^{\pm}(\Gamma,\rho)$ and in this case we shall write $a(t)\in\mathfrak{F}_p(\Gamma,\rho)$.

It is easy to prove the following proposition. If $\chi_1(z)$ and $\chi_2(z)$ are normal matrices (in particular canonical) of the problem (1.2) of one and the same class then

$$\chi_1(z)=\chi_2(z)P(z).$$

where $P(z)$ is a polynomial matrix with constant and nonzero determinant.

Consequently the determinants of all normal (canonical) matrices of the given class of the boundary problem (1.2) have the same orders at infinity.

Definition 17.1.1 *We call the index* (*or the total index*) *of the problem* (1.2) *of the class* $E_p^{\pm}(\Gamma,\rho)$ (*or the index of class* $E_p^{\pm}(\Gamma,\rho)$ *of the matrix* $a(t)$) *the order at infinity of the determinant of the normal* (*canonical*) *matrix of the given class* $E_p^{\pm}(\Gamma,\rho)$ *taken with the opposite sign.*

Having the normal matrix $\chi(z)$ of some class we may obtain the canonical matrix multiplying $\chi(z)$ from the right on corresponding polynomial matrix with the constant nonzero determinant.

Let $\chi(z)$ be a canonical matrix (of the given class) for the matrix $a(t)$. Denote by $-\varkappa_1,\cdots,-\varkappa_n$ the orders of the columns of $\chi(z)$ at infinity. The integers $\varkappa_1,\cdots,\varkappa_n$ are called the partial indices of the matrix $a(t)$ or of the boundary problem (1.2) (of the given class). The sum of the partial indices $\varkappa_1+\varkappa_2+\cdots+\varkappa_n$ is equal to the index of $a(t)$ (or of the problem (1.2) of the given class).

Note that if $\chi(z)$ is a canonical matrix of $E_p^{\pm}(\Gamma,\rho)$ of the matrix $a(t)$ then the matrix $[\chi'(z)]^{-1}$ will be a canonical matrix of the class $E_p^{\pm}(\Gamma,\rho^{1-q})$ of the matrix $[a'(t)]^{-1}$.

It is easy to prove the following lemmas.

Lemma 17.1.1 *Let* $\chi(z)$ *be a normal* (*canonical*) *matrix of the class* $E_p^{\pm}(\Gamma,\rho)$ *of the problem* (1.2). *If* (1.2) *is solvable for the given matrix* $b(t)\in L_p(\Gamma,\rho)$ *then all solutions of the problem* (1.2) *of the class* $E_p^{\pm}(\Gamma,\rho)$ *are given by the following formula*

$$\Phi(z)=\frac{\chi(z)}{2\pi i}\int_\Gamma\frac{[\chi^+(t)]^{-1}b(t)dt}{t-z}+\chi(z)P(z),$$

where $P(z)$ *is an arbitrary polynomial* $(n\times l)-$ *matrix. In particular the solutions of the homogeneous problem* $(b(t)\equiv 0)$ *have the form*

$$\chi(z)P(z).$$

Lemma 17.1.2 *Let $\chi(z)$ be a normal (canonical) matrix of the class $E_p^{\pm}(\Gamma,\rho)$ of the problem (1.2) and let the angular boundary values of the matrix of the form $\Phi(z)=f(z)\varphi(z)g(z)$ ($\varphi(z)\in E_p^{\pm}(\Gamma,\rho)$, $f(z),g(z)$ be the piecewise-meromorphic matrices which are continuously extendable from the both sides, everywhere on Γ) are satisfying the boundary problem (1.2) for the given $b(t)\in L_p(\Gamma,\rho)$; then the boundary problem (1.2) has the solution of the class $E_p^{\pm}(\Gamma,\rho)$.*

Let us prove the following propositions.

Lemma 17.1.3 *If the boundary problem (1.2) is solvable for an arbitrary $b(t)\in L_p(\Gamma,\rho)$ and there exists the normal (canonical) matrix χ of the class $E_p^{\pm}(\Gamma,\rho)$ then the expressions*

$$L_1 b \equiv \chi^{+}(t)\int_{\Gamma}\frac{[\chi^{+}(\tau)]^{-1}b(\tau)}{\tau-t}d\tau.$$

$$L_2 b \equiv \chi^{-}(t)\int_{\Gamma}\frac{[\chi^{-}(\tau)]^{-1}b(\tau)}{\tau-t}d\tau$$

are the linear bounded operators in the space $L_p(\Gamma,\rho)$.

Proof Indeed, let $b_m(t)\to b(t)$ and $L_1b_m\to g$ with respect to the norm of the space $L_p(\Gamma,\rho)$. It is known that from $b_m(t)\to b(t)$ it follows that $L_1b_m\to L_1b$ with respect to the measure (see [114]); therefore $g=L_1b$ and the operator L_1 is closed operator; as $L_p(\Gamma,\rho)$ is a Banach space then L_1b will be the bounded operator.

Lemma 17.1.4 *The partial indices $\varkappa_1,\cdots,\varkappa_n$ of the problem (1.2) of the class $E_p^{\pm}(\Gamma,\rho)$ are not depending on the choice of a canonical matrix.*

Proof (cf.[108], [138]). Let $\chi(z)$ be a canonical matrix of the class $E_p^{\pm}(\Gamma,\rho)$, D^+ and D^- are finite and infinite domains bounded by Γ. We have

$$\chi(z)=\chi_0(z)\Lambda(z),\quad z\in D^-,$$

$$\Lambda(z)=diag[(z-c)^{-\varkappa_1},\cdots,(z-c)^{-\varkappa_n}],\quad c\in D^+,\quad det\chi_0(\infty)\neq 0,$$

Rewrite the boundary condition of the homogeneous problem (1.2) in the following form

$$[\chi^{+}(t)]^{-1}\Phi^{+}(t)=\Lambda^{-1}(t)[\chi_0^{-}(t)]^{-1}\Phi^{-}(t),$$

from which it follows that

$$[\chi(z)]^{-1}\Phi(z)=P(z),\quad z\in D^+,\quad [\chi_0(z)]^{-1}\Phi(z)=\xi(z)P(z)$$
$$\xi(z)=diag[(z-c)^{-\varkappa_1},\cdots,(z-c^{-\varkappa_n})],\quad P(z)=(p_1,\cdots,p_n) \tag{1.3}$$

$P_j(z)$ is an arbitrary polynomial of order j, $P_j(z) = 0$ when $j < 0$.

Denoting by λ the number of linear independent solutions of the homogeneous problem (1.2) of the class $E^{\pm}_{p,0}(\Gamma, \rho)$, from the equalities (1.3) we obtain

$$\lambda = \sum_{\varkappa_k \geqslant 0} \varkappa_k.$$

It is evident that the number μ of linear independent solutions of the conjugate homogeneous problem of the class $E^{\pm}_{q,0}(\Gamma, \rho^{1-q})$

$$\Phi^+(t) = [a'(t)]^{-1}\Phi^-(t)$$

is equal to

$$\mu = -\sum_{\varkappa_k \leqslant 0} \varkappa_k.$$

Obviously λ and μ are the invariant values.

Let $\chi_1(z)$ and $\chi_2(z)$ be the canonical matrices of the problem (1.2) of the class $E^{\pm}_p(\Gamma, \rho)$. Denote by $-\varkappa_k^{(i)}$ $(i = 1, 2, k = 1, 2, \cdots, n)$. the orders of the columns of $\chi_i(z)$ at infinity. Let

$$\varkappa_1^{(i)} \geqslant \varkappa_2^{(i)} \geqslant \cdots \geqslant \varkappa_n^{(n)}, \ \ \varkappa_1^{(1)} \geqslant \varkappa_1^{(2)}.$$

Consider the matrix

$$a_0(t) = (t-c)^{-\varkappa_1^{(2)}} a(t)$$

and for this matrix as a canonical matrix we may take the matrix

$$\chi_i^0(z) = \begin{cases} \chi_i(z), & z \in D^+, \\ (z-c)^{\varkappa_1^{(2)}} \chi_i(z), & z \in D^-. \end{cases} \quad i = 1, 2.$$

Remarking that the orders of columns of the matrix $\chi_1^0(z)$ at infinity are equal to

$$-\varkappa_k^{(1)} + \varkappa_1^{(2)},$$

we get

$$\varkappa_1^{(1)} - \varkappa_1^{(2)} \leqslant 0, \ \ \varkappa_1^{(1)} = \varkappa_1^{(2)}.$$

If we continue the arguments then it occurs that

$$\varkappa_k^{(1)} = \varkappa_k^{(2)}, \ \ k = 2, \cdots, n.$$

17.2 Boundary value problem of linear conjugation with continuous coefficients

Consider the following boundary value problem

$$\Phi^+(t) = a(t)\Phi^-(t) + b(t), \ \ t \in \Gamma, \tag{2.1}$$

where $a(t), b(t)$ are given $(n \times n)$ and $(n \times l)$ matrices on Γ, respectively, $b(t) \in L_p(\Gamma),\ \ p > 1,\ \ a(t) \in C(\Gamma),\ \ deta(t) \neq 0$.

For an arbitrary $\varepsilon > 0$ there exists the rational matrix $r(z)$ satisfying the conditions; $r(z)$ has no poles on $\Gamma,\ \ detr(t) \neq 0$ when $t \in \Gamma$ and

$$\|a(t)r^{-1}(t) - I\|_{C(\Gamma)} \leqslant \varepsilon,\ \ \|a^{-1}(t)r(t) - I\|_{C(\Gamma)} \leqslant \varepsilon, \tag{2.2}$$

where I is a unit matrix.

Let us consider the sequence of matrices

$$\varphi_m(z) = \frac{1}{2\pi i}\int_\Gamma \frac{a_0(t)\varphi^-_{m-1}(t)}{t - z}dt + \frac{1}{2\pi i}\int_\Gamma \frac{b(t)dt}{t - z}, \tag{2.3}$$

$$a_0 = ar^{-1} - I, m = 1, 2, \cdots, \varphi^-_0(t) = 0.$$

It is evident that $\varphi^-_m(t) \in L_p(\Gamma)$.

Using the Sokhotsky-Plemely formulas from 2.3 we obtain

$$\varphi^-_{m+1}(t) - \varphi^-_m(t) = \frac{1}{2\pi i}\int_\Gamma \frac{a_0(\tau)[\varphi^-_m(\tau) - \varphi^-_{m-1}(\tau)] - a_0(t)[\varphi^-_m(t) - \varphi^-_{m-1}(t)]}{\tau - t}d\tau.$$

Hence

$$\|\varphi^-_{m+1} - \varphi^-_m\|_{L_p(\Gamma)} \leqslant A_p\varepsilon\|\varphi^-_m - \varphi^-_{m-1}\|_{L_p(\Gamma)}. \tag{2.4}$$

From the inequality (2.4) it follows that if $A_p\varepsilon < 1$ then the sequence $\varphi^-_m(t)$ converges by the norm of $L_p(\Gamma)$ to some matrix $\varphi^-(t) \in L_p(\Gamma)$. Whence it follows that for every $z \notin \Gamma$ there exists the $\lim \varphi_m(z) = \varphi(z)$ representable by the following formula

$$\varphi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{a_0(t)\varphi^-(\tau)}{\tau - t}d\tau + \frac{1}{2\pi i}\int_\Gamma \frac{b(t)dt}{\tau - t}. \tag{2.5}$$

The matrix $\varphi(z)$ defined by the formula (2.5) belongs to the class $E^\pm_{p,0}(\Gamma)$ and satisfies the boundary value condition

$$\varphi^+(t) = a(t)r^{-1}(t)\varphi^-(t) + b(t).$$

If we take $b(t)$ equal to $a(t)r^{-1}(t)$ then $\varphi(z)$ will satisfy the following boundary condition

$$\varphi^+(t) = a(t)r^{-1}(t)[\varphi^-(t) + I]. \tag{2.6}$$

Substituting the matrix $a(t)$ onto $a'^{-1}(t)$ and $r(t)$ onto $r'^{-1}(t)$, (it is possible by virtue of (2.2)) we obtain that there exists the matrix $\psi(z) \in E^\pm_{p,0}(\Gamma)$ such that

$$\psi^+(t) = a'^{-1}(t)r'(t)[\psi^-(t) + I]$$

or

$$\psi'^+(t) = [\psi'^-(t) + I]\tau(t)a^{-1}(t). \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$\psi'^{+}(t)\varphi^{+}(t) = [\psi'^{-}(t) + I][\varphi^{-}(t) + I]. \tag{2.8}$$

Let $p \geqslant 2$; the matrix defined by the following formula

$$\chi(z) = \begin{cases} \psi'(z)\varphi(z), & z \in D^{+}, \\ (\psi'(z) + I)(\varphi(z) + I), & z \in D^{-} \end{cases}$$

belongs to the class $E_1^{\pm}(\Gamma)$ and from (2.8) we have $\chi(z) \equiv I$, i.e.

$$[\varphi(z)]^{-1} = \psi'(z), \;\; z \in D^{+}, \;\; [\varphi(z) + I]^{-1} = \psi'(z) + I, \;\; z \in D^{-}.$$

Consider now the matrix

$$\chi(z) = \begin{cases} \varphi(z)R(z), & z \in D^{+}, \\ r^{-1}(z)[\varphi(z) + I]R(z), & z \in D^{-}, \end{cases}$$

where $R(z)$ is a rational matrix chosen in the following way: it liquidates the zeros of $detr^{-1}(t)$ in the domain D^{-} and the poles of $r^{-}(z)$ in the same domain and gives to $\chi(z)$ the normal form at infinity; there exists such a matrix [[15], [16], [44], [108]]. It is easy to see that $\chi(z) \in E_p^{\pm}(\Gamma)$, $\chi^{-1}(z) \in E_p^{\pm}(\Gamma)$; therefore for an arbitrary continuous nonsingular matrix $a(t)$ there exists a canonical matrix of the class $E_p^{\pm}(\Gamma)$ for an arbitrary $p \geqslant 2$.

Let $\chi_1(z)$ and $\chi_2(z)$ be the canonical matrices of the classes $E_{p_1}^{\pm}(\Gamma)$, $E_{p_2}^{\pm}(\Gamma)$ respectively, $2 \leqslant p_1 < p_2$.

We obtain

$$\chi_1(z) = \chi_2(z)P_1(z),$$

$$[\chi_1'(z)]^{-1} = [\chi_2'(z)]^{-1}P_2(z),$$

where $P_1(z), P_2(z)$ are some polynomial matrices. From the last equalities it follows that

$$\chi_1(z) \in E_{p_2}^{\pm}(\Gamma), \;\; [\chi_1(z)]^{-1} \in E_{p_2}^{\pm}(\Gamma).$$

Consequently the canonical matrix of an arbitrary class $E_p^{\pm}(\Gamma)$ $(p \geqslant 2)$ has the property:

$$\chi(z) \in E_{\infty}^{\pm}(\Gamma), \;\; \chi^{-1}(z) \in E_{\infty}^{\pm}(\Gamma).$$

It is evident that these matrices are the canonical matrices also for $1 < p < 2$. So it comes from these arguments that the boundary value problem (2.1) is solvable for an arbitrary $b(t) \in L_p(\Gamma, \rho)$ in the class $E_p^{\pm}(\Gamma)$ and all solutions of this class are given by the following formula

$$\Phi(z) = \frac{\chi(z)}{2\pi i} \int_{\Gamma} \frac{[\chi^{+}(\tau)]^{-1} b(\tau) d\tau}{\tau - t} + \chi(z)P(z),$$

where $P(z)$ is an arbitrary polynomial $(n \times l)-$ matrix.

Let now the matrix $a(t)$ be a Holder-continuous. Then the canonical matrix $\chi(z)$ is continuously extendable for all points of the curve Γ form the both sides and the matrices $\chi^{+}(t)$ and $\chi^{-}(t)$ are Holder-continuous, $\det\chi^{\pm}(t) \neq 0$.

Indeed if we construct again the sequence $\varphi_m(\Gamma)$ by the formula (2.3), however the rational matrix $\chi(z)$ we take such that the inequalities (2.2) will be fulfilled by the norm of the space $H_{\beta}(\Gamma), \quad 0 < \beta < \alpha. \quad (a(t) \in H_{\alpha}(\Gamma))$. Then the sequence $\varphi_m^{-}(t)$ converges by the norm of $H_{\beta}(\Gamma)$, $\varphi^{-}(t) \in H_{\beta}(\Gamma)$ and the matrix $\varphi(z)$ defined by the formula (2.5) will be Holder-continuous in closures $\bar{D}^{+}, \bar{D}^{-}$. This proves the above formulated proposition.

17.3 Boundary value problems with piecewise continuous coefficients

17.3.1 The scalar case

In this subsection we shall consider the case when $n = l = 1$. First let us consider the homogeneous problem

$$\Phi^{+}(t) = a(t)\Phi^{-}(t), \quad a(t) \in C_0(\Gamma; c_1, \cdots, c_m) \tag{3.1}$$

Now we make the substitution: ([136], [116]).

$$\Phi(z) = \prod_{k=1}^{m} \chi_k^1(z)\varphi(z), \quad z \in D^{+}, \quad \Phi(z) = \prod_{k=1}^{m} \chi_k(z)\varphi(z), \quad z \in D^{-}, \tag{3.2}$$

where

$$\chi_k^1(z) = (z - c_k)^{\tau_k}, \quad \chi_k(z) = \left(\frac{z - c_k}{z - z_0}\right)^{\tau_k}, \quad z_0 \in D^{+},$$

$$\tau_k = \frac{1}{2\pi i} ln\lambda_k, \quad \lambda_k = \frac{a(c_k - 0)}{a(c_k + 0)}, \quad -1 < Re\tau_k \leqslant 0,$$

where χ_k^1, χ_k are the univalent branches of the elementary multivalued functions defined as follows: $\chi_k^1(z)$ is the univalent branch in the plane cut along the line e_k which connects the point c_k with the point $z = \infty$ and lies in the domain D^{-}, $\chi_k(z)$ is the univalent branch in the plane cut along the line ℓ_k^1 which connects the point z_0 with the point c_k and lies in the domain D^{+}, $\chi_k(\infty) = 1$. With respect to the function $\varphi(z)$ we obtain the following boundary condition

$$\varphi^{+}(t) = g(t)\varphi^{-}(t),$$

where

$$g(t) = a(t)\left[\prod_{k=1}^{r} \chi_k^{1+}(t)\right]^{-1} \prod_{k=1}^{r} \chi_k^{-}(t) = a(t)\prod_{k=1}^{r}(t - z_0)^{-\tau_k},$$

$g(t)$ is a continuous function, $g(t) \neq 0$.

In previous section we proved that for the continuous function $g(t)$ there exists the canonical function $A(z) \in E^{\pm}_{\infty}(\Gamma),\ \ A^{-1}(z) \in E^{\pm}_{\infty}(\Gamma)$.

Consider the function

$$\chi_0(z) = \begin{cases} A(z) \displaystyle\prod_{k=1}^{r} X^1_k(z), & z \in D^+, \\ A(z) \displaystyle\prod_{k=1}^{r} X_k(z), & z \in D^-. \end{cases}$$

It is evident that $\chi_0^{-1}(z) \in E^{\pm}_{\infty}(\Gamma),\ \ \chi_0(z) \in E^{\pm}_{\varepsilon}(\Gamma)$ for some $\varepsilon > 1$.

Let $\Phi(z)$ be some solution of the problem (3.1) of the class $E^{\pm}_{\delta}(\Gamma), \delta > 1$. Consider the following function

$$\Phi_1(z) = \Phi(z)/\chi_0(z).$$

Obviously $\Phi_1(z) \in E^{\pm}_{\delta_1}(\Gamma), \delta_1 > 1$ and

$$\Phi_1^+(t) = \Phi_1^-(t),\ \ t \in \Gamma.$$

Consequently $\Phi_1(z)$ is the polynomial $P(z)$ and

$$\Phi(z) = \chi_0(z)P(z).$$

Let there exists the canonical function of the problem (3.1) of the class $E^{\pm}_p(\Gamma, \rho)$, $\rho(t) = \prod_{K=1}^{r} |t - a_k|^{\nu_k}, -1 < \nu_k < p - 1$. Then it will have the following form

$$\chi(z) = \chi_0(z)Q(z),$$

where $Q(z)$ is some polynomial; in addition

$$\chi_0(z)Q(z) \in E^{\pm}_p(\Gamma, \rho),\ \ [\chi_0(z)Q(z)]^{-1} \in E^{\pm}_q(\Gamma, \rho^{1-q}). \tag{3.3}$$

One can see from (3.3) that the polynomial $Q(z)$ may have zeros only in the points c_k and

$$\begin{aligned} &A^+(t) \prod_{k=1}^{r} (t - c_k)^{\tau_k} Q(t) \in L_p(\Gamma, \rho), \\ &\left[A^+(t) \prod_{k=1}^{r} (t - c_k)^{\tau_k} Q(t)\right]^{-1} \in L_q(\Gamma, \rho^{1-q}). \end{aligned} \tag{3.4}$$

Denote by $m_k (m_k \geqslant 0)$ order of zero of the polynomial $Q(z)$ at the point c_k. The following relations

$$|A^+(t)|^p |Q_s(t)|^p \prod_{k=1}^{r} |t - c_k|^{m_k p + \nu_k} |(t - c_k)^{\tau_k}|^p \in L_1(\Gamma),$$

$$|A^{+}(t)|^{q}|Q_{s}(t)|^{q}\prod_{k=1}^{r}|t-c_{k}|^{-m_{k}q+\nu_{k}(1-q)}|(t-c_{k})^{-\tau_{k}}|^{q}\in L_{1}(\Gamma),$$
$$Q_{k}(z)=(z-c_{k})^{-m_{k}}Q(z).$$

holds.

From these relations it follows that

$$\prod_{k=1}^{r}|t-c_{k}|^{m_{k}p+\nu_{k}}|(t-c_{k})^{\tau_{k}}|^{p}\in L_{1-\lambda}(\Gamma),$$
$$\prod_{k=1}^{r}|t-c_{k}|^{-m_{k}q+\nu_{k}(1-q)}|(t-c_{k})^{-\tau_{k}}|^{q}\in L_{1-\lambda}(\Gamma). \tag{3.5}$$

where λ is an arbitrary small positive number.

Denoting by $\tau_{k}=\alpha_{k}+i\beta_{k}$, from (3.5) we obtain

$$\prod_{k=1}^{r}|t-c_{k}|^{(\alpha_{k}+m_{k})p+\nu_{k}}\in L_{1-\lambda}(\Gamma),$$
$$\prod_{k=1}^{r}|t-c_{k}|^{-(\alpha_{k}+m_{k})q+\nu_{k}(1-q)}\in L_{1-\lambda}(\Gamma),$$

from which it follows that

$$(\alpha_{k}+m_{k})p+\nu_{k}>-1,-(\alpha_{k}+m_{k})q+\nu_{k}(1-q)>1$$

or

$$-\alpha_{k}-\frac{1}{p}-\frac{\nu_{k}}{p}<m_{k}<-\alpha_{k}+\frac{1}{q}-\frac{\nu_{k}}{p}.$$

Denote by $|\alpha_{k}|=\mu_{k}$ and let us call this number the parameter of the function $a(t)$ at the point c_{k}. The parameter μ_{k} may be defined also by the following relations:

$$\mu_{k}=\operatorname{Re}\frac{1}{2\pi i}\frac{a(c_{k}+0)}{a(c_{k}-0)},\quad 0\leqslant\arg\frac{a(c_{k}+0)}{a(c_{k}-0)}<2\pi.$$

Denote also by $\mu_{k}-\dfrac{1+u_{k}}{p}=\varepsilon_{k}$. Evidently $-\dfrac{1+\nu_{k}}{p}<\varepsilon_{k}<\nu_{k}$ and therefore

$$-1<\varepsilon_{k}<1.$$

So we have

$$\varepsilon_{k}<m_{k}<1+\varepsilon_{k}.$$

If $\varepsilon_{k}=0$ then the inequality is unrealizable, if $\varepsilon_{k}>0$ then $m_{k}=1$; if $\varepsilon_{k}<0$ then $m_{k}=0$. Hence we get the following proposition:

Theorem 17.3.1 *If $\mu_k p = 1 + \nu_k$ for some k then a canonical function of the corresponding class doesn't exist.*

If $\mu_k p \neq 1+\nu_k, k = 1, 2, \cdots, r$, then the canonical function of the class $E_p^{\pm}(\Gamma, \rho)$ exists and is given by the formula

$$\chi(z) = \chi_0(z)Q(z),$$

where $Q(z) = \prod_{k=1}^{r}(z - c_k)^{m_k}$

$$m_k = \begin{cases} 1, & \text{if} \quad \mu_k - \dfrac{1+\nu_k}{p} > 0, \\ 0, & \text{if} \quad \mu_k - \dfrac{1+\nu_k}{p} < 0. \end{cases}$$

The index of the class of the $E_p^{\pm}(\Gamma, \rho)$ of the function $a(t)$ (or the problem (3.1)) is given by the formula $\varkappa = ind\ g(t) - \sum_{k=1}^{r} m_k$ or by the formula

$$\varkappa = \frac{1}{2\pi}\left[\arg\frac{a(t)}{\prod\limits_{k=1}^{r}(t - z_0)^{s_k}}\right]_{\Gamma}, \tag{3.6}$$

where $s_k = \dfrac{1}{2\pi i}\ln\lambda_k$,

$$-1 < \text{Re } s_k \leqslant 0 \ \text{ if } \ \mu_k < \frac{1+\nu_k}{p} \ \ (\text{i.e. } \ s_k = \tau_k),$$

$$0 \leqslant \text{Re } s_k < 1 \ \text{ if } \ \mu_k > \frac{1+\nu_k}{p} \ \ (\text{i.e. } \ s_k = \tau_{k+1}).$$

Note that the condition

$$\mu_{kp} \neq 1 + \nu_k$$

is trivially fulfilled if the point c_k is singular, because in this case $\mu_k = \alpha_k = 0$; $s_k = \tau_k$, $\text{Re}s_k = 0$.

Remark *If $\chi_i(z) (i = 1, 2)$ are the canonical functions of the classes $E_{p_i}^{\pm}(\Gamma, \rho_i)$ (ρ_i are the functions of the form (1.1)), then $\chi_2(z) = \chi_1(z)\prod_{k=1}^{r}(z - c_k)^{m_k}$, where $m = +1, -1$ or 0.*

In particular if $\chi_1(z)$ and $\chi_2(z)$ are the canonical functions of the classes $E_{1+\varepsilon}^{\pm}(r)$ and $E_p^{\pm}(r)$ (ε is a sufficiently small positive number, p is a sufficiently large number) then in the last equality $m_k = 0$ for singular points and $m_k = 1$ for nonsingular points. Between the indices of these classes the following relation

$$\varkappa_p = \varkappa_{1+\varepsilon} - \tau_0,$$

holds, where τ_0 is the number of nonsingular points (see [108], p.78).

Consider now the nonhomogeneous problem

$$\phi^+(t) = a(t)\phi^- + b(t),\ b(t) \in L_p(\Gamma, \rho) \tag{3.7}$$

and make the substitution (3.2).

Instead of $-1 < \text{Re}\ \tau_k \leqslant 0$ suppose

$$\frac{-1+\nu_k}{p} < \text{Re}\ \tau_k < 1 - \frac{1+\nu_k}{p}. \tag{3.8}$$

Since Re τ_k is defined to within an integer, then the inequalities

$$\frac{1+\nu_k}{p} \leqslant \text{Re}\ \tau_k < 1 - \frac{1+\nu_k}{p}$$

are always fulfilled.

But the equality

$$\frac{-1+\nu_k}{p} = \text{Re}\ \tau_k$$

is eliminated, therefore the inequalities (3.7) are satisfiable.

We obtain the nonhomogeneous problem

$$\phi^+(t) = g(t)\phi^-(t) + f(t),\ f(t) = b(t)\left(\prod_{k=1}^{r} \chi_k^1(t)\right)^{-1}. \tag{3.9}$$

It is evident that $f(t) \in L_p(\Gamma, \rho_1),\ \rho_1(t) = \prod_{k=1}^{r} |(t-c_k)|^{\nu_k^1}, \nu_k^1 = \alpha_k p + \nu_k, \alpha_k =$ Reτ_k. It is easy to see that

$$-1 < \nu_k^1 < p - 1$$

as this inequality coincides with (3.7).

We shall construct the solution of (3.8) in the class $E_p^{\pm}(\Gamma, \rho_1)$.

Take the rational function $R(z)$ such that

$$\|g(t) - R(t)\|_{C(\Gamma)} \leqslant \varepsilon,$$

where ε is a sufficiently positive number and consider the following sequence

$$\psi_{m+1}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{g_0(t)\psi_m^-(t)}{t-z}dt + \frac{1}{2\pi i}\int_\Gamma \frac{f(t)dt}{t-z},\ \ \psi_0^-(t) = 0, \tag{3.10}$$

$$g_0 = gR^{-1} - I.$$

It is evident that

$$\psi_m(z) \in E_{\rho,0}^{\pm}(r, \rho_1).$$

From (3.10) we have

$$\psi^-_{m+1}(t_0)-\psi^-_m(t_0) = -\frac{1}{2}g_0(t)[\psi^-_m(t_0)-\psi^-_{m-1}(t_0)]+\frac{1}{2\pi i}\int_\Gamma \frac{g_0(t)[\psi^-_m(t)-\psi^-_{m-1}(t)]}{t-z}dt.$$

Consequently the sequence $\psi^-_m(t)$ converges by the norm of the space $L_p(\Gamma,\rho)$ to some function $h(t)\in L_p(\Gamma,\rho_1)$.

From (3.10) we have also

$$h(t_o) = -\frac{1}{2}[g_0(t_0)h(t_0)+f(t_0)]+\frac{1}{2\pi i}\int_\Gamma \frac{g_0(t)h(t)+f(t)}{t-t_0}dt.$$

Hence $h(t_0)$ is a boundary value of some analytic function on Γ in the domain D^- vanishing at infinity. Finally we obtain

$$\psi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{g_0(t)\psi^-(t)}{t-z}dt + \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}dt$$

(here $\psi^-(t)$ denotes h(t)).

From the last equality we get

$$\psi^+(t)-\psi^-(t) = g_0(t)\psi^-(t)+f(t)$$

or

$$\psi^+(t) = gR^{-1}\psi^-(t)+f(t). \tag{3.11}$$

Comparing (3.8) and (3.10) we can see that the function

$$\phi(z) = \begin{cases} \psi(z), & z\in D^+, \\ R^{-1}(z)\psi(z), & z\in D^- \end{cases}$$

is a solution of the problem (3.9). As far as the problem (3.8) has a canonical function of the class $E^\pm_p(\Gamma,\rho)$ then it is solvable in this class for any $f(t)\in L_p(\Gamma,\rho)$ and the initial problem (3.7) is solvable in $E^\pm_p(\Gamma,\rho)$ for an arbitrary function $b(t)\in L_p(\Gamma,\rho)$. Whence by virtue of the lemma 1.3 the expressions

$$\chi^+(t_0)\int_r \frac{[\chi^+(t)]^{-1}b(t)}{t-t_0}dt, \chi^-(t_0)\int_r \frac{[\chi^+(t)]^{-1}b(t)}{t-t_0}dt$$

are the linear bounded operators in $L_p(\Gamma,\rho)$.

17.3.2 The case of a triangular matrix

Consider now the following boundary value problem

$$\phi^+(t) = a(t)\phi^-(t)+b(t), t\in\Gamma \tag{3.12}$$

where $a(t)$ is a triangular piecewise-continuous nonsingular matrix $a = (a_{ik}), a_{ik} = 0$ when $i < k$, $b \in L_p(\Gamma, \rho)$. Denote by $c_1, \cdots c_r$ all discontinuity points of the functions $a_{ii}(t)(i = 1, \cdots, n)$. By μ_{ik} denote the parameters of the functions $a_{ii}(t)$ at the points $c_k(k = 1, \cdots, r)$. It is evident that $\mu_{ik} = 0$ if the function $a_{ii}(t)$ is continuous at the point c_k. Let us assume that the inequalities

$$\frac{1+\nu_k}{p} \neq \mu_{ik}, k = 1, \cdots, r, i = 1, \cdots, n \tag{3.13}$$

are valid and show that in this case there exists a canonical matrix of the problem (3.12) of the corresponding class. Obviously if the inequalities (3.13) are fulfilled then every function $a_{kk}(t)$ has canonical function of the class $E_p^{\pm}(\Gamma, \rho)$. Denote it by $\chi_k(z)$.

Consider the triangular matrix $\chi(z) = (\chi_{ik}), i, k = 1, \cdots, n; \chi_{ik} = 0$ when $i < k$, $\chi_{ik}(z) = \chi_k(z)$ and the remaining elements are defined by the formulas

$$\chi_{s1}(z) = \frac{\chi_s(z)}{2\pi i} \int_\Gamma \frac{\sum\limits_{i=1}^{s-1} a_{si}(t)\chi_{i1}^-(t)dt}{\chi_s^+(t)(t-z)} \quad s = 2, \cdots, n,$$

$$\chi_{s2}(z) = \frac{\chi_s(z)}{2\pi i} \int_\Gamma \frac{\sum\limits_{i=1}^{s-1} a_{si}(t)\chi_{i2}^-(t)dt}{\chi_s^+(t)(t-z)}, \qquad s = 3, \cdots, n,$$

$$\chi_{n,n-1}(z) = \frac{\chi_s(z)}{2\pi i} \int_\Gamma \frac{\sum\limits_{i=1}^{s-1} a_{n,n-1}(t)\chi_{n-1,n-1}^-(t)dt}{\chi_n^+(t)(t-z)}.$$

It can be easily seen that constructed in this manner matrix belongs to the class $E_p^{\pm}(\Gamma, \rho)$ and satisfies the following relation

$$\chi^+(t) = a(t)\chi^-(t).$$

Moreover det $\chi(z) = \prod_{k=1}^n \chi_k(z)$

Construct now the same matrix $\chi_*(z)$ for the matrix $[a'(t)]^{-1}$ as above.

$$\chi_*(z) \in E_q(\Gamma, \rho^{1-q}), \ \det\chi_*(z) = \prod_{k=1}^n [\chi_k(z)]^{-1},$$

$$\chi_*^+(t) = [a'(t)]^{-1}\chi_*^-(t).$$

Consider the matrix $\chi_*'(t)\chi(z) = \chi_0(z)$.

We have $\chi_0(z) \in E_1^{\pm}(\Gamma)$, $\chi_*'^+(t) = \chi_*'^-((t)[a(t)]^{-1}, \chi_*'^+(t)\chi^+(t) = \chi_+'^-(t)\chi^-(t)$.

Whence $\chi_0(z) = P(z)$, where $P(z)$ is some polynomial matrix. But det $P(z) = 1$ and $P^{-1}(z)$ is also a polynomial matrix.

Thus

$$P^{-1}(z)\chi'_*(z)\chi(z) = I;$$

$\chi(z)$ has a inverse matrix equal to $P^{-1}\chi_*^{'}(z) \in E_q^{\pm}(\Gamma, \rho^{1-q})$. Finally we have proved, that $\chi(z)$ is a normal matrix for $a(t)$ of the class $E_p^{\pm}(\Gamma, \rho)$.

It is easy to see that the boundary problem (3.12) is solvable for an arbitrary vector $b(t) \in L_p(\Gamma, \rho)$ and therefore the operators

$$\chi^+(t_0)\int_\Gamma \frac{[\chi^+(t)]^{-1}b(t)dt}{t-t_0}, \chi^-(t_0)\int_\Gamma \frac{[\chi^-(t)]^{-1}b(t)dt}{t-t_0}$$

are the linear bounded operators in $L_p(\Gamma, \rho)$.

The index of the problem (3.12) of the class $E_p^{\pm}(\Gamma, \rho)$ is equal to the sum of indices of the boundary problems $\varphi_k^+(t) = a_{kk}(t)\varphi_k^-(t)$, i.e. $\kappa = \sum_{k=1}^n \kappa_k$, κ_k is calculated by the formula (3.6):

$$\kappa_k = \frac{1}{2\pi}\left\{\arg\frac{a_{kk}(t)}{\prod\limits_{j=1}^{r}(t-z_0)^{s_{kj}}}\right\}_\Gamma,$$

where $s_{kj} = \dfrac{1}{2\pi i}\ln\lambda_{kj}$, $\lambda_{kj} = \dfrac{a_{kk}(c_j-0)}{a_{kk}(c_j+0)}$;

$-1 < \operatorname{Re} s_{kj} \leqslant 0$ if $\mu_{kj} < \dfrac{1+\nu_j}{p}$; $0 \leqslant \operatorname{Re} s_{kj} < 1$ if $\mu_{kj} > \dfrac{1+\nu_j}{p}$.

17.3.3 General case

Consider now the following problem

$$\Phi^+(t) = a(t)\Phi^-(t) + b(t), b(t) \in L_p(\Gamma, \rho), \tag{3.14}$$

where $a(t)$ is an arbitrary piecewise-continuous matrix, $inf|deta(t)| > 0$. Let us represent the matrix $a(t)$ in the following form

$$a(t) = a_1(t)\Lambda(t)a_2(t),$$

where $a_1(t), a_2(t)$ are continuous nonsingular matrices, $\Lambda(t)$ is piecewise-continuous nonsingular triangular matrix. This is possible by virtue of the lemma, proved in [38].

Take the rational matrices $R_1(z)$, $R_2(z)$ such that

$$\|a_k(t) - R_k(t)\| \leqslant \varepsilon, k = 1, 2,$$

where ε is a sufficiently small positive number.

Rewrite the boundary condition (3.14) in the following form:

$$\Phi^+ = R_1(t)\Lambda(t)R_2(t)\Phi^-(t) + [a(t) - R_1(t)\Lambda(t)R_2(t)]\Phi^-(t) + b(t).$$

Introduce the following notations

$$R_1^{-1}(z)\Phi(z) = \varphi(z), z \in D^+, R_2(z)\Phi(z) = \varphi(z), z \in D^-, R_1^{-1}(t)b(t) = B(t);$$

we have $\varphi^+(t) = \Lambda(t)\varphi^-(t) + [R_1^{-1}(t)a_1(t)\Lambda(t)a_2(t)R_2^{-1}(t) - \Lambda(t)]\varphi^-(t) + B(t)$

It is evident that $a_0(t) = R_1^{-1}(t)a(t)R_2^{-1}(t) - \Lambda(t)$ is a piecewise-continuous matrix and

$$\sup_{t\in\Gamma} |a_0(t)| < C_1\varepsilon, \ \ C_1 \text{ is a constant.}$$

Consider now a sequence of the matrices:

$$\varphi_{m+1}(z) = \frac{\chi(z)}{2\pi i}\int_\Gamma \frac{[\chi^+(t)]^{-1}a_0(t)\varphi_m^-(t)}{t-z}dt + \frac{\chi(z)}{2\pi i}\int_\Gamma \frac{[\chi^+(t)]^{-1}B(t)}{t-z}dt, \tag{3.15}$$

where $\varphi_0^-(t) = 0$, $\chi(z)$ is a canonical matrix of the class $E_p^{\pm}(\Gamma,\rho)$ of the matrix $\Lambda(t)$. It is evident that $\varphi_m(z) \in E_p^{\pm}(\Gamma,\rho), m \geqslant 1$. From (3.15) we have

$$\varphi_{m+1}^-(t) - \varphi_m^-(t_0) = -\frac{1}{2}a^{-1}(t)a_0(t)[\varphi_m^-(t_0) - \varphi_{m-1}^-(t_0)]$$

$$+\frac{\chi(t_0)}{2\pi i}\int_\Gamma \frac{[\chi^+(t)]^{-1}a_0(t)[\varphi_m^-(t) - \varphi_{m-1}^-(t)]}{t-z}dt.$$

Whence

$$\|\varphi_{m+1}^- - \varphi_m^{-1}\|_{L_p(\Gamma,\rho)} \leqslant C_2\varepsilon,$$

where C_2 is a constant. Therefore, if $C_1C_2\varepsilon < 1$ then the sequence φ_m^- converges in the space $L_p(\Gamma,\rho)$. It follows from (3.15) that φ_m^+ also converges in the space $L_p(\Gamma,\rho)$. The limit matrix $\varphi(z) \in E_{p,0}^{\pm}(\Gamma,\rho)$ and satisfies the following boundary condition

$$\varphi^+(t) = R_1^{-1}(t)a(t)R_2^{-1}(t)\varphi^-(t) + R_1^{-1}(t)b(t).$$

Consequently the matrix

$$\Phi(z) = \begin{cases} R_1(z)\varphi(z), & z \in D^+, \\ R_2^{-1}(z)\varphi(z), & z \in D^- \end{cases} \tag{3.16}$$

will be the solution of the boundary problem which may have poles in some points of the domains D^+ and D^-.

Consider now the adjoint boundary value problem i.e. the problem.

$$\Psi^+(t) = [a^{'}(t)]^{-1}\Psi^-(t) + g(t), g \in L_q(\Gamma,\rho^{1-q}). \tag{3.17}$$

Substituting in previous arguments the matrices R_1 and R_2 correspondingly by the matrices $R_1^{'-1}$ and $R_2^{'-1}$, we construct the solution of the form:

$$\Psi(z) = \begin{cases} [R_1'(z)]^{-1}\psi(z), & z \in D^+, \\ R_1'(z)\psi(z), & z \in D^-. \end{cases}$$

Take now $b = aR_2^{-1}\chi^-$, $g = a^{'-1}R_2^1(\chi'^-)^{-1}$. We obtain

$$\Phi^+(t) = a(t)[\Phi^-(t) + R_2^{-1}(t)\chi^-(t)],$$

$$\Psi^+(t) = [a^{'}(t)]^{-1}[\Psi^-(t) + R_2'(t)(\chi^{1-}(t))^{-1}].$$

It follows from these equalities, that

$$\Psi^{'+}(t)\Phi^+(t) = [\Psi^{'-}(t) + (\chi^-(t))^{-1}][\Phi^-(t) + \chi^-(t)].$$

Consider the matrix

$$Q(z) = \begin{cases} \psi'(z)\varphi(z), & z \in D^+, \\ \left[\psi'(z) + \chi^{-1}(z)\right]\left[\varphi(z) + \chi(z)\right], & z \in D^-. \end{cases}$$

It is evident that $Q(z) \in E_1^{\pm}(\Gamma)$; $Q(\infty) = I$. Therefore $Q(z) \equiv I$ and

$$[\varphi(z)]^{-1} = \psi^{'}(z), z \in D^+,$$

$$[\varphi(z) + \chi(z)]^{-1} = \psi^{'}(z) + \chi^{-1}(z), z \in D^-.$$

Consequently the matrix

$$\omega(z) = \begin{cases} \varphi(z), z \in D^+, \\ \varphi(z) + \chi(z), z \in D^- \end{cases}$$

has the following properties

$$\omega(z) \in E_p^{\pm}(\Gamma, \rho), \quad \omega^{-1}(z) \in E_q^{\pm}(\Gamma, \rho^{1-q})$$

and the matrix

$$\Phi(z) = \begin{cases} R_1(z)\omega(z), & z \in D^+, \\ R_2^{-1}(z)\omega(z), & z \in D^- \end{cases} \tag{3.18}$$

is suitable for the "preparation" of the canonical matrix.

Now we shall show this. First cite the following auxiliary propositions.

Lemma 17.3.1 *Let $\varphi_1(z)$ be a quadratic matrix of order n and has the following form*

$$\varphi_1(z) = P(z)\varphi(z)[\varphi(c)]^{-1}P^{-1}(z), c \in \Gamma,$$

where $P(z)$ is a diagonal matrix, $P_{kk}(z) = 1$, $k = 1, \cdots, s$, $P_{kk}(z) = z - c$, $k = s+1, \cdots, n$(or all $P_{kk}(z) = z - c$), $\varphi \in E_p^{\pm}(\Gamma, \rho)$, $\varphi^{-1} \in E_q(\Gamma, \rho^{1-q})$. Then $\varphi_1(z) \in E_p^{\pm}(\Gamma, \rho), \varphi_1^{-1}(z) \in E_q^{\pm}(\Gamma, \rho^{1-q})$.

From the equalities

$$\varphi_1(z) = P(z)[\varphi(z)[\varphi(c)]^{-1} - I]P^{-1}(z) + I,$$

$$\varphi_1^{-1}(z) = P^{-1}(z)[\varphi(c)\varphi^{-1}(z) - I]P(z) + I,$$

it follows immediately that the lemma is correct.

Lemma 17.3.2 *Let $\Phi(z)$ be a matrix defined by the formula*

$$\Phi(z) = \begin{cases} r_1(z)\varphi(z), & z \in D^+, \\ r_2(z)\varphi(z), & z \in D^- \end{cases}$$

(here $r_k(z), k = 1,2$ are the rational matrices poles of which aren't situated on $\Gamma, det r_k(t) \neq 0, t \in \Gamma, \varphi(z) \in E_p^{\pm}(\Gamma,\rho), \varphi^{-1}(z) \in E_q(\Gamma,\rho^{1-q}))$. If $\Phi(z)$ satisfies the condition

$$\Phi^+(t) = a(t)\Phi^-(t), t \in \Gamma, \tag{3.19}$$

where $a(t)$ is a given piecewise-continuous matrix on Γ. Then there exists the rational matrix $R(z)$ such, that $\Phi(z)R(z)$ is a canonical matrix for the matrix $a(t)$ of the class $E_p^{\pm}(\Gamma,\rho)$. The index of the matrix $a(t)$ of the class $E_p^{\pm}(\Gamma,\rho)$ is equal to

$$\varkappa = \frac{1}{2\pi i}\left[\arg\frac{\det r_1(t)}{\det r_2(t)}\right]_\Gamma - s, \tag{3.20}$$

where s is order of $\det\varphi(z)$ at infinity.

Proof Let us represent the matrices $r_k(z)$, $k = 1,2$ in the following form [46]

$$r_k(z) = P_k^{(1)}(z)Q_k(z)P_k^{(2)}(z)/\lambda_k(z),$$

where $\lambda_k(z)$ are the polynomials, $P_k^{(1)}(z), P_k^{(2)}(z)$ are the polynomial matrices with the non-zero, constant determinants. $Q_k(z)$ is diagonal polynomial matrix, the polynomial $Q_k^{s+1,s+1}$ is dividing by the polynomial $Q_k^{s,s}$.

Represent the polynomial λ_k and the matrix Q_k in the following form

$$\lambda_k(z) = \lambda_k^{(1)}(z)\lambda_k^{(2)}(z), \quad Q_k(z) = Q_k^{(1)}(z)Q_k^{(2)}(z),$$

where the polynomials $\lambda_k^1(z)$ $\left(\lambda_k^2(z)\right)$ may have the poles only in the domain $D^+(D^-)$, the elements of the main diagonal of the matrix $Q_k^{(1)}(z)$ $\left(Q_k^{(2)}(z)\right)$ may have the zeros only in the domain $D^+(D^-)$.

Give to the matrix $\Phi(z)$ the following form

$$\Phi(z) = \begin{cases} \dfrac{P_1^{(1)}(z)}{\lambda_1^{(1)}(z)} q_1(z)\Psi(z), & z \in D^+, \\[2ex] \dfrac{P_2^1(z)}{\lambda_2^{(2)}(z)} q_2(z)\Psi(z) & z \in D^-, \end{cases}$$

where the following notations are introduced

$$\Psi(z) = \begin{cases} \dfrac{Q_1^{(2)}(z)P_1^{(2)}(z)}{\lambda_1^{(2)}(z)}\varphi(z), & z \in D^+, \\ \dfrac{Q_2^{(1)}(z)P_2^{(2)}(z)}{\lambda_2^{(1)}(z)}\varphi(z), & z \in D^-, \end{cases}$$

$$q_k(z) = Q_k^{(k)}(z) = \mathrm{diag}(q_k^1, \cdots, q_k^n), k = 1, 2.$$

It is evident that $\Psi(z) \in E_p^{\pm}(\Gamma, \rho), \Psi^{-1}(z) \in E_q^{\pm}(\Gamma, \rho^{1-q})$.
Consider the matrix

$$\Phi_1(z) = \frac{\lambda_1^{(1)}(z)\lambda_2^{(2)}(z)}{q_1'(z)q_2'(z)},$$

$$\Phi(z) = \begin{cases} P_1(z)[q_1'(z)]^{-1}q_1(z)\Psi(z), & z \in D^+, \\ P_2(z)[q_2'(z)]^{-1}q_2(z)\Psi(z), & z \in D^-. \end{cases}$$

$$P_1 = \lambda_2^2 P_1^{(1)}/q_2^1(z),$$
$$P_2 = \lambda_1^1 P_2^{(1)}/q_1^1(z).$$

It is clear that $\Phi(z)$ satisfies the boundary condition (3.19).

Denote by c a zero of the polynomial $q_1^2(z)/q_1^1(z)$ (if such exists) and consider the matrix

$$\Phi_2(z) = \Phi_1(z)[\Psi(c)]^{-1}M^{-1}(z) = \begin{cases} P_1(z)(q_1'(z))\Psi(z)[\Psi(c)]^{-1}M^{-1}(z), & z \in D^+, \\ P_2(q_2'(z))^{-1}q_2(z)\Psi(z)[\Psi(c)]^{-1}M^{-1}(z), & z \in D^-. \end{cases}$$

where $M(z) = \mathrm{diag}[1, z - c, \cdots, z - c]$. It is evident that $\Phi(z)$ also satisfies the boundary condition (3.19). If we continue this process, then we will get the solution of the homogeneous problem (3.19), the determinant of which is not equal to zero in the domains D^+, D^-. Consequently we obtain the normal matrix of the class $E_p^{\pm}(\Gamma, \rho)$. Giving to this matrix normal form at infinity (for this we shall multiple it on the corresponding polynomial matrix from the right) we get the canonical matrix.

Tracing the construction of the normal matrix, it is not difficult to be convinced that the formula (3.20) is valid. If we apply this formula to the matrix (3.16) we will obtain the index for the problem (3.15) of the class $E_p^{\pm}(\Gamma, \rho)$ (if the corresponding conditions are fulfilled) is equal to

$$\varkappa = \frac{1}{2\pi}\{\arg\det[a_1(t)a_2(t)]\}_\Gamma + \varkappa_\Lambda,$$

where $\varkappa_\Lambda$ is the index of the matrix $\Lambda(t)$ of the class $E_p^{\pm}(\Gamma, \rho)$. Thus we have the following theorem:

Theorem 17.3.2 *Let $a(t)$ be a piecewise-continuous nonsingular matrix with the points of discontinuity t_k $(k = 1, \cdots, r)$ and let λ_{kj} $(k = 1, \cdots r, j = 1, \cdots, n)$ are the roots of the equation*

$$\det[a^{-1}(t_{k-0})a(t_{k+0}) - \lambda I] = 0$$

Denote by $\mu_{kj} = \arg\lambda_{kj}/2\pi, 0 \leqslant \arg\lambda_{kj} < 2\pi$.

If the inequalities

$$\frac{1+\nu_k}{p} \neq \mu_{kj}, \tag{3.21}$$

are fulfilled, then there exists the canonical matrix of the problem (3.15) *of the class $E_p^{\pm}(\Gamma, \rho)$, and the index of the matrix $a(t)$ is calculated by the formula*

$$\varkappa = \frac{1}{2\pi}\left[\arg\frac{\det a(t)}{\prod\limits_{k=1}^{r}(t - z_0)^{\sigma_k}}\right]_{\Gamma}, \tag{3.22}$$

where $\sigma_k = \sum_{j=1}^{r} \rho_{kj}$

$$\begin{aligned} 1 < \text{Re } \rho_{kj} \leqslant 0 \quad &\text{if} \quad \mu_{kj} < \frac{1+\nu_k}{p}, \\ 0 \leqslant \text{Re } \rho_{kj} < 1 \quad &\text{if} \quad \mu_{kj} > \frac{1+\nu_k}{p}, \end{aligned} \qquad \rho_{kj} = -\frac{1}{2\pi}\ln\lambda_{kj}.$$

The formula (3.22) is analogous to the formula mentioned in the book of [136], §18. Consider now the non-homogeneous problem. Denote by $\chi(z)$ the canonical matrix of the class $E_p^{\pm}(\Gamma, \rho)$. By virtue of the lemmas 17.1.1 and 17.1.2 the problem (3.14) is solvable in the class $E_p^{\pm}(\Gamma, \rho)$ and solutions of this class are given by the formula

$$\Phi(z) = \frac{\chi(z)}{2\pi i}\int \frac{[\chi^+(t)]^{-1}b(t)dt}{t - z} + \chi(z)P(z), \tag{3.23}$$

where $P(z)$ is an arbitrary polynomial vector.

Now look for the solutions of (3.15) vanishing at infinity. Without the loss of generality it is possible to assume that the partial indices $\varkappa_1, \varkappa_2, \cdots, \varkappa_n$ are situated in the decreasing order: $\varkappa_1 \geqslant \varkappa_2 \geqslant \cdots \geqslant \varkappa_n$. For this purpose it is enough to change the position of the columns, i.e. to multiply $\chi(z)$ from the right on the constant nonsingular matrix. Let $\varkappa_1 \geqslant \cdots \geqslant \varkappa_m \geqslant 0 > \varkappa_{m+1} \geqslant \cdots \geqslant \varkappa_n$, $\lambda = \varkappa_1 + \varkappa_2 + \cdots + \varkappa_n$, $\mu = -(\varkappa_{m+1} + \cdots + \varkappa_n)$.

Introduce the following notations

$$[\chi^+(t)]^{-1}b(t) = (b_1, \cdots, b_n),$$

$$P(z) = (P_1, \cdots, P_n);$$

denote also the columns of the canonical matrix by $\chi^1(z), \cdots, \chi^n(z)$ It is possible to write the formula (3.12) in the form

$$\Phi(z) = \sum_{k=1}^{n} \chi^k(z) \left[\frac{1}{2\pi i} \int_\Gamma \frac{h_k(t)dt}{t-z} + P_k(z) \right], \tag{3.24}$$

Expanding the Cauchy type integral in (3.24) in the neighborhood of the point $z = \infty$:

$$\int_\Gamma \frac{h_k(t)dt}{t-z} = -\sum_{s=0}^{\infty} \frac{1}{z^{s+1}} \int_\Gamma t^s h_k(t)dt,$$

we obtain that for the existence of the desired solution it is necessary and sufficient that the free term $b(t)$ have to satisfy the $\mu = -\sum_{k=m+1}^{n} \varkappa_k$ conditions

$$\int_\Gamma t^s h_k(t)dt = 0, \quad (s = 0, 1, \cdots, -\varkappa_{k-1}, k = m+1, \cdots, n) \tag{3.25}$$

and when these conditions are fulfilled the general solution of the desired form is given by the formula (3.23) in which we assume, that

$$P_k(z) = P_{\varkappa_{k-1}}(z),$$

where $P_\alpha(z)$ denotes an arbitrary polynomial of order α; $P_\alpha(z) \equiv 0$ it $\alpha < 0$. The union of the conditions (3.24) we may write in the form of one relation:

$$\int_\Gamma q(t)h(t)dt = 0 \text{ or } \int_\Gamma q(t)[\chi^+(t)]^{-1}h(t)dt = 0, \tag{3.26}$$

where $q(t)$ is defined by the formula

$$q(t) = (q_{-\varkappa_1-1}, \cdots, q_{-\varkappa_n-1}).$$

q_α are the arbitrary polynomials of order $\alpha (q_\alpha = 0$ in case $\alpha < 0)$. the condition (3.26) we may rewrite in the form

$$\int_\Gamma h'(t)[\chi'^+(t)]^{-1}q'(t)dt = 0. \tag{3.27}$$

Note that the expression $[\chi'^+(t)]^{-1}q'(t)$ in (3.26) is a boundary value of the general solution from the domain D^+ of the adjoint homogeneous problem

$$\Psi^+(t) = [a'(t)]^{-1}\Psi^-(t) \tag{3.28}$$

of the class $E^\pm_{q,0}(\Gamma, \rho^{1-q})$. Therefore we get the following theorem.

Theorem 17.3.3 *If the conditions* (3.18) *are fulfilled then for the problem* (3.15) *to be solvable in the class* $E^{\pm}_{p,0}(\Gamma,\rho)$ *it is necessary and sufficient the fulfillment of the conditions*

$$\int_{\Gamma} h(t)\Psi^{+}(t)dt = 0,$$

where $\Psi(z)$ *is an arbitrary solution of the adjoint homogeneous problem* (3.26) *of the class* $E^{\pm}_{q,0}(\Gamma,\rho^{1-q})$.

Let $l(l')$ be a number of linear independent solutions of the homogeneous problem (3.14) (of the homogeneous problem (3.26)) of the class $E^{\pm}_{p}(\Gamma,\rho)$ (of the class $E^{\pm}_{q}(\Gamma,\rho^{1-q})$. Then $l - l' = \varkappa$, where $\varkappa$ is the index of the matrix $a(t)$ of the class $E^{\pm}_{p}(\Gamma,\rho)$.

Remark 1 *If* $\chi(z)$ *is a canonical matrix of the problem* (3.14) *of the class* $E^{\pm}_{p}(\Gamma,\rho)$, *then* $\chi(z)$ *is a canonical matrix of the same problem of the class* $E_{p+\varepsilon}(\Gamma,\rho_\eta)$, $\rho_\eta = \Pi|t-t_k|^{\nu_k+\eta_k}$. *if* ε, η_k *are sufficiently small numbers.*

Remark 2 *For the boundary problem* (3.14) *the following proposition is valid: if* $a(t), b(t) \in H(\Gamma)$ *then the solution of this problem of an arbitrary class* $E^{\pm}_{p}(\Gamma,\rho)$ *are the Hölder-continuous in the closures* $\overline{D}^{+}, \overline{D}^{-}$ (*except perhaps the point* $z=\infty$, *if the solution have the pole there.*) *If* $a(t), b(t) \in H_0(\Gamma)$). *then the solution of the problem of an arbitrary class are the piecewise-holomorphic vectors; they are continuously extendable on all points of* Γ, *except perhaps the points of discontinuity of* $a(t), b(t)$.

17.3.4 Stability of partial indices

The partial indices of the continuous matrix are unstable values in general. The necessary and sufficient stability condition is the following condition

$$\varkappa_1 - \varkappa_n \leqslant 1,$$

where $\varkappa_1(\varkappa_n)$ is the greatest (the smallest) among the partial indices. [see.[20], [136], [51]].

Consider the problem of stability of the partial indices of piecewise-continuous matrix. Let the matrix $a(t) \in C_0(\Gamma, t_1, \cdots, t_2)$, inf $|deta(t)| > 0$.

Let the matrix $g(t)$ of the class $C_0(\Gamma, t_1, \cdots, t_r)$ satisfies the following conditions.

a) $g(c \pm 0) = a(c \pm 0)$, c is an arbitrary singular point of the matrix a,

b) $sup|a(t) - g(t)| \leqslant \varepsilon$; for small ε we shall say, that $g(t)$ is close to $a(t)$.

It is evident, that if the Noetherity conditions (3.18) for the matrix $a(t)$ are fulfilled then these conditions are fulfilled also for matrix $g(t)$ and we may speak about the partial indices of $g(t)$.

Let $\chi(z)$ be a canonical matrix of the class $E_p^{\pm}(\Gamma,\rho)$ and let the vector Φ be some solution of the class $E_{p,0}^{\pm}(\Gamma,\rho)$ of the boundary problem

$$\Phi^+(t) = g(t)\Phi^-(t), \quad t \in \Gamma. \tag{3.29}$$

Rewrite (3.29) in the form

$$[\chi^+(t)]^{-1}\Phi^+(t) = [\chi^-(t)]^{-1}\Phi^-(t) + F(t),$$

$$F(t) = [\chi^+(t)][g(t) - a(t)]\Phi(t). \tag{3.30}$$

If the partial indices of the matrix $a(t)$ are nonpositive then it follows from (3.30) that

$$[\chi(z)]^{-1}\Phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{F(t)dt}{t-z},$$

$$\Phi^-(t_0) = -\frac{1}{2}a^{-1}(t_0)[g(t_0)-a(t_0)]\Phi^-(t_0) + \frac{\chi^-(t_0)}{2\pi i}\int_\Gamma \frac{[\chi^+(t)]^{-1}[g(t)-a(t)]\Phi^-(t)dt}{t-t_0}.$$

It follows from the last equality that

$$\|\Phi^-\|_{L_p(\Gamma,\rho)} \leqslant B \sup|g(t)-a(t)| \|\Phi^-(t)\|_{L_p(\Gamma,\rho)}, \tag{3.31}$$

where B is constant.

If $\sup|g(t)-a(t)|$ is sufficiently small, then from the inequality (3.31) it follows that $\Phi^-(t) \equiv 0$, $\Phi(z) \equiv 0$.

Therefore, if the matrix has the non-positive partial indices then the boundary problem (3.29) have the nontrivial solutions of the class $E_{p,0}^{\pm}(\Gamma,\rho)$ for close to matrix $a(t)$ matrix $g(t)$ and hence such matrices $g(t)$ have also non-positive indices.

Let now the matrix $a(t)$ have arbitrary partial indices

$$\varkappa_1 \geqslant \cdots \geqslant \varkappa_n$$

and $g(t)$ is the matrix close to $a(t)$ with the partial indices

$$\eta_1 \geqslant \cdots \geqslant \eta_k.$$

It is clear that the matrix $a_1(t)(t-b)^{-\varkappa_1}a(t)[g_1(t) = (t-b)^{-\varkappa_1}g(t)]$, where b is a fixed point inside of Γ, has the numbers $\varkappa_k - \varkappa_1 \leqslant O(\eta_k - \eta_1)$ as the partial indices.

Hence, when the matrices $a(t)$ and $g(t)$ are sufficiently close, then the partial indices of the matrix $g_1(t)$ will be non-positive and therefore $\eta_1 \leqslant \varkappa_1$.

Going over from the matrices a and g to the matrices $(a')^{-1}$ and $(g')^{-1}$ and to the classes $E_p^{\pm}(\Gamma,\rho)$, $E_q^{\pm}(\Gamma,\rho^{1-q})$ we get $\eta_n \geqslant \varkappa_k$,

$$\varkappa_1 \geqslant \eta_1 \geqslant \cdots \geqslant \eta_n \geqslant \varkappa_n. \tag{3.32}$$

It implies from the relations (3.32) that if the partial indices of the matrix $a(t)$ satisfies the condition $\varkappa_1 - \varkappa_n \leqslant 1$, then for all sufficiently close matrices

$$\eta_k = \varkappa_k \quad (k = 1, \cdots, n).$$

Due to [20] prove that if $\varkappa_1 - \varkappa_n \geqslant 2$, then the partial indices are unstable.

Let

$$\varkappa_1 = \cdots = \varkappa_s > \varkappa_{s+1} \geqslant \cdots \geqslant \varkappa_n$$

be the partial indices of the matrix $a(t)$ of the class $E_p^{\pm}(\Gamma, \rho)$.

Consider the case when the matrix $a(t)$ has only one point of discontinuity $c \in \Gamma$, this restriction is not essential and is made because of the simplicity of the formulas.

Construct the sequence of the matrices $a_m(t) \in H_0^1(\Gamma, C)$, $a_m(c \pm 0) = a(c \pm 0)$ convergent to the matrix $a(t)$

$$\sup_t |a_m(t) - a(t)| \to 0, \quad m \to 0.$$

Consider two possible cases:

a) the partial indices of $a_m(t)$ coincide with the partial indices starting from some m_0;

b) when the case a) is not possible.

In the case b) the partial indices are unstable. Therefore, we consider the case a).

As it is known the partial indices of the matrix $a(t)$ of the class $E_p^{\pm}(\Gamma, \rho)$ $(\rho = |t - c|^{\nu})$ coincide with the partial indices of the Hölder-continuous matrix

$$A_m(t) = Y_+^{-1} a_m Y_-(t),$$

where

$$\begin{aligned}
&Y_+(z) = AU[u_1]\chi_1(z), && z \in D^+,\\
&Y_-(z) = BU[u]\chi(z), && z \in D^-,\\
&\chi_1(z) = \mathrm{diag}[(z-c)^{\rho_1}, \cdots, (z-c)^{\rho_n}], && \chi = \chi_1 \chi_0^{-1},\\
&\chi_0(z) = \mathrm{diag}[(z-z_0)^{\rho_1}, \cdots, (z-z_0)^{\rho_n}], && z_0 \in D^+,\\
&-\frac{1+\nu}{p} < \mathrm{Re}\rho_n < 1 - \frac{1+\nu}{p}, && \rho_k = \frac{1}{2\pi i} \ln \lambda_k,
\end{aligned}$$

A, B are the constant non-singular matrices.

λ_k are the roots of the equation $\det(a^{-1}(c+0)a(c-0) - \lambda I) = 0$,

$$u_1 = \frac{1}{2\pi i} \ln(z - c), \quad u_2 = \frac{1}{2\pi i} \ln \frac{z - c}{z - z_0},$$

$u(\xi)$ is definite polynomial matrix of ξ. These matrices are defined in the book [136] §18.

Represent the matrix A_m in the form (see [136], §7)

$$A_m = \chi_m^+ \Lambda \chi^-,$$

where $\chi_m^{\pm}(t)$ are the Hölder-continuous matrices

$$\Lambda(t) = \mathrm{diag}[t^{\varkappa_1}, t^{\varkappa_2}, \cdots, t^{\varkappa_{n-1}}].$$

(We suppose that $\mathrm{O} \in D^+$).

Consider the matrix

$$A_m^\varepsilon = A_m(t) + \varepsilon(t-c)q(t),$$

$$q(t) = \begin{pmatrix} 0 & t^{\varkappa_2} & \cdots & 0 \\ t^{\varkappa_2 - 2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & . \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{3.33}$$

Let $\tilde{\varkappa}_1 \geqslant \tilde{\varkappa}_2 \geqslant \cdots \geqslant \tilde{\varkappa}_n$ be the partial indices of A_m^ε. It is not difficult to check that for sufficiently small ε for the matrix $A_m^\varepsilon(t)$ we will have

$$\tilde{\varkappa}_s = \varkappa_s - 1.$$

It follows from (3.33) that

$$a_m^\varepsilon = Y_+ A_m^\varepsilon Y_-^{-1} = a_m + \varepsilon(t-c) Y_+ q Y_-^{-1}$$

and hence

$$a_m^\varepsilon(c \pm 0) = a_m(c \pm 0),$$
$$\sup |a_m^\varepsilon - a_m| \to 0, \text{ when } \varepsilon \to 0.$$

The sequence $a_m^{\varepsilon_m}(t)$ $(\varepsilon_m \to 0)$ converges to the matrix $a(t)$ with respect to the above mentioned norm; therefore the condition $\varkappa_1 - \varkappa_n \leqslant 1$ is not only sufficient but also necessary condition for the partial indices to be stable.

17.4 Systems of singular Integral equations

Consider first the so called characteristic system of singular integral equations

$$\sum_{\beta=1}^{n} \left[A_{\alpha\beta}(t_0)\varphi_\beta(t_o) + \frac{B_{\alpha\beta}(t_0)}{\pi i} \int_\Gamma \frac{\varphi_\beta(t)dt}{t - t_0} \right] = f_\alpha(t_0), \quad \alpha = 1, \cdots, n, \tag{4.1}$$

where $A_{\alpha\beta}$, $B_{\alpha\beta}$ are given piecewise-continuous functions on Γ, f_α are the given functions on Γ of the class $L_p(\Gamma, \rho)$ We look for the solution of the system (4.1) in the class $L_p(\Gamma, \rho)$.

Introducing the examined matrices and vectors

$$A = (A_{\alpha\beta}), \quad B = (B_{\alpha\beta}), \quad \varphi = (\varphi_1, \cdots, \varphi_n), \quad f = (f_1, \cdots, f_n).$$

we may rewrite (4.1) in the form

$$K^0\varphi \equiv A(t_0)\varphi(t_0) + \frac{B(t_0)}{\pi i}\int \frac{\varphi(t)}{t-t_0}dt = f(t_0). \tag{4.2}$$

Let φ be a solution of the equation (4.2). Denote by

$$\Phi(z) = \frac{1}{2\pi i}\int \frac{\varphi(t)}{t-z}dt. \tag{4.3}$$

We have

$$\varphi(t) = \Phi^+(t) - \Phi^-(t), \quad \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)dt}{t-t_0} = \Phi^+(t_0) + \Phi^-(t_0). \tag{4.4}$$

If we substitute these values in the equation (4.2) we get

$$S(t)\Phi^+(t) = D(t)\Phi^-(t) + f(t), \tag{4.5}$$

where $S = A + B, \quad D = A - B$.

Let

$$\inf|\det S(t)| > 0, \inf|\det D(t)| > 0, \quad t \in \Gamma. \tag{4.6}$$

Then we may rewrite (4.5) in the following form

$$\Phi^+(t) = a(t)\Phi^-(t) + b(t), \tag{4.7}$$

where $a = S^{-1}D, \quad b = S^{-1}f$.

Therefore the equation (4.2) is reduced to the boundary problem (4.7): to every solution of (4.2) of the class $L_p(\Gamma, \rho)$ corresponds the solution of the problem (4.3) of $L^{\pm}_{p,0}(\Gamma, \rho)$ by the formula (4.3), and to every such solution of (4.7) corresponds the solution of the equation (4.2) of the class $L_p(\Gamma, \rho)$ by the formula (4.4).

This connection between the equations and boundary problem gives us the possibility to establish the following proposition (see [136], [108]).

Theorem 17.4.1 *Let the conditions* (4.6), (3.18) *be fulfilled. For the equation* (4.2) *to be solvable in the class* $L_p(\Gamma, \rho)$ *it is necessary and sufficient that*

$$\int_\Gamma f(t)\psi(t)dt = 0, \tag{4.8}$$

where ψ *is an arbitrary solution of the class* $L_q(\Gamma, \rho^{1-q})$ *of the adjoint homogeneous equation*

$$K^{0\prime}\psi = A'(t_0)\psi(t_0) - \frac{1}{\pi i}\int_\Gamma \frac{B'(t)\psi(t)}{t-t_0}dt = 0. \tag{4.9}$$

In case when the conditions (4.8) *are fulfilled all solutions of the equation* (4.2) *of the class* $L_p(\Gamma,\rho)$ *are given by the formula*

$$\varphi(t_0)=A^*(t_0)f(t_0)-\frac{B^*(t_0)Z(t_0)}{\pi i}\int_L\frac{[Z(t)]^{-1}f(t)}{t-t_0}dt+B^*(t_0)Z(t_0)P(t_0),$$

$$A^*(t_0)=\frac{1}{2}[S^{-1}(t)+D^{-1}(t)],\ \ B^*(t)=-\frac{1}{2}[S^{-1}(t)-D^{-1}(t)],$$

$$Z(t_0)=S(t)\chi^+(t)=D(t)\chi^-(t) \tag{4.10}$$

$\chi(z)$ *is a canonical matrix of the class* $E_p^{\pm}(\Gamma,\rho)$ *for the matrix* $a(t)=S^{-1}D, P(t)$ *is a vector*

$$P(t)=(P_{\varkappa_1-1},\cdots,P_{\varkappa_n-1}),$$

$P_\alpha(t)$ *denotes the arbitrary polynomials of order not more then* $\alpha, P_\alpha(t)=0$ *when* $\alpha<0$.

The difference between the number l *linearly independent solutions of the homogeneous equation* $K^0\varphi=0$(*in* $L_p(\Gamma,\rho)$) *and the number* l' *linearly independent solutions of the adjoint homogeneous equation* $K^{0\prime}\psi=0$ (*in* $L_q(\Gamma,\rho^{1-q})$) *is equal to the index of the matrix* $a=S^{-1}D$ *of the class* $E_p^{\pm}(\Gamma,\rho)$:

$$l-l'=\varkappa.$$

Let us consider the equation of more general form

$$K\varphi=f, \tag{4.11}$$

where $K\varphi=K^0\varphi+k\varphi,\ k\varphi\equiv\int_\Gamma\frac{h(t_0,t)}{|t-t_0|^\alpha}\varphi(t)dt,\quad 0\leqslant\alpha<1,\ h(t_0,t)$ is a measurable bounded matrix.

$k\varphi$ is a completely continuous operator in any space $L_p(\Gamma,\rho)$ [79]; basing on the well-known theorems of functional analysis (see for example [105]), we obtain that the formulated above theorem 17.4.1 is valid also for the equation (4.10) substituting the operators $K^0\varphi$ and $K^{0\prime}\psi'$ by the operators $K\varphi$ and $K'\psi$ correspondingly. $K'\psi=K^{0\prime}\psi+k'\psi$,

$$k'\psi\equiv\int_\Gamma\frac{h'(t,t_0)}{|t-t_0|^\alpha}\psi(t)dt.$$

The equation of the form

$$A_1(t_0)\varphi(t_0)+\sum_{k=1}^{s}\frac{B_k(t_0)}{\pi i}\int_\Gamma\frac{D_k(t)\varphi(t)}{t-t_0}dt+\int_\Gamma K_1(t_0,t)\varphi(t)dt=f(t_0) \tag{4.12}$$

is reducing to the equation of the form (4.9); thus we may obtain for this equation the theorem analogous to the theorem 17.4.1.

The equation (4.10) is equivalent to the following equation (see [136], §27):

$$A(t_0)\Phi(t_0) + \frac{B(t_0)}{\pi i}\int_\Gamma \frac{\Phi(t)dt}{t-t_0} + \int_\Gamma K(t_0,t)\Phi(t)dt = F(t_0), \tag{4.13}$$

where A, B, K are the block matrices

$$A = \begin{pmatrix} A_1\delta_1 & 0 & \cdots & 0 \\ \delta_1 & \delta_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & .. \\ \delta_1 & 0 & \cdots & \delta_s \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & B_2 & \cdots & B_s \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & .. \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$K(t_0,t) = \begin{pmatrix} K_1(t_0,t)\delta_1(t) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots\cdots\cdots & \cdots & \cdots & .. \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$\delta_1 = D_1^{-1}, \quad \delta_1 = -D_k^{-1}, \quad k = 2, \cdots, s,$$

$F(t)$ is $(s \times n)-$dimensional vector

$$F(t) = (f(t), 0, \cdots, 0),$$

$\Phi(t)$ is a desired vector. Reducing the equation (4.12) to (4.13) it is supposed that D_k are nonsingular matrices but it is not essential; substituting in case of necessity the matrices $D_k(t)$ on the matrices $D_k + cI$ and $-cI$ (c is sufficiently large with respect to the modulus constant) and the number s on $2s'$ we obtain the equation of the form (4.11) for which corresponding condition is fulfilled.

In applications it may occur very frequently the following singular integral equation of the form

$$K_1\varphi + \overline{K_2\varphi} = f, \tag{4.14}$$

where

$$K_s\varphi \equiv A_s(t_0)\varphi(t_0) + \frac{B_s(t_0)}{\pi i}\int_\Gamma \frac{\varphi(t)dt}{t-t_0} + \int_\Gamma \frac{h_s(t_0,t)}{|t-t_0|^\alpha}\varphi(t)dt.$$

$A_s, B_s (s = 1, 2)$ are given piecewise-continuous quadratic matrices of order n, $h_s(t_0,t)$ are measurable bounded matrices, $f(t) = (f_1, \cdots, f_n)$ is given vector of the class $L_p(\Gamma, \rho)$, $\Gamma \in H^1_\mu$.

We may reduce the equation (4.12) also to the equation of the form (4.10); the role of the matrix S and D in case of the equation (4.12) play the following block matrices

$$\begin{pmatrix} A_1 + B_1 & \overline{A_2} - \overline{B_2} \\ A_2 + B_2 & \overline{A_1} - \overline{B_1} \end{pmatrix}, \quad \begin{pmatrix} A_1 - B_1 & \overline{A_2} + \overline{B_2} \\ A_2 - B_2 & \overline{A_1} + \overline{B_1} \end{pmatrix}.$$

Remark 1 *Using the properties of the solutions of the boundary problem of linear conjugation we get the following proposition: if the coefficients and the free terms* *of the equations mentioned in this section are Hölder continuous then the solutions of any class are also Hölder-continuous and if the coefficients and the free terms belong to the class $H_0(\Gamma)$ then the solutions of any class belong to the class $H^*(\Gamma)$.*

Indeed, let in the equation (4.11) $A(t), B(t), f(t) \in H(\Gamma)$, $h(t_0, t) \in H(\Gamma \times \Gamma)$, $det(A+B) \neq 0$, $det(A-B) \neq 0$, $\varphi(t) \in L_p(\Gamma)(p > 1)$ is a solution of this equation and $\varphi(t)$ satisfies also the equation

$$K^0\varphi = f_0,$$
$$f_0(t_0) = f(t_0) - \int_\Gamma \frac{h(t_0), t}{|t - t_0|^\alpha}\varphi(t)dt.$$

The vector f_0 belongs to the class $L_{p_1}(\Gamma)$, $p_1 > p$ (see [79], §8).

By virtue of the formula (4.10), as

$$A^*(t), B^*(t), Z(t) \in H(\Gamma),$$

we obtain $\varphi(t) \in L_{p_1}(\Gamma)$; reasoning in such a manner we may conclude that $\varphi(t) \in L_\infty(\Gamma)$ and $\varphi(t) \in H(\Gamma)$.

Let now $A(t), B(t), f(t) \in H_0(\Gamma)$ and $h(t_0, t) \in H(\Gamma \times \ \Gamma)$ (the last means that $h(t_0, t)$ belongs to the class H_0 with respect to t for the fixed t_0 and also with respect to t for the fixed t_0); besides, let

$$inf|det(A+B)| > 0, \quad inf|det(A-B)| > 0$$

and the inequalities (3.18) are fulfilled. In this case $A^*(t), B^*(t) \in H_0(\Gamma)$, $Z(t) \in H^*(\Gamma)$; reasoning as above, we obtain that the solution of the equation of the class $L_p(\Gamma, \rho)$ belongs to the class $H^*(\Gamma)$.

Remark 2 *It constitutes no principal difficulty to consider the cases when in singular integral equations the integration domain is a finite union of simple piecewise-smooth curves or when in the linear conjugation problems the boundary is the same union of the curves. Whereas, if we can construct a canonical matrix in the case of one simple curve then me may construct a canonical matrix for the finite union of the curves.* (*see* [108] §129).

In fact let $\Gamma = \bigcup_{k=1}^m \Gamma_k, a(t)$ be a given piecewise-continuous nonsingular matrix on Γ and let for every separate curve Γ_k exists a canonical matrix of the given class. Denote by $\chi_k(z)$ normal (or canonical) matrix for the problem

$$\Phi_k^+(t) = a_k(t)\Phi_k^-, t \in \Gamma_k,$$

where

$$\begin{aligned}
&a_1(t) = a(t), \quad t \in \Gamma_1,\\
&a_2(t) = [\chi_1]^{-1}a(t)\chi_1(t), t \in \Gamma_2,\\
&\cdots\\
&a_m(t) = [\chi_1(t)\cdots\chi_{m-1}(t)]^{-1}a(t)\chi_1(t)\cdots\chi_{m-1}(t), t \in \Gamma_m.
\end{aligned}$$

It is easy to be convinced that the product

$$\chi(z) = \chi_1(z) \cdots \chi_m(z)$$

is a normal matrix for the union of the curves $\Gamma = \bigcup_{k=1}^{m} \Gamma_k$.

17.5 Differentiability of solutions and singular integral equations

In researching the problems of mathematical physics with the help of singular integral equations (or with the help of the problems of linear conjugation of the analytic functions) sometimes it is necessary to study the problem of existence of the derivatives of the desired solution or the behavior of the solutions in the neighborhood of the discontinuity points of the coefficients. In the case of one unknown function this problem is comparatively easily solved because in this case it is possible to use the effective (explicit) solutions of singular integral equations or the solutions of linear conjugation problems. But in the case of the systems of the equations or linear conjugation problems for several unknown functions the situation is rather different.

Consider the boundary value problem of linear conjugation

$$\Phi^+ = a(t)\Phi^-(t) + b(t). \tag{5.1}$$

The boundary condition should be fulfilled across the simple smooth curve Γ; $a(t)$ is a given nonsingular quadratic matrix of order n, $b(t)$ is a given $(m \times l)$-matrix, $a(t), b(t) \in H^\mu(\Gamma), \Phi(z)$ is a desired piecewise-holomorphic matrix , it is continuously extendable on Γ from the domains D^+, D^- (D^+ is a finite domain bounded by Γ, D^- is an exterior domain).

The solution of the problem (5.1) belongs to the classes $H^\mu(D^\pm)$ for $\mu < 1$ and to the classes $H^{1-\varepsilon}(D^\pm)$ for $\mu = 1$, ε is an arbitrary small positive number (see.[108], §133); it is clear that when $\Phi(z)$ has the pole in the point $z = \infty$ then

$$\Phi(z) - P(z) \in H^\mu(D^-);$$

where the polynomial matrix P(z) is a principal part of $\Phi(z)$ at the point $z = \infty$.

Let now $a(t)$, $b(t) \in H^s_\mu(\Gamma)$, $s \geqslant 1$.

Choose the rational matrix $r(z)$ such that

$$\left\| \frac{d^k a(t)}{dt^k} - \frac{d^k r(t)}{dt^k} \right\|_{H^\nu} \leqslant \varepsilon, k = 0, \cdots, s, \nu < \mu.$$

Consider the sequence of piecewise-holomorphic matrices

$$\varphi_{m+1}(z) = \frac{1}{2\pi i} \int_\Gamma \frac{a_0 \varphi^-_m(t)}{t - z} dt + \frac{1}{2\pi i} \int_\Gamma \frac{b(t)dt}{t - z},\ a_0 = ar^{-1} - I,\ \varphi^-_0(t) = 0. \tag{5.2}$$

For sufficiently small ε the boundary values $\varphi_m^+(t), \varphi_m^-(t)$ are converging with respect to the norm of the space $H^0(\Gamma)$ to the boundary values of piecewise-holomorphic matrix $\varphi(z)$ which satisfies the boundary condition

$$\varphi^t = a(t)r^{-1}(t)\varphi^-(t) + b(t).$$

If we differentiate (5.1), we get

$$\varphi'_{m+1} = \frac{1}{2\pi i}\int_\Gamma \frac{a_0(t)\varphi_m^{'-}(t) + a_0'(t)\varphi_m^-(t)}{t-z}dt + \frac{1}{2\pi i}\int_\Gamma \frac{b'(t)}{t-z}dt. \tag{5.3}$$

From the equality (5.3) we conclude that the boundary values φ'_m are converging to the boundary values of the matrix $\varphi'(z)$ with respect to the norm of the space $H^\nu(\Gamma)$; continuing this argument we get that the boundary values of the derivatives $\varphi_m^{(s)}(z)$ are also converging to the boundary values $\varphi^{(s)}(z)$. The boundary values of the matrix

$$\Phi(z) = \begin{cases} \varphi(z), & z \in D^+, \\ r^{-1}(z)\varphi(z), & z \in D^-, \end{cases}$$

are satisfying the boundary condition (5.1).

The similar argumentation implies, that for an arbitrary solution $\Phi(z)$ of the problem (5.1) the following inclusions:

$$\Phi(z) \in H_\mu^s(D^+), \Phi(z) - P(z) \in H_\nu^s(D^-)$$

are valid. P(z) is a principal part of $\Phi(z)$ at the point $z = \infty$. It is easy to show that, when $\mu < 1$ in these inclusions we may take $\nu = \mu$.

Let now there exist the derivatives $da^s/dt^s, d^s b/dt^s$, satisfying the condition $H(\mu)$ on the open arc $\sigma \subset \Gamma$; then for an arbitrary solution of the problem (5.1) we get

$$\frac{d^s\Phi^\pm(t)}{dt^s} \in H(\sigma).$$

Take some arc $c_1c_2 \subset \sigma$ and construct the rational matrix $R(z)$, having the properties:

$$detR(t) \neq 0 \text{ on } \Gamma; \ R(c_j)a(c_j) = I, \ j = 1, 2.$$

$$\frac{d^k}{dt^k}[R(t)a(t)] = 0 \text{ when } t = c_1, \ t = c_2, \ k = 1, \cdots, s.$$

Let

$$a_1(t) = \begin{cases} R(t)a(t), & t \in c_1c_2, \\ I, & t \in \Gamma \backslash c_1c_2. \end{cases}$$

It is evident that $deta_1(t) \neq 0$, $a_1(t) \in H_\mu^s\Gamma$.

By virtue of the just proved proposition, $d^s\chi^\pm/dt^s \in H(\Gamma)$ ($\chi_1(z)$ is a canonical matrix of the matrix $a_1(t)$). Consider the matrix

$$a_2(t) = [\chi_1^+(t)]^{-1}R(t)a(t)\chi_1^-(t);$$

it is easy to see that $a_1(t) = I$ when $t \in c_1c_2$.

Let $\chi_2(z)$ be a canonical matrix of the matrix $a_2(t)$:

$$\chi_2^+(t) = a_2(t)\chi_2^-(t) \tag{5.4}$$

when $t \in c_1c_2$, $\chi_2^+(t) = \chi_2^-(t)$ and $\chi_2(z)$ is a holomorphic matrix in the neighborhood of the arc c_1c_2.

We may rewrite the equality (5.4) in the following form

$$R^{-1}(t)\chi_1^+(t)\chi_2^+(t) = a(t)\chi_1^-(t)\chi_2^-(t).$$

Hence the matrix

$$X(z) = \begin{cases} R^{-1}(z)\chi_1(z)\chi_2(z)R_1(z), & z \in D^+, \\ \chi_1(z)\chi_2(z)R_1(z), & z \in D^-, \end{cases} \tag{5.5}$$

where $R_1(z)$ is the rational matrix chosen in the corresponding manner, is a canonical matrix of the matrix $a(t)$.

If follows from the formulas (5.4) that

$$d^s\chi \pm (t)/dt^s \in H(\Gamma).$$

Consider now the problem (5.1) under the following assumptions $a(t),\ b(t) \in H_0(\Gamma, c_1, \cdots, c_r)$ and on the closed arcs c_kc_{k+1} the matrices $a(t),\ b(t) \in H_\mu^s$ $(a(c_k) = a(c_k+0), a(c_{k+1}) = a(c_{k+1}-0), b(c_k) = b(c_k+0), b(c_{k+1}) = b(c_{k+1}-0))$.

It is easy to prove, that $a(t)$ may be represented as the following (see 17.3):

$$a(t) = R_1(t)\Lambda(t)h(t),$$

$R_1(t), \Lambda(t), h(t)$ are the nonsingular matrices, R_1 is a rational matrix, $h(t) \in H_\mu^s, \Lambda(t)$ is a lower triangular matrix, $\Lambda(t)$ belongs to the class C^∞ on the closed arcs c_kc_{k+1}.

Construct the rational matrix R_2, satisfying the conditions

$$R_2(c_k) = h(c_k), R_2'(c_k) = h'(c_k), \cdots, R_2^{(s)}(c_k) = h^{(s)}(c_k),$$

$$\|R_2 - h\|_e \leqslant \varepsilon, \cdots, \|R_2^{(s)} - h^{(s)}\|_e \leqslant \varepsilon.$$

For a sufficiently small ε, $detR(t) \neq 0, t \in \Gamma$.

Construct the sequence of the matrix $\varphi_m(z)$ by the formula

$$\varphi_{m+1}(z) = \frac{\chi(z)}{2\pi i}\int_\Gamma \frac{[\chi^+(t)]^{-1}g(t)\varphi_m^-(t)}{t-z}dt + \frac{\chi(z)}{2\pi i}\int_\Gamma \frac{[\chi^+(t)]^{-1}b_0(t)}{t-z}dt. \tag{5.6}$$

where $\chi(z)$ is a canonical matrix of the matrix $\Lambda(t)$ of the class $E_p^{\pm}(\Gamma)$ (p is a sufficiently large number),

$$b_0(t) = R_1^{-1}(t)b(t), \quad g(t) = \Lambda(t)[h(t) - R_2(t)]R_2^{-1}(t),$$

$$\varphi_0^-(t) = 0.$$

It is not difficult to see that the norms

$$\|g\|_C, \|g'\|_C, \cdots, \|g^{(s)}(t)|_C$$

are the values of order ε; besides

$$g(c_k) = g'(c_k) = \cdots = g^{(s)}(c_k) = 0.$$

From the equality (5.5) we can see that the sequences $\varphi_m^+(t), \varphi_m^-(t)$ for the sufficiently small ε are converging to the boundary values of the matrix $\varphi(z) \in E_p^{\pm}(\Gamma)$ under the norm of the space $L_p(\Gamma)$ and the angular boundary values of $\varphi(z)$ are satisfying the relation

$$\varphi^+(t) = \Lambda(t)h(t)R_2^-(t)\varphi^-(t) + R_1^{-1}(t)b(t) \tag{5.7}$$

almost everywhere on Γ.

Based on the Subsection 17.3.1, we may assert that $\varphi^+(t), \varphi^-(t)$ are the Hölder-continuous functions on every arc which is not containing the discontinuity points c_k. But in the case, mentioned below one may assume that $\varphi^+(t)$, $\varphi^-(t) \in H(\Gamma)$.

Let λ_{kj} $(k = 1, \cdots, r;\ j = 1, \cdots, n)$ be the roots of the equation

$$\det[a(c_k - 0) - \lambda a(c_k + 0)] = 0.$$

Assume that among the numbers

$$\tau_{kj} = \mathrm{Re}\frac{1}{2\pi i}\ln\lambda_{kj},$$

we have no integers. While constructing the canonical matrix $\chi(z)$ of the class $E_p^{\pm}(\Gamma)$, corresponding $a(t)$, the numbers

$$\tau_{kj} \in \left(-\frac{1}{p}\ ,\ \frac{p-1}{p}\right);$$

if p is sufficiently large, then $\tau_{kj} > 0$. Therefore the sequences $\varphi_m^+(t)$, $\varphi_m^-(t)$ will converge with respect to the norm of the space $H^\delta(\Gamma)$, if δ is a sufficiently small positive number; thus,

$$\varphi(z) \in H^\delta(D^+), \quad \varphi(z) \in H^\delta(D^-).$$

The case when some of the numbers $\tau_{kj} = 0$, reduces to the case considered above, if we multiply the matrix by the piecewise-continuous function with the discontinuity points $c_1, \cdots, c_r$ chosen in the corresponding manner. That is why in the general case we have

$$\varphi^+(t),\ \varphi^-(t) \in H_\varepsilon^*(\Gamma).$$

Consider now the case when $s \geqslant 1$.

From the formulas mentioned in Subsection 17.3.3, it is easy to see that for the sufficiently large p for the arbitrary matrix $\chi(z)$ we get:

$$\Pi(t)\frac{d\chi^\pm(t)}{dt} \in H_\varepsilon^*(\Gamma),\ \ \Pi(z) = \prod_{k=1}^{r}(z - z_k).$$

Based on this inclusion we obtain the following formula:

$$\chi'(z) = \frac{\chi(z)}{2\pi i}\int_\Gamma \frac{[\chi^+(t)]^{-1}\Lambda_1(t)\chi^-(t)}{t-z}dt + \frac{\chi(z)}{\Pi(z)}P(z), \tag{5.8}$$

where $P(z)$ is some polynomial matrix, $\Lambda_1(t) = d\Lambda(t)/dt$; it is easy to see that the matrix $\Lambda(t)$ may be chosen in such a manner that $\Lambda_1(t) = 0$ in the neighborhood of the points c_k $(k = 1, \cdots, r)$. After differentiating the equality (5.6) by z and multiplying by $\Pi(z)$ and using the formulas (5.8) one may conclude that for sufficiently small ε the boundary values of the sequences

$$\Pi(t)\varphi_m^{'+}(t), \quad \Pi(t)\varphi_m^{'-}(t)$$

are converging, with respect to the norm of $L_p(\Gamma)$, to

$$\Pi(t)\varphi^{'+}(t), \quad \Pi(t)\varphi^{'-}(t)$$

and these matrices belong to the class $H_\varepsilon^*(\Gamma)$. Continuing these arguments, we obtain analogously that

$$\{\Pi(t)\}^k\,\varphi^{(k)+}(t),\ \ \{\Pi(t)\}^k\,\varphi^{(k)-}(t) \in H_\varepsilon^*(\Gamma),\ \ k = 2, \cdots, s. \tag{5.9}$$

If for some nonsingular point c_k, $\tau_{kj} > 0$ $(j = 1, \cdots, n)$, then

$$\{\Pi(t)\}^{s-1}\,\frac{d^s\varphi^\pm(t)}{dt^s} \in H^*(\gamma_k), \tag{5.10}$$

where γ_k is an arc containing from the points of discontinuity only the point c_k.

From the formula (5.7) one can see, that the matrix

$$\Phi(z) = \begin{cases} R_1(z)\varphi(z), & z \in D^+, \\ R_2(z)\varphi(z), & z \in D^- \end{cases}$$

is a piecewise-meromorphic solution of the problem (5.1) i.e., is a solution which may have only finite number of the poles (different from the point $z = \infty$).

As it was mentioned above, from such a solution it is possible to construct a piecewise-holomorphic solution. Consequently the relations (5.9), (5.10) are valid for an arbitrary solution of the class $E_p^{\pm}(\Gamma)$ (p is a sufficiently large number) of the problem (5.1).

Find the necessary and sufficient conditions of the existence of a piecewise-holomorphic solution ϕ of the problem (5.1), vanishing at infinity with the derivatives almost bounded up to s order at the points of discontinuity $c_j (j = 1, \cdots, r)$ of the matrices $a(t)$ and $b(t)$, i.e., $\phi(z)$ satisfies the condition

$$\lim_{z \to c_j} |z - c_j|^{\varepsilon} \Phi^{(s)}(z) = 0, \quad j = 1, \cdots, r,$$

for any $\varepsilon > 0$. For the simplicity we'll assume that $\varepsilon = 1$.

Together with the problem (5.1) we consider the problem of linear conjugation

$$\Psi^{+}(t) = A(t)\Psi^{-}(t) + F(t), \tag{5.11}$$

where $A(t)$ is a block matrix, $A = (A_{ik})$, $i, k = 1, \cdots, s+1$,

$$A_{ik} = \begin{pmatrix} i & -1 \\ k & -1 \end{pmatrix} \frac{d^{i-k}a}{dt^{i-k}}, \quad k \leqslant i, \quad A_{ik} = 0, \quad k > i,$$

$F(t)$ is a block vector, $F = (a, da/dt, \cdots, d^s a/dt^{(s)})$, $\Psi(z)$ is a desired block vector, $\Psi(z) = (\Psi_1, \cdots, \Psi_{s+1})$.

For the formulated problem to be solvable the fulfillment of the following conditions is necessary and sufficient

a) the problem (5.11) has the vanishing at infinity solution $\Psi(z)$, which is almost bounded in the neighborhoods of the discontinuity points c_k;

b) $\Psi(z)$ has the property

$$\frac{d\Psi_k(z)}{dz} = \Psi_{k+1}(z), \quad k = 1, \cdots, s.$$

Note that the singular (nonsingular) points of the problem (5.1) are the singular (nonsingular) points of the problem (5.11); besides, if $\varkappa_k$ $(k = 1, \cdots, n)$ and η_k $(k = 1, \cdots, n(s+1))$ are the partial indices of some class of the problems (4.14) and (5.11) respectively then

$$\varkappa_1 = \eta_k \ \ (k = 1, \cdots, s+1), \quad \varkappa_2 = \eta_k \ \ (k = s+2, \cdots, 2s+2),$$

$$\varkappa_n = \eta_k \ \ (k = n(s+1) - n, \cdots, n(s+1)).$$

The conditions a) are expressed in the following form [136], §19

$$\int_{\Gamma} q(t)[\chi^{+}(t)]^{-1} F(t) dt = 0, \tag{5.12}$$

where $\chi(z)$ is a canonical matrix of the problem (5.11), $q(t)$is a polynomial vector of the form $q = (q_1, \cdots, q_N)$, $N = n(s+1)$, $q_k(z)$ is an arbitrary polynomial of the order - $\eta_k - 1 (k = 1, \cdots, N), \eta_k$ are the partial indices of the problem (5.11) of the considered class.

Change the conditions b) by the conditions

$$\frac{d^{j+1}\Psi_k}{dz_0^{j+1}} = \frac{d^{\nu}\Psi_{k+1}}{dz_0^{\nu}}, \quad k = 1, \cdots, s; \quad j = 0, \cdots, M, \tag{5.13}$$

where M is some natural number, z_0 is a fixed point, $z_0 \notin \Gamma$.

This is weakening the conditions b). Show that if M is sufficiently large number then the conditions (5.12), (5.13) are sufficient (and therefore necessary and sufficient) for the existence of the desired solution.

Let the conditions (5.12) be fulfilled then the problem (5.11) has the vanishing at infinity solution $\Psi = (\Psi_1, \cdots, \Psi_{s+1})$.

From the boundary condition (5.11) using the differentiation we get

$$\Omega_k^+(t) = a(t)\Omega_k^-(t), \quad \Omega_k = \Psi_{k+1} - d\Psi_k/dz, \quad k = 1, \cdots, s,$$

and hence,

$$\Omega_k(z) = [\Pi(z)]^{-1}\chi(z)P_k(z), \tag{5.14}$$

where $P_k(z)$ is some polynomial vector, $\chi(z)$ is a canonical matrix of the problem (5.1) of the class $E_p^{\pm}(\Gamma)$ (p is a sufficiently large number), $\chi(z)$ is almost bounded in the neighborhoods of the points c_k.

In the equality (5.14) the left hand side order is not more then (-1), that is why the order of $P_k(z)$ might be not more then $\varkappa_1 + r - 1$; therefore, if we take in the conditions (5.13) $M \geqslant \varkappa_1 + r - 1$, where $\varkappa_1$ is the maximal among the partial indices $\varkappa_k$ then the equality

$$\frac{d\Psi_k}{dz} = \Psi_{k+1}$$

is fulfilled.

Note also that if $\varkappa_1 \leqslant -m$, then the conditions (5.12) are sufficient for the existence of the desired solution. If the partial indices $\varkappa_k \geqslant 0$ then we haven't the conditions (5.12); the conditions (5.13) may be fulfilled at the expense of the arbitrary constants entering in the general solution of the problem (5.11).

In the general case if we substitute the general solution of the problem (5.11) in the conditions (5.13), we get the linear system of algebraic equations with respect to the constants entering in the general solution. The necessary and sufficient conditions for this linear system to be solvable are the following

$$\int_{\Gamma} F(t)H_k(t)dt = 0, \quad k = 1, \cdots, L,$$

where H_k are definite linearly independent vectors (which are depending only on the matrix $a(t)$).

Hence, for the problem mentioned at the beginning of this section to be solvable it is necessary and sufficient, that

$$\int_\Gamma F(t)Q_k(t)dt = 0, \quad k = 1, \cdots, L^*, \tag{5.15}$$

where Q_k are the linearly independent vectors which are constructed in the above mentioned manner. When $n = 1$ the vectors are constructed in quadratures effectively.

Note that if we have the characteristic system of singular integral equations

$$K^0\varphi \equiv A(t_0)\varphi(t_0) + \frac{B(t_0)}{\pi i}\int_\Gamma \frac{\varphi(t)dx}{t - t_0} = f(t_0), \tag{5.16}$$

then reducing it to the boundary problem of the linear conjugation of the form (5.1) we obtain the following conclusion, that for the system (5.16) to have the solutions, derivatives of which are almost bounded at the points of discontinuity of the coefficients up to order s inclusively, the fulfillment of the following conditions is necessary and sufficient:

$$\int_\Gamma F(t)S_k(t)dt = 0, \quad k = 1, \cdots, \delta, \tag{5.17}$$

where $F = (f, df/dt, \cdots, d^s f/dt^s), S_k(t)$ are the linear independent vectors, depending only on the matrices A and B and the vector f satisfies the corresponding differentiability conditions.

Consider now the system of singular integral equations of the general form

$$\begin{aligned} K^0\varphi + l\varphi &= f, \\ l\varphi &\equiv \int_\Gamma l(t_0, t)\varphi(t)dt, \end{aligned} \tag{5.18}$$

$$d^i A/dt^i, \;\; d^i B/dt^i, \;\; d^i f/dt^i, \;\; \partial^i l/\partial t_0^i \in H_0(\Gamma), \;\; i = 0, \cdots, s.$$

Rewriting the system (5.17) in the following form

$$K^0\varphi = f_0, \quad f_0 = f - l\varphi,$$

we may determine the behavior of the solution and it's derivatives.

Reasoning as above we get that for the system (5.17) the existence of the solution derivatives of which are almost bounded up to some order in the neighborhoods of the discontinuity points it is necessary and sufficient the fulfillment of the conditions of the form (5.15), where the vectors $S_k(t)$ are depending on $A, B, l(t_0, t)$.

Completing the short presentation of basic aspects of the theory of linear conjugation problems of analytic functions and one-dimensional singular integral equations note that at present there are numerous published researches in this theory and its applications in the case of two-dimensional problems of mathematical physics (see for example the following monographs [108], [134], [137], [45], [136], [72], [17], [18], [53], [35], [105], [119], [139], [32], [88]).

In this chapter the papers of the author [93], [96], [97] and also the work [101] were used.

Chapter 18

Linear Conjugation with Displacement for Analytic Functions

by Giorgi F. Manjavidze

18.1 Introduction and auxiliary propositions

Let Γ_1 and Γ_2 be simple closed curves on the plane of the complex variable $z = x+iy$. Denote by $D_k^+ (D_k^-)$ the domain situated inside (outside) of the curve Γ_k $(k = 1, 2)$.

Let

$$\omega^+[\alpha(t)] = \omega^-(t),\ \ t \in \Gamma_1, \tag{1.1}$$

where $\alpha(t)$ is a continuous function transferring the curve Γ_1 onto Γ_2 in one-to-one manner keeping the orientation, $\omega^-(z) = \omega_0^-(z) + Az$, $A = const \neq 0$, the function $\omega_0^-(z)$ is holomorphic in D_1^- and is continuous in the closure $\bar{D}_1^-$, the function $\omega^+(z)$ is holomorphic in D_2^+ and is continuous in the closure $\bar{D}_2^+$; the set of points of the curve $\gamma : z = \omega^-(t), t \in \Gamma_1$ (or the curve $z = \omega^+(t), t \in \Gamma_2$) has no (inner) points.

Under these considerations the following statement holds.

Lemma 18.1.1 *The functions $\omega^+(z)$ and $\omega^-(z)$ are schlicht in D_2^+ and in D_1^- respectively.*

Proof Denote by $n_a^+ (n_a^-)$ the number of a-points of the functions $\omega^+(z)$ $(\omega^-(z))$ in the domain $D_2^+(D_1^-)$ and let $a \notin \gamma$. From the boundary condition (1.1) we have

$$[\arg\ (\omega^-(t) - a]_{\Gamma_1} = [\arg\ (\omega^+(\alpha(t) - a)]_{\Gamma_1} = [\arg\ (\omega^+(t)) - a]_{\Gamma_2}. \tag{1.2}$$

If follows from this equality that

$$1 - n_a^- = n_a^+ \tag{1.3}$$

and therefore

$$n_a^+ = 1, n_a^- = 0 \ \text{ or } \ n_a^+ = 0, n_a^- = 1.$$

Let now $\omega^+(z_1) = \omega(z_2) = c, z_1 \neq z_2, z_1, z_2 \in D_2^+$. then it is evident[1] that $c \in \gamma$. Take in D_2^+ disjoint (non-intersecting) vicinities u_k of the points $z_k, k = 1, 2$, the domains $\omega^+(u_k)$ which are containing the point c and the intersection $M = \omega^+(u_1) \cap \omega^+(u_2)$ are not empty. It is easy to see that it is possible to find the point $c_1 \in M$ such that $c_1 \notin \gamma$. The value c_1 is assumed by the function $\omega^+(z)$ at least at two points, that is impossible. Hence, $\omega^+(z)$ is schlicht in D_2^+ and analogously $\omega^-(z)$ is schlicht in D_1^-.

Assume that $\omega^+(z_2) = \omega^-(z_1) = c$, $z_1 \in D_1^-$, $z_2 \in D_2^+$. Consider disjoint neighborhoods u_k of the point z_k and find a point $b \in w^+(u_2) \cap \omega^+(u_1), c \notin \gamma$. We get

$$n_b^+ \geqslant 1, n_b^- \geqslant 1$$

and it is impossible. The domains $\delta_2 = \omega^+(D_2^+)$ and $\delta_1 = \omega^-(D_1^-)$ are not intersecting.

Let $c \in \gamma$. It is evident that $c \in \overline{\delta_2} \cap \overline{\delta_1}$ because of $c = \omega^-(t_1) = \omega^+(t_2)$, $t_k \in \Gamma_k$. Now show that $c \in F_r\delta_2 \cap F_r\delta_1 = \Gamma$. Indeed assume $c \in \delta_2$ and consider some vicinity $u(c) \in \delta_2$. Take the sequence $z_n \in D_1^-$, $z_n \to t_1$ then $\omega^-(z_n) \to \omega^-(t_1) = c$. For large n, $\omega^-(z_n) \in u(c)$ and consequently the domains δ_2 and δ_1 are intersecting. Obtained contradiction proves that $c \notin \delta_2$, analogously $c \notin \delta_1$.

Let $d \in \Gamma$, it is evident that $d \in \gamma$. Therefore the set of the points of the curve γ coincides with the intersection of the boundaries of the domains δ_1 and δ_2.

Take an arbitrary point $c \in \Gamma$. Then one can find the point $t^1 \in \Gamma_1$, such that

$$c = \omega^-(t^1) = \omega^+(t^2), t^2 = \alpha(t^1) \in \Gamma_2.$$

The point $t^1(t^2)$ is accessible, from the domain $D_1^-(D_2^+)$, because $\Gamma_1(\Gamma_2)$ is a simple closed curve. Take the point $z_1(z_2)$ in $D_1^-(D_2^+)$ and connect it with the point $t^1(t^2)$ by the Jordan curve $\gamma_1(\gamma_2)$, lying entirely in the domain $D_1^-(D_2^+)$ except the end point $t^1(t^2)$. It is evident, that $z = \omega^-(t), t \in \gamma_1 [z = \omega^+(t), t \in \gamma_2]$ will be a Jordan arc, lying in $\gamma_1(\gamma_2)$ and ending in c. Point c is accessible form the both domains δ_1 and δ_2 and curve γ is a simple closed curve by virtue of Jordan inverse theorem [82]. Therefore, $\omega^+(z)$ and $\omega^-(z)$ are schlicht in the closures $\overline{D_2^+}$ and $\overline{D_1^-}$ because the function conformally mapping a Jordan domain onto the Jordan domain sets determines the one-to-one continuous correspondence between the boundary points.

If in the Lemma 18.1.1 we remain all conditions except the condition $A \neq 0$ and require $A = 0$ then the following statement holds:

Lemma 18.1.2 *Under mentioned considerations $\omega^+(z) = C, w^-(z) = C$ where C is constant.*

[1]There exist holomorphic functions for which the mentioned set has interior points; (see [8]).

Indeed, instead of (1.3) in considered case we shall have

$$n_a^- = n_a^+,$$

i.e., $n_a^- = n_2^+ = 0$ that proves the statement.

In this chapter we often deal with the integral operator

$$\int_\Gamma k(t_0,t)\phi(t)dt, \quad k(t_0,t) = \frac{\alpha'(t)}{\alpha(t)-\alpha'(t_0)} - \frac{1}{t-t_0}, \tag{1.4}$$

where $\alpha(t)$ is a continuous function mapping smooth Jordan curve Γ in one-to-one manner keeping the orientation onto itself or onto another smooth curve.

In case $0 \neq \alpha'(t) \in H(\Gamma)$ the kernel of the operator (1.4) has the form

$$k(t_0,t) = \frac{k_0(t_0,t)}{|t-t_0|^\alpha}, \quad k_0(t_0,t) \in H(\Gamma\times\Gamma), 0 \leqslant \alpha < 1.$$

In case $0 \neq \alpha'(t) \in C(\Gamma)$ the operator is a completely continuous operator in the space $L_p(\Gamma,\rho)$, $\rho > 1$, $\rho(t) = \prod_{k=1}^m |t-t_k|^{\nu_k}$, $t_k \in \Gamma$, $-1 < \nu_k < p-1$, (see [57], [58]).

18.2 Linear conjugation with displacement in case of continuous coefficients

Let $\Gamma_k (k = 1,2)$ be a simple smooth curve bounding finite and infinite domains D_k^+ and D_k^- on the plane of the complex variable $z = x + iy$.

Consider the following boundary problem.

Find a vector $\varphi(z) = (\varphi_1, \cdots, \varphi_n) \in E_p^\pm(\Gamma_1,\Gamma_2)$ satisfying the boundary condition

$$\varphi^+[\alpha(t)] = \alpha(t)\varphi^-(t) + b(t) \tag{2.1}$$

almost everywhere on Γ; where $a(t)$ is a given continuous non-singular quadratic matrix of order n; $b(t) = (b_1, \cdots, b_n)$ is a given vector on Γ of the class $L_p(\Gamma_1), p > 1, \alpha(t)$ is a function mapping Γ_1 onto the Γ_2 in one-to-one manner keeping the orientation; $\alpha(t)$ has non-zero continuous derivative $\alpha'(t)$. We call a quadratic matrix of order n the canonical matrix of the boundary problem (2.1), if the following properties hold:

1) $\chi(z), \chi^{-1}(z) \in E_\infty^\pm(\Gamma_1,\Gamma_2)$,

2) satisfies the homogeneous boundary condition

$$\chi^+[\alpha(t)] = \chi^-(t), \quad t \in \Gamma_1;$$

3) has a normal form at infinity along the columns.

We call the orders at infinity of the columns of the canonical matrix taken with opposite sign by the partial indices and the sum of the partial indices by the index (cf. 19.1).

It is easy to see that if the boundary problem (2.5) is solvable in $E^{\pm}_{p,0}(\Gamma_1,\Gamma_2)$ then the solution admits the following representation (cf. [136]):

$$\varphi(z) = -\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\mu(t)dt}{t-z},\quad z\in D_1^-,\quad \mu(t)\in L_p(\Gamma_1),$$
$$\varphi(z) = \frac{1}{2\pi i}\int_{\Gamma_2}\frac{\alpha[\beta(t)]\mu[\beta(t)]+b[\beta(t)]}{t-z}dt,\quad z\in D_2^+, \tag{2.2}$$

where $\beta(t)$ is an inverse function to $\alpha(t)$.

From the boundary condition (2.5) we obtain that the vector $\mu(t)$ will satisfy the following equation

$$K(a)\mu \equiv \frac{1}{\pi i}\int_{\Gamma_1}\frac{a(t)+a(t_0)}{t-t_0}\mu(t)dt + M(a)\mu = \widetilde{b(t_0)},\quad t_0\in\Gamma_1.$$
$$M(a)\mu \equiv \frac{1}{\pi i}\int_{\Gamma_1}k(t_0,t)a(t)\mu(t)dt, \tag{2.3}$$
$$\widetilde{b(t_0)} = lb = b(t_0) - \frac{1}{\pi i}\int_{\Gamma_1}\frac{b(t)dt}{t-t_0} - \int_{\Gamma_1}k(t_0,t)b(t)dt.$$

The operator $M(a)$ is a completely continuous linear operator in $L_p(\Gamma,\rho)$, l is a linear bounded operator in $L_p(\Gamma,\rho)$.

The solvability conditions of the problem (2.5) in $L_p(\Gamma_1)$ have the form

$$\int_{\Gamma_1}\widetilde{b(t)}v(t)dt = 0, \tag{2.4}$$

where $v(t)\in L_q(\Gamma_1)(q=p/p-1)$ is an arbitrary solution of adjoint homogeneous equation

$$K'(a)v \equiv -\frac{1}{\pi i}\int_{\Gamma_1}\frac{a'(t)+a'(t_0)}{t-t_0}v(t)dt + \frac{a'(t_0)}{\pi i}\int_{\Gamma_1}k'(t,t_0)v(t)dt = 0. \tag{2.5}$$

The condition (2.8) we may rewrite in the following form

$$\int_{\Gamma_1}b(t)\lambda(t)dt = 0;$$
$$\lambda(t) = v(t) + \frac{a'(t)}{\pi i}\int_{\Gamma_1}\frac{v(t_1)dt_1}{\alpha(t_1)-\alpha(t)}. \tag{2.6}$$

The condition (2.6) is equivalent to the condition

$$\int_{\Gamma_2}b[\beta(t)]\psi^+(t)dt = 0, \tag{2.7}$$

where $\psi(z)$ is an arbitrary solution of the class $E^{\pm}_{q,0}(\Gamma_1,\Gamma_2)$ of the homogeneous problem

$$\alpha'(t)\psi^+[\alpha(t)] = [a'(t)]^{-1}\psi^-(t),\quad t\in\Gamma_1. \tag{2.8}$$

Consider the boundary problem (2.5) for $a(t) = I$ (I is an unit matrix). In this case the problem (2.12) takes the form

$$\alpha'(t)\psi^+[\alpha(t)] = \psi^-(t), \quad t \in \Gamma_1. \tag{2.9}$$

Let us show that the problem (2.13) has the only trivial solution in any class $E^{\pm}_{\tau,0}(\Gamma_1, \Gamma_2)$.

It is sufficient to consider the case $n = 1$. Denote by $F^+(z)$ and $F^-(z)$ the primitive functions of the functions $\psi^+(z)$ and $\psi^-(z)$ in the domains D_2^+ and D_1^- respectively (by virtue of the equality $\int_{\Gamma_1} \psi^-(t)dt = 0$ the primitive $F^-(z)$ is singlevalued in D_1^-). These primitives are continuous in closed domains and absolutely continuous on Γ_2 and Γ_1 with respect to the arc [55], [118].

From (2.9) we have

$$F^+[\alpha^-(t)] = F^-(t), \quad t \in \Gamma_1.$$

From the last equality by the lemma 18.2.2 we obtain that $F(z) = \text{const}$ and $\psi(z) = 0$. Consequently the boundary problem (1.1) for $a(t) = I$ is solvable for a certain vector $b(t) \in L_p(\Gamma_1)(p > 1)$ in $E^{\pm}_{p,0}(\Gamma_1, \Gamma_2)$.

Now show that the equation (2.1) in case $a = 1$ has the only trivial solution in $L_p(\Gamma_1)(p > 1)$. Let $v \in L_p(\Gamma_1)$ be a solution of this equation.

Consider the vector

$$N(z) = \begin{cases} -\dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_1} \dfrac{v(t)dt}{t-z}, & z \in D_1^-. \\ \dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_2} \dfrac{\beta(t)v[\beta(t)]dt}{t-z}, & z \in D_2^+. \end{cases} \tag{2.10}$$

From (2.10) we have

$$\alpha'(t)N^+[\alpha(t)] = N^-(t), \quad t \in \Gamma_1, \quad N(z) \equiv 0.$$

Therefore, there exists the vector $q(z) \in E^{\pm}_{p,0}(\Gamma_1, \Gamma_2)$ such, that

$$v(t) = q^+(t), \quad t \in \Gamma_1, \quad \beta'(t)\psi[\beta(t)] = q^-(t), \quad t \in \Gamma_2.$$

It follows from the last relations, that

$$\beta'(t)f^+[\beta(t)] = f^-(t), \quad t \in \Gamma_2,$$
$$f(z) \equiv 0, \quad v(t) \equiv 0.$$

Remarking that the index of the operator $K(I)$ is equal to zero for any space $L_p(\Gamma_1), (p > 1)$, we get, that the operators $K(I)$ and $K'(I)$ are invertible in space $L_p(\Gamma_1)$ and also in every space $L_p(\Gamma.\rho)$.

So we get the following proposition

Theorem 18.2.1 *The boundary problems*

$$\varphi^+[\alpha(t)] = \varphi^-(t) + b(t), \;\; \varphi^+[\alpha(t)] = \frac{1}{\alpha'(t)}\varphi^-(t) + b(t), \;\; t \in \Gamma_1$$

have the solution (unique) in $E_{p,0}(\Gamma_1, \Gamma_2)$ *for every* $b(t) \in L_p(\Gamma_1), p > 1$.

Remark *The solution of the boundary problem*

$$\varphi^+[\alpha(t)] = \varphi^-(t) + b(t), \;\; t \in \Gamma_1.$$

may be constructed also by the following way (see. [81], [84]).

We look for the solution in the class $E^{\pm}_{p,0}(\Gamma_1, \Gamma_2, \rho)$ of the following form:

$$\begin{aligned} \varphi(z) &= \frac{1}{2\pi i}\int_{\Gamma_2} \frac{\mu[\beta(t)]dt}{t - z}, \;\; z \in D_2^+, \\ \varphi(z) &= \frac{1}{2\pi i}\int_{\Gamma_1} \frac{\mu(t)dt}{t - z}, \;\; z \in D_1^-. \end{aligned} \tag{2.11}$$

We obtain the Fredholm integral equation with respect to the function $\mu(t) \in L_p(\Gamma_1, \rho)$

$$R\mu \equiv \mu(t_0) = \frac{1}{2\pi i}\int_{\Gamma_1} K(t_0, t)\mu(t)dt = b(t_0).$$

Let us show, that the equation $R\mu = 0$ has the only trivial solution. Let $\mu_0(t)$ be a solution of this equation; construct the function $\varphi_0(z)$ of the class $E^{\pm}_{p,0}(\Gamma_1, \Gamma_2, \rho)$ by the formulas (2.11).

We obtain

$$\varphi_0^+[\alpha(t)] = \varphi_0^-(t), \;\; t \in \Gamma_1.$$

Therefore $\varphi_0(z) \equiv 0$. Then from the formulas (2.15) we have $\mu(t) = S^+(t), t \in \Gamma_1, \mu[\beta(t)] = S^-(t), t \in \Gamma_2, S^+(z) \in E_\lambda(D_1^+), S^-(z) \in E_\lambda(D^-)$. Since

$$S^+[\beta(t)] = S^-(t), \;\; t \in \Gamma_2,$$

we get $S = 0, \mu_0 = 0$. Because of why the equation $R\mu = b$ is solvable (by the only possible way) for any $b(t) \in L_p(\Gamma, \rho)$ and the operator R has an inverse operator.

Substituting $\mu = R^{-1}b$ in the formulas (2.11), we obtain

$$\varphi^-(t) = L_1 b, \;\; \varphi^+ = L_2 b.$$

where L_1, L_2 are linear bounded operators in the spaces $L_p(\Gamma_1, \rho_1)$ and $L_p(\Gamma_2, \rho_2), \rho_2 = \prod_{k=1}^m |t - \alpha(t_k)|^{\nu_k}$ respectively.

Consider the solution of the boundary problem

$$\alpha'(t)\varphi^+[\alpha(t)] = \varphi^-(t) + 1, \;\; t \in \Gamma_1, \tag{2.12}$$

in $E^{\pm}_{p,0}(\Gamma_1, \Gamma_2), p > 1$. Denote by $\omega^+(z)[\omega^-(z)]$ the primitive of the function $\varphi^+(z)[\varphi^-(z)+1]$ in $D_2^+[D_1^-]$.

We may choose the primitives such that

$$\omega^+[\alpha(t)] = \omega^-(t), \quad t \in \Gamma_1.$$

$\omega^+(z)$ and $\omega^-(z)$ are Holder-continuous in closures $\overline{D}_2^+$ and $\overline{D}_1^-$ (with an exponent arbitrarily close to 1) and are absolutely continuous on Γ_2 and Γ_1 with respect to the arc (see [55], [118]). By virtue of lemma 18.1.1, $\omega^+(z)$ and $\omega^-(z)$ are schlicht in $\overline{D}_2^+$ and $\overline{D}_1^-$.

If in addition we require that the derivative $\alpha'(t)$ satisfies the Holder-condition, then the solution of the problem (2.16) of any class $E^{\pm}_{p,0}(\Gamma_1, \Gamma_2)$ as it is easy to see will be Hölder-continuous in the closures $\overline{D}_2^+$ and $\overline{D}_1^-$ and hence the derivatives $d\omega^+(z)/dz, d\omega^-(z)/dz$ will be also Hölder-continuous in these closures.

Thus the following theorem is proved.

Theorem 18.2.2 *Let Γ_1, Γ_2 be a simple closed smooth curves, $D_k^+(D_k^-)$ is a domain lying inside (outside) $\Gamma_k, \alpha(t)$ is a function mapping Γ_1 onto Γ_2 in one-to-one manner keeping the orientation, $\alpha(t)$ has non-zero continuous derivative $\alpha'(t)$. There exists the solution of the boundary problem*

$$\omega^+[\alpha(t)] = \omega^-(t), \quad t \in \Gamma_1 \tag{2.13}$$

having the following properties: $\omega^+(z)(\omega^-(z))$ is continuous and schlicht function in $\overline{D}_2^+(\overline{D}_1^-), \omega^+(z)$ is a holomorphic function in $D_2^+, \omega^-(z)$ has the form

$$\omega^-(z) = z + \omega_0^-(z),$$

where $\omega_0^-(z)$ is holomorphic in D_1^-, the curves $\omega^+(\Gamma_2)$ and $\omega^-(\Gamma_1)$ are simple closed rectifiable curves, if additionally is required that $\alpha'(t)$ satisfies Hölder-condition, then $\omega^+(\Gamma_2), \omega^-(\Gamma_1)$ will be smooth curves.

The following proposition holds.

Lemma 18.2.1 *Let p be any number more then 2.*

It can be found such $\varepsilon(p) > 0$ that if

$$\|a - I\|_c < \varepsilon,$$

then the boundary problem

$$\varphi^+[\alpha(t)] = a(t)\varphi^-(t)$$

has the solution $\chi(z) \in E_p^{\pm}(\Gamma_1, \Gamma_2)$ with the property $\chi^{-1}(z) \in E_p^{\pm}(\Gamma_1, \Gamma_2)$.

Take ε small as much as the operators $K(a)$ and $K(a^{'-1})$ have the inverse operators in $L_p(\Gamma_1)$ and using them we construct the solutions of the boundary problems

$$\begin{aligned}\varphi^+[\alpha(t)] &= a(t)\varphi^-(t) + a(t),\\ \psi^+[\alpha(t)] &= [a'(t)]^{-1}\psi^-(t) + [a'(t)]^{-1}, \quad t \in \Gamma_1\end{aligned} \tag{2.14}$$

in the class $E_p^{\pm}(\Gamma_1, \Gamma_2)$.

From the equalities (2.18) we have

$$\psi'(z)\varphi(z) = I, \quad z \in D_2^+, \quad [\psi'(z) + I][\varphi'(z) + I], \quad z \in \overline{(}D_1).$$

These equalities show that the desired solution exists.

By virtue of the last lemma one can prove the following proposition.

Lemma 18.2.2 *All solutions of the problem*

$$\varphi^+[\alpha(t)] = a(t)\varphi^-(t) \tag{2.15}$$

of the class $E_\lambda^{\pm}(\Gamma_1, \Gamma_2)$, where λ is an arbitrary number more then 1, $a(t)$ is an arbitrary continuous non-singular matrix, belongs to the class $E_\infty^{\pm}(\Gamma_1, \Gamma_2)$.

Rewrite (2.15) in the following form

$$\varphi^+[\alpha(t)] = a_0(t)r(t)\varphi^-(t), \quad a_0 = ar^{-1}, \tag{2.16}$$

where r is a rational matrix chosen such that $\|a_0 - I\|_c < \varepsilon(p)$, $p > 2$, $p > \lambda/(\lambda - 1)$; $\varepsilon(p)$ is number mentioned in Lemma 18.2.1 By this lemma there exists the matrix $\chi_0(z)$ such that

$$\chi_0(z)\chi_0^{-1}(z) \in E_p^{\pm}(\Gamma_1, \Gamma_2), \quad \chi_0^+[\alpha(t)] = a_0(t)\chi^-(t), \quad t \in \Gamma_1.$$

We may rewrite (2.20) in the following form

$$\left\{\chi_0^+[\alpha(t)]\right\}^{-1}\varphi^+[\alpha(t)] = [\chi_0(t)]^{-1}r(t)\varphi^-(t), \quad t \in \Gamma_1$$

or using the function $w(t)$, constructed above, we get

$$\left\{\chi_0^+[\omega_1^+(\tau)]\right\}^{-1}\varphi^+[\omega_1^+(\tau)] = \left\{\chi_0^-[\omega_1^-(\tau)]\right\}^{-1}r[\omega_1^-(\tau)]\varphi^-[\omega_1^-(\tau)], \tag{2.17}$$

where ω_1^+ and ω_1^- are the functions inverse of the functions ω^+ and ω^-; the equality is valid almost everywhere along the simple closed rectifiable curve $\gamma : r = \omega(t)$, $t \in \Gamma_1$. Denote by D^+ and D^- the domains bounded by γ.

As $w'(z) \in E_\lambda(D_2^+)$ for every λ, it is easy to check, that $\varphi^+[\omega_1^+(\zeta)] \in E_{\lambda-\varepsilon}(D^+)$, where ε is an arbitrary small positive number and

$$\left\{\chi_0^+[\omega_1^+(\xi)]\right\}^{-1}\varphi^+[\omega_1^+(\xi)] \in E_1(D^+). \tag{2.18}$$

We may set analogously, that

$$\left\{\chi_0^-[\omega_1^-(\zeta)]\right\}^{-1} r[\omega_1^-(\zeta)]\varphi^-[\omega_1^-(\zeta)] = \frac{\varphi_1^-(\zeta)}{P_1(\zeta)} + P_2(\zeta),\ \ \xi \in D^-, \tag{2.19}$$

where $\varphi_1^-(\zeta) \in E_1(D^-)$, $P_1(\zeta)$ is some polynomial having no zeros on γ, and $P_2(\zeta)$ is also a polynomial vector. From the last relations it implies that

$$\varphi(z) = \chi_0(z)R[\omega(z)],\ \ z \in D_2^+,\ \ \varphi(z) = r(z)\chi_0(z)R[\omega(z)],\ \ z \in D_1^-,$$

where $R(\zeta)$ is a vector, components of which are rational functions and so $\varphi(z) \in E_p^{\pm}(\Gamma_1, \Gamma_2)$. Since we may take p as large as desired, then $\varphi \in E_\infty^{\pm}(\Gamma_1, \Gamma_2)$.

By virtue of Lemmas 18.2.1 and 18.2.2 we conclude, that if the norm of the difference $a(t) - I$ is sufficiently small then for the boundary problem there exists the canonical matrix.

Having established this fact we may prove the following proposition.

Theorem 18.2.3 *There exists a canonical matrix for the boundary problem* (2.5) *for every continuous non-singular matrix $a(t)$.*

First choose the rational matrix $r(t)$ such that for the matrix $\alpha_0(t) = a(t)r(t)$ there exists a canonical matrix; denote it by $\chi_0(z)$.

Then construct the matrix

$$\chi(z) = \begin{cases} \chi_0(z)R[\omega(z)], & z \in D_2^+, \\ r(z)\chi_0(z)R[\omega(z)], & z \in D_1^-, \end{cases} \tag{2.20}$$

where $R(\xi)$ is correspondingly chosen rational matrix. $R[\omega(z)]$ will liquidate the zeros of det $r(z)$ in D_1^- and the poles in the same domain; moreover, it will give the normal form to $\chi(z)$ at infinity with respect to the columns; such matrix exists [108]. The formula (2.16) gives one of the canonical matrices.

If $\chi(z)$ is one of the canonical matrices of the problem (2.5) then all canonical matrices of this problem are given by the formula

$$\chi(z)P[\omega(z)],$$

where $P(\zeta)$ is a polynomial with non-zero constant determinant which is constructed by the special way [see [136] §5)].

The following statement holds:

Theorem 18.2.4 *All solutions of the class $E_p^{\pm}(\Gamma_1, \Gamma_2)$ of the problem* (2.5) *are given by the formula*

$$\varphi(z) = \chi(z)[\varphi_0(z) + P(\omega(z))], \tag{2.21}$$

where $\chi(z)$ is a canonical matrix, $P(z)$ is an arbitrary polynomial vector, $\varphi_0(z)$ is a solution of the class $E_{p-\varepsilon,0}^{\pm}(\Gamma_1, \Gamma_2)$($\varepsilon$ is a small positive number) of the boundary problem

$$\varphi_0^+[\alpha(t)] = \varphi_0^-(t) + b_0(t),\ \ t \in \Gamma_1,$$

$$b_0(t) = \left\{\chi^+[\alpha(t)]\right\}^{-1} b(t); \tag{2.22}$$

the solutions vanishing at infinity are given by the same formula (2.25) *in which*

$$P(\zeta) = (P_{\varkappa_1 - 1}(\zeta), \cdots, P_{\varkappa_n - 1}(\zeta));$$

$\varkappa_1 \geqslant \cdots \geqslant \varkappa_n$ *are the partial indices of the problem* (2.5)*,* $P_j(\zeta)$ *denotes an arbitrary polynomial of order* $j, P_j(\zeta) = 0$ *if* $j < 0$*. If all partial indices are non-negative then such solutions exist for every* $b(t) \in L_p(\Gamma_1)$*; if*

$$0 > \varkappa_{s+1} \geqslant \cdots \geqslant \varkappa_n.$$

then the vector $b(t)$ *will satisfy the condition* (2.10)*, which we may write in the form*

$$\int_{\Gamma_1} t^k \rho_j^0(t) dt = 0, \;\; j = s+1, \cdots, n; \;\; k = 0, \cdots, |\varkappa_j| - 1, \tag{2.23}$$

where the vector $(\rho_1^0, \cdots, \rho_n^0) = \rho_0$ *is a solution of the equation* $K(I)\rho_0 = \tilde{b_0}$ *(or* $\rho_0 = \mathcal{L}_1 b_0$*).*

Proof Let $\varphi_*(z)$ be some solution of the homogeneous problem of the class $E_p^\pm(\Gamma_1, \Gamma_2)$, then the vector

$$\varphi_1(z) = \chi^{-1}(z)\varphi_*(z)$$

will belong to the class $E_s^\pm(\Gamma_1, \Gamma_2)$, $1 < s < p$ and

$$\varphi_1[\alpha(t)] = \varphi_1^-(t), \;\; t \in \Gamma_1.$$

From the last equality it follows that

$$\varphi_2^+(\tau) = \varphi_2^-(\tau), \;\; \tau \in \gamma,$$

where $\varphi_2^+(\zeta) = \varphi_1[\omega^+(\zeta)]$, $\zeta \in D^+$, $\varphi_2^-(\zeta) = \varphi_2[\omega_1^-(\zeta)]$, $\zeta \in D^-$, $\varphi_2(\zeta) \in E_{s_1}^\pm(\gamma)$, $1 < s_1 < p$; and so

$$\varphi_2(\zeta) = P(\zeta),$$

where $P(\zeta)$ is an arbitrary polynomial vector

$$\varphi_*(z) = \chi(z)P[\omega(z)].$$

Now show, that non-homogeneous boundary problem is solvable for $b(t) \in L_p(\Gamma_1)$ in the class $E_p^\pm(\Gamma_1, \Gamma_2)$.

Choose the rational matrix $r(z)$ in such a way that $\|a_0(t) - I\|_c$ will be small ($a_0(t) = a(t)r^{-1}(t)$) and consider the boundary problem

$$\psi^+[\alpha(t)] = a_0(t)\psi^-(t) + b(t), \;\; t \in \Gamma_1.$$

This problem is solvable in the class $E^{\pm}_{p,0}(\Gamma_1, \Gamma_2)$ if the above mentioned norm is sufficiently small.

A vector of the form

$$\widetilde{\varphi(z)} = \begin{cases} \psi(z), & z \in D_2^+, \\ r(z)\psi(z), & z \in D_1^- \end{cases}$$

satisfies the boundary condition (2.5).

Consider the following vector

$$\widetilde{\psi(\zeta)} = \chi^{-1}[\omega_1^-(\zeta)]r[\omega_1^-(\zeta)]\psi[\omega_1^-(\zeta)]$$

in the domain D^- and denote by $R_k(\zeta)$ $(k = 1, \cdots, m)$ the principal parts of this vector for the singular points ζ_k (or poles) different from the point $z = \infty$. Then the vector

$$\widetilde{\psi(\zeta)} + R(\zeta), \quad R(\zeta) = -\sum_{k=1}^{m} R_k(\zeta)$$

will not have poles in the domain D^- (except possibly the point $z = \infty$); the vector

$$r(z)\psi(z) + \chi(z)R[\omega(z)]$$

will not have poles in D^- (except possible the point $z = \infty$); the vector

$$\varphi(z) = \widetilde{\varphi(z)} + \chi(z)R[\omega(z)] \tag{2.24}$$

will be a solution of the problem (2.5) of the class $E^{\pm}_p(\Gamma_1, \Gamma_2)$. On the other hand from the boundary condition (2.1) we have

$$\varphi_0[\alpha(t)] = \varphi_0(t) + b_0(t),$$

where

$$\varphi_0(z) = \chi(z)\varphi(z),$$
$$b_0(t) = \left\{\chi^+[\alpha(t)]\right\}^{-1} b^-(t) \in L_{p_1}(\Gamma_1); \;\; 1 < p_1 < p.$$

It is evident, that the vector

$$\varphi_*(z) = \chi(z)\varphi_0(z) \tag{2.25}$$

is a solution of the problem (2.1) of the class $E^{\pm}_{p_2}(\Gamma_1, \Gamma_2)$, $1 < p_2 < p$.

The difference between the vector defined by the formulas (2.24) and (2.25)

$$\varphi_*(z) - \varphi(z)$$

is a vector of the class $E^{\pm}_p(\Gamma_1, \Gamma_2)$ and satisfies the homogeneous boundary condition; so $\varphi_*(z) = \varphi(z) + \chi(z)Q(\omega(z))$, Q is a polynomial vector, the vector $\varphi_*(z) \in E^{\pm}_p(\Gamma_1, \Gamma_2)$ and the formula (2.21) gives us the general solution of the problem (2.5) of the class $E^{\pm}_p(\Gamma_1, \Gamma_2)$. Other statements of the theorem we may obtain from the formulas (2.21) by the well-known way (see [136] §5)

The following lemma we need in the sequel.

Lemma 18.2.3 *Let $c_k \in \Gamma_k, k = 1, 2$. For the function $\omega(z)$ the following relations are valid*

$$\begin{aligned} &\omega^+(z) - \omega^+(c_2) = (z - c_2)\Omega^+(z), \quad z \in D_2^+, \\ &\omega^-(z) - \omega^-(c_1) = (z - c_1)\Omega^-(z), \quad z \in D_1^-, \\ &\Omega^+(z), 1/\Omega^+(z) \in E_\infty(D_2^+), \quad \Omega^-(z), 1/\Omega^-(z) \in E_\infty(D_1^-). \end{aligned} \tag{2.26}$$

Proof From the boundary condition

$$\omega^+[\alpha(t)] = \omega^-(t), \tag{2.27}$$

we have

$$\Omega^+[\alpha(t)] = h(t)\Omega^-(t), \quad t \in \Gamma_1, \tag{2.28}$$

where $\Omega^\pm(z)$ are defined by the formula (2.30), $c_2 = \alpha(c_1)$,

$$h(t) = \frac{t - c_1}{\alpha(t) - \alpha(c_1)},$$

$h(t)$ is a continuous function with the index 0.

We have from (2.28), taking into account the equality $\Omega^-(\infty) = 1$,

$$\Omega(z) = \chi(z), \tag{2.29}$$

where $\chi(z)$ is a canonical function of the problem

$$\chi^+[\alpha(t)] = h(t)\chi^-(t).$$

The equality (2.29) proves the validity of the lemma.

18.3 Linear conjugation with displacement in case of piecewise continuous coefficients

In this section preserving the above mentioned notations, we consider the problem

$$\varphi^+[\alpha(t)] = a(t)\varphi^-(t) + b(t), t \in \Gamma_1, \tag{3.1}$$

but we shall assume that $a(t)$ is a piecewise-continuous matrix

$$\inf |\det\, a(t)| > 0,$$

$b(t) \in L_p(\Gamma_1, \Gamma_2, \rho)$ $p > 1$, $\rho(t) = \prod_{k=1}^m |t - t_0|^{\nu_k}$, $t_k \in \Gamma_1$, $-1 < \nu_k < p - 1$, the set $\{t_k\}$ contains all points of discontinuity of the matrix $a(t)$.

Begin with the consideration of the case $n = 1$ and first consider the homogeneous problem

$$\varphi^+[\alpha(t)] = a(t)\varphi^-(t), a(t) \in C_0(\Gamma_1, c_1, \cdots, c_m). \tag{3.2}$$

Make the substitution

$$\varphi(z) = \prod_{k=1}^{m} \chi_n^1(z)\Phi(z), z \in D_2,$$

$$\varphi(z) = \prod_{k=1}^{m} \chi_k(z)\Phi(z), \quad z \in \overline{D_1}, \tag{3.3}$$

$$\chi_k^1(z) = [z - \alpha(c_k)]^{\tau_k}, \quad \chi_k(z) = \left(\frac{z - c_k}{z - z_0}\right)^{\tau_k}, \quad z_0 \in D_1^+,$$

$$\tau_k = \frac{1}{2\pi i}\ln\lambda_k, \quad \lambda_k = \frac{a(c_k - 0)}{a(c_k + 0)}, \quad -1 \leqslant \mathrm{Re}\tau_k \leqslant 0,$$

$\chi_k^1(z)$ is a single-valued branch, defined on the whole plane cut along the curve l_k which connects the point $\alpha(k)$ with the point ∞ and lies in $D_2^-, \chi_k(z)$ is a singlevalued branch, defined on the whole plane, cut along the curve l_k' which connects the point z_0 with the point c_k and lies in $D_1^+, \chi_k(\infty) = 1$.

With respect to $\Phi(z)$ we obtain the following boundary condition

$$\Phi^+[\alpha(t)] = g(t)\Phi^-(t), t \in \Gamma_1, \tag{3.4}$$

$$g(t) = a(t)\left(\prod_{k=1}^{m} \chi_k^1(\alpha(t))\right)^{-1} \prod_{k=1}^{m} \chi_k^-(t) = a(t)\prod_{k=1}^{m}(t - z_0)^{-\tau_k}\prod_{k=1}^{m}\left(\frac{t - c_k}{\alpha(t) - \alpha(c_k)}\right)^{\tau_k},$$

$g(t)$ is a continuous function, $g(t) \neq 0$.

Let $A(z)$ be a canonical function of the problem (3.4), $A(z), A^{-1}(z) \in E_\infty^\pm$ (Γ_1, Γ_2).

Consider the function $\chi_0(z)$ constructed by the formula (3.2)

$$\chi_0(z) = \begin{cases} A(z)\prod_{k=1}^{m}\chi_k(z), & z \in D_2^+, \\ A(z)\prod_{k=1}^{m}\chi_k(z), & z \in D_1^-. \end{cases}$$

It is evident, that $\chi_0^{-1}(z) \in E_\infty^\pm(\Gamma_1, \Gamma_2)$, $\chi_0(z) \in E_\lambda^\pm(\Gamma_1, \Gamma_2)$ for some $\lambda > 1$. Any solutions of the problem (3.35) of the class $E_s^\pm(\Gamma_1, \Gamma_2), s > 1$ has the form

$$\varphi(z) = \chi_0(z)Q[\omega(z)],$$

where $Q(z)$ is some polynomial. By virtue of the lemma 18.2.3 and applying the scheme of proving the theorem 18.2.3 we obtain the theorem analogous to this theorem.

Theorem 18.3.1 *Let $\mu_k = |Re\tau_k|$. If $\mu_k p = 1+\nu_k$ for some k then the canonical function of the corresponding class doesn't exist. If $\mu_k p \neq 1+\nu_k, k = 1,\cdots,m$ then the canonical function of the class $E_p^{\pm}(\Gamma_1,\Gamma_2,\rho)$ exists and is defined by the formula*

$$\chi(z) = \chi_0(z)Q[\omega(z)], \tag{3.5}$$

where

$$Q(z) = \prod_{k=1}^{m}[z-\omega^-(c_k)]^{m_k},$$

$$m_k = \begin{cases} 1, & if \quad \mu_k > \dfrac{1+\nu_k}{p}, \\ 0, & if \quad \mu_k < \dfrac{1+\nu_k}{p}; \end{cases} \tag{3.6}$$

the index of the class $E_p^{\pm}(\Gamma_1,\Gamma_2,\rho)$ of the problem (3.2) is given by the formula $\varkappa = \operatorname{indg}(t) - \sum_{k=1}^{m} m_k$ or by the formula

$$\varkappa = \frac{1}{2\pi i}\left\{\arg\frac{a(t)}{\prod\limits_{k=1}^{m}(t-z_0)^{s_k}}\right\}_{\Gamma_1}, \tag{3.7}$$

where $s_k = \dfrac{1}{2\pi i}\lambda_k$, $-1 < \mathrm{Re}s_k \leqslant 0$, if $\mu_k < \dfrac{1+\nu_k}{p}$, (i.e. $s_k = \tau_k$), $0 \leqslant \mathrm{Re}s_k < 1$, if $\mu_k > \dfrac{1+\nu_k}{p}$ (i.e $s_k = \tau_{k+1}$)

On the basis of the formulas (3.5), (3.6) it is easy to see that the following proposition is valid

Lemma 18.3.1 *Let $\chi(z)$ be a canonical function of some class $E_p^{\pm}(\Gamma_1,\Gamma_2,\rho)$. Then the operators $\{\chi^+[\alpha(t)]\}^{-1}b(t)$ and $[\chi^-(t)]^{-1}b(t)$ are the linear bounded operators from the space $L_p(\Gamma_1,\rho)$ into the space $L_r(\Gamma_1)$ for some $r > 1$.*

Consider now the non-homogeneous problem

$$\varphi^+[\alpha(t)] = a(t)\varphi^-(t) + b(t), t \in \Gamma_1, \tag{3.8}$$

$$b(t) \in L_p(\Gamma_1,\rho), \quad \rho(t) = \prod_{k=1}^{m}|t-t_k|^{\nu_k}$$

and make the substitution (3.3), subordinating $Re\tau_k$ by the following restriction

$$-\frac{1+\nu_k}{p} < Re\tau_k < 1 - \frac{1+\nu_k}{p}. \tag{3.9}$$

We obtain the following problem

$$\Phi^+[\alpha(t)] = g(t)\Phi^-(t) + f(t),$$

$$f(t) = b(t)\left(\prod_{k=1}^{m}\chi_k^1\Phi(t)\right)^{-1} = b(t)\left(a(t)\prod_{k=1}^{m}\chi_k(t)\right)^{-1}, \tag{3.10}$$

$$f(t) \in L_p(\Gamma_1,\rho_1),\quad \rho_1 = \prod_{k=1}^{m}|t-c_k|^{\nu_k^1},\quad \nu_k^1 = \alpha_k\rho + \nu_k,\quad \alpha_k = \mathrm{Re}\tau_k.$$

We look for the solution of this problem in $E_p^{\pm}(\Gamma_1,\Gamma_2,\rho)$.

Take the rational function $R(z)$ such that

$$\|g(t) - R(t)\|_C < \varepsilon,$$

where ε is a sufficiently small positive number.

Construct the sequence of the functions $\Phi_m(z)$ as follows: $\Phi_{m+1}(z)$ is a solution of the class $E_\rho^{\pm}(\Gamma_1,\Gamma_2,\rho)$ of the following problem

$$\Phi_{m+1}^+[\alpha(t)] - \Phi_{m+1}^-(t) = b_m(t), t \in \Gamma_1,$$

$$g_0(t) = g(t)R^{-1}(t) - 1, b_m(t) = g_0(t)\Phi_m^-(t) + f(t), \Phi_0^-(t) = 0.$$

Convergence of this sequence for sufficiently small ε one can prove similarly as the convergence of the sequence (3.21) from section 19.3 the limit function will satisfy the boundary condition

$$\Phi^+[\alpha(z)] = gR^{-1}\Phi^-(t) + f(t).$$

The boundary values of the function

$$\varphi(z) = \begin{cases} \Phi(z), & z \in D_2^+, \\ R^{-1}(z)\Phi(z), & z \in D^-, \end{cases}$$

will satisfy the boundary condition (3.10). Since the problem (3.10) has a canonical function of the class $E_p^{\pm}(\Gamma_1,\Gamma_2,\rho)$then it is solvable in this class for any $f(t) \in L_p(\Gamma,\rho)$ and the considered problem is solvable in $E_p^{\pm}(\Gamma_1,\Gamma_2,\rho)$ for any function $b(t) \in L_p(\Gamma_1,\rho)$. Consequently, the expressions $\chi^-(t_0)\mathcal{L}_1[(\chi^+(\alpha(t)))^{-1}b(t)]$, $\chi^-(t_0)\mathcal{L}_1[(\chi^{-1}(t))^{-1}b(t)]$, are the linear bounded operators in $L_p(\Gamma,\rho)$.

Consider now the following problem

$$\varphi^+[\alpha(t)] = a(t)\varphi^-(t) + b(t),\quad t \in \Gamma_1, \tag{3.11}$$

where $a(t)$ is a triangular piecewise-continuous matrix $a = (a_{ik}), a_{ik} = 0$, when $i < k$, $\inf|\det a(t)| > 0, b(t) \in L_p(\Gamma_1,\rho)$. Denote the discontinuity points of the function $a_{ii}(t), i = 1,\cdots,n$, by $c_1,\cdots,c_r$. By μ_{ik} denote the parameter of the function a_{ii} in the point $c_k(k = 1,\cdots,r)$. Assume that the inequalities

$$\frac{1+\nu_k}{p} \neq \mu_k,\quad k = 1,\cdots,r,\quad i = 1,\cdots,n \tag{3.12}$$

are fulfilled.

Let us show that there exists the canonical matrix for the problem (3.11) of the corresponding class.

It is evident that when the inequalities (3.12) are fulfilled, then every function a_{ii} has the canonical function of the class $E_p^{\pm}(\Gamma_1, \Gamma_2, \rho)$; denote it by $\chi_i(\lambda)$. Consider the triangular matrix $\chi(z) = (\chi_{ik}), i, k = 1, \cdots, n; \chi_{ik} = 0$, when $i < k, \chi_{kk}(z) = \chi_k(z)$ and other elements are defined by the formulas

$$\begin{aligned} \chi_{s1}^- &= \chi_s^- \mathcal{L}_1 \left[f_s \sum_{i=1}^{s-1} a_{si} \chi_{i1}^- \right], \quad s = 2, \cdots, n, \\ \chi_{s2}^- &= \chi_s^- \mathcal{L}_1 \left[f_s \sum_{i=2}^{s-1} a_{si} \chi_{i2}^- \right], \quad s = 3, \cdots, n, \\ &\cdots\cdots \\ \chi_{n,n-1}^- &= \chi_n^- \mathcal{L}_1 [f_n a_{n,n+1} \chi_{n-1,n-1}^-], \quad f_s = \{\chi_s^+[\alpha(t)]\}^{-1}. \end{aligned} \tag{3.13}$$

It is clear that the constructed matrix belongs to the class $E_p^{\pm}(\Gamma_1, \Gamma_2, \rho)$ and

$$\chi^+[\alpha(t)] = a(t)\chi^-(t), \quad t \in \Gamma_1. \tag{3.14}$$

Construct now the analogous matrix $\chi_*(z)$ for the matrix $[\alpha'(t)a'(t)]^{-1}$ of the class $E_q^{\pm}(\Gamma_1, \Gamma_2, \rho^{1-q})$:

$$\chi_*^+[\alpha(t)] = [\alpha'(t)a'(t)]^{-1}\chi_*^-(t), \quad t \in \Gamma_1.$$

We have

$$\alpha'(t)\chi_*'^+[\alpha(t)]\chi^+[\alpha(t)] = \chi_*'^-(t)\chi^-(t). \tag{3.15}$$

Integrating this equality , we get

$$\chi_0[\alpha(t)] = \chi_0^-(t), \quad t \in \Gamma_1,$$

where $\chi_0(z)$ is a primitive for $\chi_*'(z)\chi(z)$. Therefore

$$\chi_0(z) = P[\omega(z)],$$

where P is a polynomial matrix and

$$\chi_*'(z)\chi(z) = \omega'(z)Q[\omega(z)], \quad Q(z) = dP(z)/dz. \tag{3.16}$$

From the equalities (3.15) and (3.16) we have

$$\{\omega'^+[\alpha(t)]\}^{-1}\chi_*'^+[\alpha(t)]\chi^+[\alpha(t)] = \{\omega'^-(t)\}^{-1}\chi_*'^-(t)\chi^-(t) = Q[\omega^-(t)].$$

It follows from the last equality, that

$$\det Q[\omega(z)] \equiv 1$$

and thus $\chi(z)$ has an inverse matrix of the class $E_q^{\pm}(\Gamma_1, \Gamma_2, \rho^{1-q})$ and χ is a normal matrix of the class $E_p^{\pm}(\Gamma_1, \Gamma_2, \rho)$.

It is easy to see that the boundary problem (3.11) is solvable for any $b(t) \in L_p(\Gamma, \rho)$ and that's why the operators

$$\chi^-(t_0)\mathcal{L}\{[\chi^+(\alpha(t))]^{-1}b(t)\}, \;\; \chi(t_0)\mathcal{L}\{[\chi^-(t)]^{-1}b(t)\}$$

are the linear bounded operators in $L_p(\Gamma, \rho)$.

Index of the problem (3.11) is equal to the sum of the indices of boundary problems

$$\varphi_k^+[\alpha(t)] = a_{kk}(t)\varphi_k^-(t), \;\; t \in \Gamma_1,$$

i.e. $\varkappa = \sum_{k=1}^n \varkappa_k, \varkappa_k$ is calculated by the formula (3.9) of chapter 19.1.

Now consider the problem in general case.

Represent the matrix $a(t)$ as in chapter 1 in the following form

$$a(t) = a_1(t)\Lambda(t)a_2(t),$$

where $a_1(t), a_2(t)$ are nonsingular continuous matrices, $\Lambda(t)$ is a piecewise-continuous triangular matrix, inf $|\det \Lambda(t)| > 0$.

Take the rational matrices $R_1(z)$ and $R_2(z)$ such that

$$|a_1(t) - R_1(\alpha(t))|_C \leqslant \varepsilon, \;\; |a_2(t) - R_2(t)|_C \leqslant \varepsilon,$$

ε is sufficiently small positive number.

Introduce the following notations

$$R_1^{-1}(z)\Phi(z) = \varphi(z), \;\; z \in D_2^+, \;\; R_2(z)\Phi(z) = \varphi(z), \;\; z \in D_1^-,$$
$$R_1^{-1}[\alpha(t)]b(t) = b_0(t),$$

We obtain

$$\varphi^+[\alpha(t)] = \Lambda(t)\varphi^-(t) + a_0(t)\varphi^-(t) + b_0(t), \;\; t \in \Gamma_1.$$

It is clear that

$$\sup|a_2(t)| < c\varepsilon, \;\; c - \text{is a constant.}$$

Consider a sequence of matrices of the class $E_p^{\pm}(\Gamma_1, \Gamma_2, \rho)$, defined by the formulas

$$\varphi_{m+1}^+[\alpha(t)] = \Lambda(t)\varphi_{m+1}^-(t) + a_0(t)\varphi_m^-(t) + b_0(t), \;\; t \in \Gamma_1, \;\; \varphi_0^-(t) = 0,$$
$$\varphi_{m+1}^-(t_0) = \chi^-(t_0)\mathcal{L}_1\{[\chi^+(\alpha(t))]^{-1}[a_0(t)\varphi_m^-(t) + b_0(t)]\},$$

where $\chi(z)$ is a canonical matrix of the matrix $\Lambda(t)$ of the class $E_p^{\pm}(\Gamma_1, \Gamma_2, \rho)$. One can prove the convergence of the sequence $\varphi_m(z)$; the limit matrix will satisfy the boundary condition (cf.19.3).

$$\varphi^+[\alpha(t)] = R_1^{-1}[\alpha(t)]a(t)R_2^{-1}(t)\varphi^-(t) + R_1^{-1}(\alpha(t))b(t),$$
$$\varphi(z) \in E_p^{\pm}(\Gamma_1, \Gamma_2, \rho)$$

Thus the boundary values of the matrix

$$\Phi(z) = \begin{cases} R_1(z)\varphi(z), & z \in D_2^+, \\ R_2^{-1}(z)\varphi(z), & z \in D_1^- \end{cases}$$

will satisfy the considered problem.

Behave similarly with the adjoint problem

$$\Psi^+[\alpha(t)] = [\alpha'(t)a'(t)]^{-1}\Psi^-(t) + g(t),$$
$$g \in L_q(\Gamma_1, \rho^{1-q}), \;\; \Psi(z) \in E_q^{\pm}(\Gamma_1, \Gamma_2, q^{1-\rho}).$$

Substituting the matrix R_1 and R_2 by the matrices $R_1'^{-1}$ and $R_2'^{-1}$, in above arguments, we construct the solution of the problem in the following form

$$\Psi(z) = \begin{cases} [R_1'(z)]^{-1}\psi(z), & z \in D^+, \\ R_2'(z)\psi(z), & z \in D^-. \end{cases}$$

Take now $b = aR_2^{-1}\chi^-$, $\;g = [\alpha' a']R_2'[\chi'^-]^{-1}$. We have

$$\Phi^+[\alpha(t)] = a(t)[\Phi^-(t) + R_2^{-1}(t)\chi^-(t)],$$
$$\alpha(t)\Psi^+[\alpha(t)] = [a'(t)]^{-1}[\Psi^-(t) + R_2'(t)(\chi'^-(t))^{-1}].$$

It follows from these equalities, that

$$\alpha'(t)\psi'[\alpha(t)]\varphi^-(t) = [\psi'^-(t) + (\chi^-(t))^{-1}][\varphi^-(t) + \chi^-(t)].$$

Integrating this equality we get

$$\chi^+[\alpha(t)] = \chi^-(t), \;\; t \in \Gamma_1, \tag{3.17}$$

where $\chi(z)$ is a primitive matrix; it follows from (3.17) that

$$\chi(z) = P[\omega(z)].$$

Reasoning similarly as in the above Section 17.3, we establish the existence of a canonical matrix and obtain analogously to Theorem 17.3.2.:

Theorem 18.3.2 *Let $a(t)$ be a piecewise - continuous matrix with the discontinuity points $t_k (k = 1, \cdots, r), \inf|\det\, a(t)| > 0$ and let $\lambda_{kj} (k = 1, \cdots, r, j = 1, \cdots, n)$ be the roots of the equation*

$$\det\, [a^{-1}(t_k - 0)a(t_k + 0) - I] = 0,$$

$$\mu_{kj} = arg\lambda_{kj}/2\pi, \;\; 0 \leqslant arg\lambda_{kj} < 2\pi.$$

If the inequalities

$$\frac{1 + \nu_k}{p} \neq \mu_{kj}$$

are valid, then there exists the canonical matrix of the problem (3.11) *of the class* $E_p^{\pm}(\Gamma_1, \Gamma_2, \rho)$; *the index of this class is calculated by the formula*

$$\varkappa = \frac{1}{2\pi}\left\{arg\left[\prod_{k=1}^{r}(t-z_0)^{-\sigma_k}\det\ a(t)\right]\right\}_{\Gamma_1},$$

where $\sigma_k = \sum_{j=1}^{n}\rho_{kj}$, $\rho_{kj} = -\dfrac{1}{2\pi i}ln\lambda_{kj}$;

$$-1 < Re\rho_{kj} \leqslant 0 \ \text{ when } \ \mu_{kj} < (1+\nu_k)/p,$$
$$0 \leqslant Re\rho_{kj} < 1 \ \text{ when } \ \mu_{kj} > (1+\nu_k)/p.$$

All solutions of the class $E_p^{\pm}(\Gamma_1, \Gamma_2, \rho)$ *of the problem* (3.11) *are given by the formula*

$$\Phi(z) = \chi(z)[\varphi_0(z) + P(\omega(z))], \tag{3.18}$$

where P is an arbitrary polynomial vector, $\varphi_0(z)$ *is a solution of the class* $E_{1+\varepsilon,0}^{\pm}(\Gamma_1, \Gamma_2)$($\varepsilon$ *is a sufficiently small positive number*) *of the problem*

$$\varphi_0^+[\alpha(t)] = \varphi_0^-(t) + b_0(t), \ \ t\in\Gamma_1, \ \ b_0(t) = \{\chi^+[\alpha(t)]\}^{-1}b(t).$$

The solutions vanishing at infinity are given by the same formula (3.18) in which $P = (P_{\varkappa_1-1,}, \cdots, P_{\varkappa_n-1})$, $\varkappa_1 \geqslant \cdots, \geqslant \varkappa_n$ are the positive indices of the problem (3.11) of the class $E_p^{\pm}(\Gamma_1, \Gamma_2, \rho)$, $P_j(z)$ is an arbitrary polynomial of order j ($P_j = 0$ in the case $j < 0$). If all partial indices are non-negative, then vanishing solutions exist for any $b(t) \in L_p(\Gamma_1, \rho)$; if $0 > \varkappa_{s+1} \geqslant \cdots \geqslant \varkappa_n$, then the vector $b(t)$ will satisfy the following conditions

$$\int_{\Gamma_k} t^k\rho_j^0(t)dt = 0, \ \ k = 0, \cdots, |\varkappa_j| - 1, \ \ j = s+1, \cdots, n,$$

where the vector $(\rho_1^0, \cdots, \rho_n^0) = \rho_0$ is a solution of the equation $K(I)\rho_0 = \tilde{b_0}$ of the class $L_q(\Gamma, \rho^{1-q})$ (or $\rho_0 = \mathcal{L}_1 b_0$).

Note some properties of the solution of the problem (3.11).

Lemma 18.3.2 *If l is an arc of the curve Γ not containing the discontinuity points of $a(t)$ and $\Phi(z)$ is a solution of homogeneous problem* (3.11) *of some class* $E_p^{\pm}(\Gamma_1, \Gamma_2, \rho)$ *then*

$$\Phi^-(t), \ \ \Phi^+[\alpha(t)] \in L_\infty(l).$$

In particular, for the canonical matrix of any class

$$\chi^-(t), \ \ [\chi^-(t)]^{-1} \in L_\infty(l).$$

Proof Construct the matrix $a_0(t)$ continuous and nonsingular on Γ_1 which coincides with the matrix $a(t)$ on l. Let $\chi_0(t)$ be a canonical matrix for $a_0(t)$. Consider the vector $\chi_0^{-1}\Phi(z) = \Phi_0(z)$. It is evident, that $\Phi_0(z) \in E_\lambda^\pm(\Gamma_1, \Gamma_2)$ for some $\lambda > 1$. Then

$$\Phi_0^+[\alpha(t)] = [\chi_0^-(t)]^{-1} a_1(t)\Phi^-(t), \quad t \in \Gamma_1, \tag{3.19}$$

$$a_1 = a_0^{-1}a, \quad a_1(t) = I, \quad t \in l.$$

By changing of variables

$$\tau = \omega^-(t) = \omega^+[\alpha(t)], \quad t = \omega_1^-(\tau), \quad \alpha(t) = \omega_1^+(\tau)$$

the equality (3.19) turns into the equality

$$\Phi_0^+[\omega_1^+(\tau)] = [\chi_0^-(\omega_1^-(\tau))]^{-1} a_1[\omega_1^-(\tau)]\Phi^-[\omega_1^-(\tau)], \quad \tau \in \gamma.$$

On the arc $l_0 = \omega^-(l)$ this equality has the form $\Phi_0^+[w_1^+(\tau)] = \Phi_0^-[w_1^+(\tau)]$. Thus

$$\Phi_0^+[\omega_1^+(\tau)] = \Phi_0^-[\omega_1^-(\tau)] = A(\tau), \quad \tau \in l_0,$$

$A(\tau)$ is a holomorphic vector in the vicinity of the arc l_0,

$$\Phi_0^+(\alpha(t)) = \Phi^-(t) = A(\omega^-(t)), \quad t \in l.$$

Consequently, $\Phi_0^-(t)$ is Hölder-continuous on the arc l. Hence the following lemma holds

Lemma 18.3.3 *If in the problem $a(t), b(t)$ and $\alpha'(t) \in H(\Gamma)$, then every solution of* (3.11) *is a Hölder-continuous in the closures $\overline{D_2^+}$ and $\overline{D_1^-}$ (except perhaps the point $z = \infty$).*

Proof follows from the equality $\Phi^+[\omega_1^+(\tau)] = a[\omega_1^-(\tau)]\Phi^-[\omega_1^-(\tau)]$, where γ is smooth curve.

18.4 Boundary value problems with displacement containing complex conjugate values of the desired functions

Consider the problem which contains the complex conjugate values of the desired vector:

Find a vector

$$\varphi(z) = (\varphi_1, \cdots, \varphi_n) \in F_p^\pm(\Gamma, \rho),$$

satisfying the boundary condition

$$\varphi^+[\alpha(t)] = a(t)\varphi^-(t) + b(t)\overline{\varphi^-(t)} + f(t), \tag{4.1}$$

on simple closed Liapunov curve Γ, where $a(t)$, $b(t)$ are given piecewise-continuous $(n \times n)$-matrices on Γ, $\inf |\det a(t)| > 0$, $f(t)$ is a given vector, $f(t) \in L_p(\Gamma, \rho)$, $\alpha(t)$ is a function, mapping Γ onto Γ in one-to-one manner keeping the orientation

$$0 \neq \alpha'(t) \in C(\Gamma), \;\; \rho(t) = \prod_{k=1}^{r} |t - t_k|^{\nu_k}, \;\; t_k \in \Gamma, \;\; -1 < \nu_k < p - 1, \;\; p > 1,$$

$F_p^{\pm}(\Gamma, \rho)$ denotes the subclass of the class $E_1^{\pm}(\Gamma)$, for which $\varphi^-(t) \in L_p(\Gamma, \rho)$.

We seek the solution of the problem (4.53) in the following form [81], [84]

$$\begin{aligned} \varphi(z) &= \frac{1}{2\pi i} \int_\Gamma \frac{\mu(\beta(t))dt}{t - z}, \;\; z \in D^+, \\ \varphi(z) &= \varphi_0(z) + P(z), \quad \varphi_0(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\mu(t)dt}{t - z}, \;\; z \in D^-, \end{aligned} \tag{4.2}$$

D^+, D^- are domains, bounded by Γ, $\beta(t)$ is an inverse function to $\alpha(t)$, $P(z)$ is a polynomial vector - the principal part of the vector $\varphi(z)$ at infinity.

In order to define the vector $\mu(t) \in L_p(\Gamma, \rho)$ we obtain the following singular integral equation

$$\begin{aligned} \mathcal{L}\mu \equiv [I + a(t_0)]\mu(t_0) &+ \frac{I - a(t_0)}{\pi i} \int_\Gamma \frac{\mu(t)dt}{t - t_0} + \frac{1}{\pi i} \int_\Gamma k(t_0, t)\mu(t)dt \\ +b(t_0) &\left[\overline{\mu(t_0)} + \frac{1}{2\pi i} \int_\Gamma \frac{\overline{\mu(t)}dt}{t - t_0} + \frac{1}{\pi i} \int_\Gamma k(t_0, t)\overline{\mu(t)}dt \right] = f_0(t_0), \end{aligned} \tag{4.3}$$

$$f_0(t) = 2 \left[f(t) + a(t)P(t) + b(t)\overline{P(t)} \right];$$

$$k_0(t_0, t) = \frac{t - t_0}{\bar{t} - \bar{t_0}} \frac{\partial}{\partial t} \frac{\bar{t} - \bar{t_0}}{t - t_0}.$$

The Noetherity condition for the equation (4.3) in the space $L_p(\Gamma, \rho)$ is defined by the matrix

$$G = S^{-1}D,$$

where S and D are block matrices:

$$S = \begin{pmatrix} I & b \\ 0 & \bar{a} \end{pmatrix}, \;\; D = \begin{pmatrix} a & 0 \\ \bar{b} & I \end{pmatrix};$$

Index of the operator $\mathcal{L}_\mu$ (of the class $L_p(\Gamma, \rho)$) is equal to the index of the matrix $G(t)$ (of the class $E_p^{\pm}(\Gamma, \rho)$).

Consider the following homogeneous problem:

Find a vector $\psi(z) \in E_{q,0}^{\pm}(\Gamma, \rho^{1-q})$ satisfying the boundary condition

$$\psi^-(t) = \alpha'(t)a'(t)\psi^+[\alpha(t)] + \overline{\alpha'(t)}\; \overline{t'^2}\; \overline{b'(t)}\; \overline{\psi^+[\alpha(t)]}, \tag{4.4}$$

which we call the adjoint problem of the problem (4.1)

Suppose that the problem (4.1) is solvable for the given $P(z)$ and compose the following expression

$$\begin{aligned}\operatorname{Re}\int_\Gamma f_0(\beta(t))\psi^+(t)dt &= \operatorname{Re}\int_\Gamma f_0(t)\psi^+[\alpha(t)]d\alpha(t)\\ &= \operatorname{Re}\int_\Gamma \left\{\varphi^+[a(t)] - a(t)\varphi_0^-(t) - b(t)\overline{\varphi_0(t)}\right\}\psi^+[\alpha(t)]d\alpha(t)\\ &= -\operatorname{Re}\int_\Gamma \varphi_0^-(t)\left[\alpha'(t)a'(t)\psi^+(\alpha(t)) + \overline{\alpha'(t)}\,\overline{b(t)}\,\overline{t'^2}\,\overline{\psi^+(\alpha(t)}\,)\right]dt\\ &= -\operatorname{Re}\int_\Gamma \varphi_0^-(t)\psi^-(t)dt = 0\end{aligned}$$

and hence for the problem (4.1) to be solvable it is necessary and sufficient the fulfillment of the following condition

$$\operatorname{Re}\int_\Gamma f_0(\beta(t))\psi^+(t)dt = 0 \text{ or } \operatorname{Re}\int_\Gamma f_0(t)\psi^+[\alpha(t)]d\alpha(t) = 0, \tag{4.5}$$

where $\psi(z)$ is an arbitrary solution of the problem (4.4) of the class $E^{\pm}_{q,0}(\Gamma, \rho^{1-q})$.

The equation (4.3) is solvable in $L_p(\Gamma, \rho)$ if and only if

$$\operatorname{Re}\int_\Gamma f_0(t)\omega_k(t)dt = 0, \tag{4.6}$$

where ω_k $(k = 1, \cdots, l')$ is a complete system of linearly independent solutions of the adjoint homogeneous equation

$$\mathcal{L}'\omega = 0 \text{ of the class } L_q(\Gamma, \rho^{1-q}).$$

It is clear, that we mean the linear independence over the real number field. The conditions (4.6) will be fulfilled automatically if the vector $f_0(t)$ has the form

$$f_0(t) = \varphi^+[\alpha(t)] - a(t)\varphi^-(t) - b(t)\overline{\varphi^-(t)},$$

where $\varphi(z)$ is an arbitrary vector of the class $E^{\pm}_{p,0}(\Gamma, \rho)$.

From here it is easy to deduce that $\omega_k(t)$ will have the form

$$\omega_k(t) = \alpha'(t)\psi_k^+(\alpha(t)),$$

the vector $\psi_k(z) \in E^{\pm}_{q,0}(\Gamma, \rho^{1-q})$ is the solution of the problem (4.4). Therefore, the number l' coincides with the number of linearly independent solutions of the homogeneous equation (4.4) of the class $E^{\pm}_{q,0}(\Gamma, \rho^{1-q})$; because of the uniqueness of the representation the number of linearly independent solutions of the homogeneous problem $\mathcal{L}\mu = 0$ (of $L_p(\Gamma, \rho)$) and the homogeneous problem of the problem (4.1) (of $E^{\pm}_{p,0}(\Gamma, \rho)$) are also coinciding.

If we summarize these arguments then we get the following theorem.

Theorem 18.4.1 *The problem* (4.1) *is solvable if and only if*

$$Re \int_{\Gamma} f_0(t)\psi_k^+[\alpha(t)]\alpha'(t)dt = 0, \quad k = 1, \cdots, .l',$$

where $\psi_k(z)(k = 1, \cdots, l')$ *is a complete system of linear independent solutions of the homogeneous problem* (4.4) *of the class* $E^{\pm}_{q,0}(\Gamma, \rho^{1-q})$;

$$l - l' = \varkappa,$$

where l *denotes the number of linear independent solutions of the homogeneous problem* (4.1) *of the class* $E^{\pm}_{p,0}(\Gamma, \rho), \varkappa$ *is the index of the operator* $\mathcal{L}\mu$ *of the class* $L_p(\Gamma, \rho)$.

Consider the set of following problems:

$$\varphi^+[\alpha_\lambda(t)] = a(t)\varphi^-(t), \quad t \in \gamma, \tag{4.7}$$

$\gamma : t = e^{i\theta}, \ \ 0 \leqslant \theta \leqslant 2\pi$ is a unit circle,

$$\alpha_\lambda(t) = exp[i\nu_\lambda(\theta)], \ \ \nu_\lambda(\theta) = (1-\lambda)\theta + \lambda\nu(\theta), \ \ \lambda \in [0, 1],$$

$\nu(\theta)$ is strongly increasing continuous function on $[0, 2\pi]$, $\nu(0) = 0$, $\nu(2\pi) = 2\pi, \nu(\theta)$ has a continuous derivative $\nu'(\theta) > 0$, $\nu'(0) = \nu'(2\pi)$; $a(t)$ is given quadratic matrix of order n on γ,

$$a(t) \in C_0(\gamma, c_1, \cdots, c_r), \ \ \inf |\det a(t)| > 0.$$

Denote by $\varkappa_k[a, \lambda]$, $k = 1, \cdots, n$ the partial indices of the class $E_p^{\pm}(\Gamma, \rho)$ of the problem (4.7); $\rho(t) = \prod_{k=1}^r |t - c_k|^{\nu_k}, \ \ -1 < \nu_k < p - 1$.

The sum of non-negative (non-positive) partial indices denote by

$$N^+[a, \lambda](N^-[a, \lambda]).$$

We are looking for the solution of the problem (4.7) in the class $E_p^{\pm}(\Gamma, \rho)$ in the following form

$$\varphi(z) = \frac{1}{2\pi i}\int_\gamma \frac{\mu[\beta_\lambda(t)]}{t - z}dt, \ \ z \in D^+, \ \ \varphi(z) = \frac{1}{2\pi i}\int_\gamma \frac{\mu(t)dt}{t - z}, z \in D^-, \tag{4.8}$$

where D^+ and D^- denote the domains $|z| < 1$ and $|z| > 1$, $\beta_\lambda(t)$ is a function inverse to $\alpha_\lambda(t)$.

In order to define the vector $\mu \in L_p(\Gamma, \rho)$ we get the following equation

$$\begin{gathered} \mathcal{L}_\lambda\mu \equiv [I + a(t)]\mu(t_0) + \frac{I - a(t_0)}{\pi i}\int_\gamma \frac{\mu(t)dt}{t - t_0} + M_\lambda\mu = 0, \\ M_\lambda\mu = \frac{1}{\pi i}\int_\gamma \left[\frac{\alpha'_\lambda(t)}{\alpha_\lambda(t) - \alpha_\lambda(t_0)} - \frac{1}{t - t_0}\right]\mu(t)dt. \end{gathered} \tag{4.9}$$

The operator M_λ is completely continuous operator in any space $L_p(\gamma,\rho)$. The operator K_λ is the Noetherian operator in the specific spaces $L_p(\gamma,\rho)$, for which $p>1$ and ν_k will satisfy the condition

$$\frac{1+\nu_k}{p}\neq\mu_{ik}, \tag{4.10}$$

where the numbers μ_{ik} are defined by the matrix $a(t)$ (see, 19.4). We assume that the conditions (4.10) are fulfilled. It is possible to rewrite the operator M_λ in the following form

$$M_\lambda\mu=\frac{1}{2}\int_0^{2\pi}M(\theta_0,\theta,\lambda)\mu(e^{i\theta})d\theta,$$

$$M(\theta_0,\theta,\lambda)=\left[ctg\frac{\nu_\lambda(\theta)-\nu_\lambda(\theta_0)}{2}+i\right]\nu_\lambda'(\theta)-ctg\frac{\theta-\theta_0}{2}-i.$$

The function $M(\theta_0,\theta,\lambda)$ is defined on [0,1] with respect to λ, however we may consider this function as the analytic function in some domain D_λ which contains this segment. Therefore, $\mathcal{L}_\lambda$ is linear bounded operator in the space $L_p(\gamma,\rho)$; analytically depending on λ; L_λ will be Noetherian operator for all $\lambda\in D_\lambda$. Hence, the equation (4.9) has the same number of linear independent solutions in $L_p(\gamma,\rho)$ for all $\lambda\in D_\lambda\backslash D_\lambda'$, where $D_\lambda'\subset D_\lambda$ is the isolated set [36, 75]. The intersection $D_\lambda'\cap$ $[0,1]$ will be either empty or finite set; the number of linear independent solutions of the equation (4.9) in the case $\lambda\in[0,1]$ is bounded, so that $N^+[a,\lambda],N^-[a,\lambda]$ and the partial indices $\varkappa_k[a,\lambda]$ are bounded.

Let the maximal partial index $\varkappa_1,[a,\lambda]$ takes the values η_1 and η_2 on two infinite subsets E_1 and E_2 of the segment [0,1], $\eta_1\leqslant\eta_2$.

Consider the matrix $a_1(t)=t^{\eta_1}a(t)$; we get:

$$\varkappa_1[a_1\lambda]=0,\lambda\in E_1,\varkappa_1[a_1,\lambda]=\eta_2-\eta_1,\lambda\in E_2,$$

$$N^+[a_1,\lambda]=0,\lambda\in E_1,N^+[a_1,\lambda]>\eta_2-\eta_1,\lambda\in E_2\quad \eta_2-\eta_1=0.$$

Carrying out the analogous arguments with respect to other partial indices. We obtain the following theorem.

Theorem 18.4.2 *The partial indices $\varkappa_k[a,\lambda]$ are constant values for all $\lambda\in[0,1]$ except the points of some finite set.*

Consider the set of the following problems:

$$\varphi^+[\alpha_\lambda(t)]=a(t)\varphi^-(t)+b(t)\overline{\varphi^-(t)},t\in\gamma, \tag{4.11}$$

where $a(t),\alpha_\lambda(t)$ satisfy the conditions mentioned in above item, $b(t)$ is an arbitrary piecewise-continuous matrix, $\Phi(z)$ is desired vector of the class $E^{\pm}_{p,0}(\Gamma,\rho)$.

We seek the solution of the problem (4.63) in the following form

$$\varphi(z) = \frac{1}{\pi i} \int_\gamma \frac{\mu[\beta_\lambda(t)]dt}{t - z}, \quad z \in D^+, \quad \varphi(z) = \frac{1}{\pi i} \int_\gamma \frac{\mu(t)dt}{t - z}, \ z \in D^-. \tag{4.12}$$

We get the following equation with respect to $\mu \in L_p(\gamma, \rho)$

$$\begin{aligned} &[I + a(t_0)]\, \mu(t_0) + \frac{I - a(t_0)}{\pi i} \int_\gamma \frac{\mu(t)dt}{t - t_0} + M_\lambda \mu \\ &+ b(t_0) \left[\overline{\mu(t_0)} + \frac{1}{2\pi i} \int_\gamma \frac{\overline{\mu(t)}dt}{t - t_0} + \frac{1}{2\pi i} \int_\gamma k_0(t_0, t)\overline{\mu(t)}dt \right] = 0. \end{aligned} \tag{4.13}$$

Together with this equation consider the following equation

$$A(t_0)\Omega(t_0) + \frac{B(t_0)}{\pi i} \int_\gamma \frac{\Omega(t)dt}{t - t_0} + N_\lambda \Omega = 0, \tag{4.14}$$

where Ω is a desired $2n$-dimensional vector,

$$A = \begin{pmatrix} I + a, & b \\ \bar{b}, & I + \bar{a} \end{pmatrix}, \quad B = \begin{pmatrix} I - a, & b \\ -\bar{b}, & \bar{a} - I \end{pmatrix},$$

$$N_\lambda \Omega = \frac{1}{\pi i} \int_\gamma N(t_0, t_\lambda)\Omega(t)dt,$$

$$N(t_0, t, \lambda) = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix},$$

$$N_{12} = \frac{\alpha'_\lambda(t)}{\alpha_\lambda(t) - \alpha_\lambda(t_0)} - \frac{1}{t - t_0}, \quad N_{12} = b(t_0)k_0(t_0, t),$$

$$N_{21} = -\overline{b(t_0)}[k_0(t_0, t) + \overline{t'^2} \ \ \overline{k_0(t_0, t)}],$$

$$N_{22} = [\overline{a(t_0)} - I]k_0(t_0, t) - \frac{t_0}{(t - t_0)t} + \frac{\nu'_\lambda(\theta)\alpha_\lambda(t_0)}{t[\alpha_\lambda(t_0) - \alpha_\lambda(t)]}.$$

If the vector $\mu = (\mu_1, \cdots, \mu_n) \in L_p(\gamma, \rho)$ is a solution of the equation (4.13), then the vector $\Omega = (\mu_1, \cdots, \mu_n, \bar{\mu}_1, \cdots, \bar{\mu}_n)$ is a solution of the equation (4.14) and if $\Omega = (\Omega_1, \cdots, \Omega_{2n}) \in L_p(\gamma, \rho)$ is a solution of the equation (4.14), then $\mu = (\Omega_1 + \overline{\Omega}_{n+1}, \cdots, \Omega_n + \overline{\Omega}_{2n})$ is a solution of the equation (4.13).

Carrying out the arguments analogous to 18.4.2, we obtain the following theorem

Theorem 18.4.3 *The number of linear independent solutions (over the real number field) of the problem (4.11) of the class $E_p^{\pm}(\Gamma, \rho)$ is constant for all $\lambda \in [0, 1]$ except perhaps the points of some finite set.*

As it was mentioned in the introduction the systematic researches in the theory of linear conjugation with displacement for analytic functions has been started after the appearance of works of Kveselava [81]-[84]. At present numerous researches concerning of many aspects of this theory are published. Referring only to works [21], [38], [141], [143], [62], [123], [82], [31], note that the main part of these researches are reflected in the monographs [136], [45], [88], [105], [108], in the survey papers [141], [88], [142] and in [30] is given the survey of works concerning the problems of linear conjugation with displacement on Riemann surfaces.

In the recent years many researches were published, but among them we refer to the following works [2], [20]- [25], [66]-[67], [86], [87], [128], [129].

Application of the problems of linear conjugation with displacement in the theory of elasticity one can find in [9], [10], [94], [126]-[127].

In this chapter there were used the works of the author [95], [99], [100] and also [101], [102].

In the present book we do not consider the closely related problem of singular integral equations with shift; in this connection we shall indicate only some works [133], [136], [61], [69], [77], [78], [80], [29] and shall note, that the corresponding references one may find in the monographs [88], [66], and in the survey paper [70].

Chapter 19

Linear Conjugation with Displacement for Generalized Analytic Functions and Vectors

by Giorgi F. Manjavidze

19.1 Definitions and notations

In the theory of generalized analytic functions the following integral operators

$$(Tf)[z] = -\frac{1}{\pi}\iint\limits_D \frac{f(\zeta)d\sigma_\zeta}{\zeta - z}, \quad (\Pi f)[z] = -\frac{1}{\pi}\iint\limits_D \frac{f(\zeta)d\sigma_\zeta}{(\zeta - z)^2}$$

play an important role, where D is some domain in the $z-$plane, $z = x + iy$, and $f(\zeta)$ is a function of the class $L_p(\bar{D}), p \geqslant 1$. The main properties of the operators T, Π are the following.

The generalized derivatives satisfy

$$\partial_{\bar{z}} Tf = f, \ \ \partial_z Tf = \Pi f.$$

If D is a bounded domain, then Tf is a linear completely continuous operator from the space $L_p(\bar{D}), p > 2$, into the space $H^\alpha(D), \alpha = (p-2)/p$.

If the boundary Γ of D is the union of a finite number of piecewise-smooth contours, then the operator T is a linear bounded operator from $L_p(\bar{D}), 1 < p \leqslant 2$, into $L_j(\Gamma), 1 < j < p/(2-p)$.

Let $D \in H_\alpha^{m+1}, f(z) \in H_\alpha^m(D), 0 < \alpha < 1, m \geqslant 0$. Then $Tf \in H_\alpha^{m+1}(D), \partial_z Tf = \Pi f \in H_\alpha^m(D)$.

Πf is a linear bounded operator in the spaces $H^\alpha(D)$ and $L_p(\bar{D}), p > 1$.

Let $q(z)$ be a measurable bounded function in the whole plane $\mathbb{C}, |q(z)| \leqslant q_0 < 1, q(z) = 0$ in a neighborhood of $z = \infty$, and let f be a solution of the equation

$$f - q\Pi f = q$$

belonging to the class $L_p(\mathbb{C}), p > 2$. Then the function

$$\omega(z) = z + Tf$$

is a fundamental homeomorphism of the Beltrami equation

$$\partial_{\bar{z}}\omega - q(z)\partial_z\omega = 0.$$

These and other properties of the operators T and Π are formulated and proved in the monograph [139].

A vector $w(z) = (w_1, \cdots, w_n)$ is called a generalized analytic vector in the domain D if it is a solution of an elliptic system of the form

$$\partial_{\bar{z}}w - Q(z)\partial_z w + A(z)w + B(z)\bar{w} = 0, \tag{1.1}$$

where $A(z), B(z)$ are given quadratic matrices of order n of the class $L_{p_0}(D), p_0 > 2$, and $Q(z)$ is a matrix of the following special form: it is quasidiagonal and every block $Q^r = (q^r_{ik})$ is a lower (upper) triangular matrix satisfying the conditions

$$\begin{gathered} q^r_{11} = \cdots = q^r_{m_r,m_s} = q^r, \ \ |q^r| \leqslant q_0 < 1, \\ q^r_{ik} = q^r_{i+s,k+s} \ \ (i+s \leqslant n, k+s \leqslant n). \end{gathered}$$

Moreover, we suppose $Q(z) \in W^1_p(C), p > 2$, and $Q(z) = 0$ outside of some circle.

The equation

$$\partial_{\bar{z}}w - \partial_z(Q'w) - A'(z)w - \overline{B'(z)w} = 0 \tag{1.2}$$

is called conjugate to the equation (1.1), an accent$'$ denotes a transposition of a matrix.

If $A(z) \equiv B(z) \equiv 0$, the equation (1.1) and (1.2) passes into

$$\partial_{\bar{z}}w - Q(z)w_z = 0, \tag{1.3}$$

$$\partial_{\bar{z}}w - \partial_z(Q'w) = 0. \tag{1.4}$$

Solutions of the equation (1.3) are called $Q-$ holomorphic vectors.

The equation (1.3) has a solution of the form

$$\zeta(z) = zI + T\omega, \tag{1.5}$$

where I is the unit matrix and $\omega(z)$ is a solution of the equation

$$\omega(z) + Q(z)\Pi\omega = Q(z)$$

belonging to $L_p(\mathbb{C}), p > 2$.

The solution (1.5) of the equation (1.3) is analogous to the fundamental homeomorphism of the Beltrami equation.

The matrix

$$V(t, z) = \partial_t\zeta(t)[\zeta(t) - \zeta(z)]^{-1} \tag{1.6}$$

is called the generalized Cauchy Kernel for the equation (1.3) and the following assertions are true [23], [110]:

$$V(t,z)=\frac{1}{t-z}\left[I+Q(z)\frac{\bar{t}-\bar{z}}{t-z}\right]^{-1}+\frac{R_1(t,z)}{|t-z|^{\alpha}},$$

$$V(t,z)=\frac{1}{t-z}\left[I+Q(z)\frac{\bar{t}-\bar{z}}{t-z}\right]^{-1}+\frac{R_2(t,z)}{|t-z|^{\alpha}},\alpha\leqslant 1,$$

$$R_1(t,z),\quad R_2(t,z)\in H(\mathbb{C}\times\mathbb{C}),\quad R-1(z,z)=0,$$

$$|V_{ik}(t,z)|\leqslant\frac{const}{|t-z|}.$$

Next consider a generalized Cauchy-type integral defined by the matrix (1.6)

$$\Phi(z)=\frac{1}{2\pi i}\int_{\Gamma}V(t,z)d_Qt\mu(t), \tag{1.7}$$

where Γ is a closed simple smooth curve, $\mu(t)\in L_1(\Gamma)$ and

$$d_Qt=Idt+Q(t)d\bar{t}.$$

If the density $\mu(t)$ in (1.7) is Hölder-continuous on Γ, the integral (1.7) is Hölder-continuous in $\bar{D}^+$ and $\bar{D}^-$ (D^+ and D^- are the domains bounded by Γ); the boundary values of Φ on Γ are given by

$$\Phi^{\pm}(t)=\pm\frac{1}{2}\mu(t)+\frac{1}{2\pi i}\int_{\Gamma}V(\tau,t)d_Q\tau\mu(\tau). \tag{1.8}$$

If $\mu(t)\in L_p(\Gamma),p>1$, then the formulas (1.8) are fulfilled almost everywhere on Γ, provided $\Phi^{\pm}$ are now understood as angular boundary values of the vector $\Phi(z)$. The analogies of the integral operators T and Π,

$$(\tilde{T}f)[z]=-\frac{1}{\pi}\iint\limits_{D}V(t,z)f(t)d\sigma_t,$$
$$(\tilde{\Pi}f)[z]=-\frac{1}{\pi}\iint\limits_{D}\partial_zV(t,z)f(t)d\sigma_t \tag{1.9}$$

play an important role while studying generalized analytic vectors.

Let $Q\in H^{\alpha_0}(\mathbb{C})$. Then $(\tilde{T}f)$is a completely continuous operator from $L_p(\bar{D}),p>2$, into $H^{\alpha}(D),\alpha=\min\{\alpha_0,(p-2)/p\}$. (see [103], [112]). Moreover the operator $\tilde{\Pi}$ is a linear bounded operator from $L_p(\bar{D})$ in $L_p(\bar{D})$, and the relations

$$(\partial_{\bar{z}}-Q\partial_z)\tilde{T}f=f,\quad \partial_z\tilde{T}f=\widetilde{\Pi}f \tag{1.10}$$

are true.

Using $Q-$ holomorphic vectors, generalized analytic vectors can be represented as follows (see [23])

$$w(z) = \Phi(z) + \iint_D \Gamma_1(z,t)\Phi(t)d\sigma_t + \iint_D \Gamma_2(z,t)\overline{\Phi(t)}d\sigma_t + \sum_{k=1}^{N} c_k w_k(z), \quad (1.11)$$

where $\Phi(z)$ is a $Q-$ holomorphic vector, and $w_k(z)(k = 1, \cdots, N)$ is a complete system of linearly independent solutions of the Fredholm equation

$$Kw \equiv w(t) - \frac{1}{\pi}\iint_D V(t,z)[A(t)w(t) + B(t)\overline{w(t)}]d\sigma_t = 0.$$

the $w_k(z)$ turn out to be continuous vectors in the whole plane vanishing at infinity, and the c_k's are arbitrary real constants; the kernels $\Gamma_1(z,t)$ and $\Gamma_2(z,t)$, finally, satisfy the system of the integral equations

$$\begin{aligned}
&\Gamma_1(z,t) + \frac{1}{\pi}V(t,z)A(t) + \frac{1}{\pi}\iint_D V(\tau,z)[A(\tau)\Gamma_1(\tau,t) + B(\tau)\overline{\Gamma_2(\tau,t)}d\sigma_t \\
&\qquad = -\frac{1}{2}\sum_{k=1}^{N}\{v_k(z), \bar{v}_k(t)\}, \\
&\Gamma_2(z,t) + \frac{1}{\pi}V(t,z)A(t) + \frac{1}{\pi}\iint_D V(\tau,z)[A(\tau)\Gamma_2(\tau,t) + B(\tau)\overline{\Gamma_1(\tau,t)}d\sigma_t \\
&\qquad = -\frac{1}{2}\sum_{k=1}^{N}\{v_k(z), \bar{v}_k(t)\},
\end{aligned} \quad (1.12)$$

where the $v_k(z) \in L_p(\bar{D})(k = 1, \cdots, N)$ form a system of linearly independent solutions of the Fredholm integral equation

$$v(z) + \frac{\overline{A'(z)}}{\pi}\iint_D \overline{V(z,t)}v(t)d\sigma_t + \frac{B'(z)}{\pi}\iint_D V'(z,t)\overline{v(t)}d\sigma_t = 0.$$

In the formulas (1.12) curly bracket $\{v, w\}$ means a diagonal product of the vectors v and w: $\{v, w\}$ is a quadratic matrix of order n, whose elements $\{v, w\}_{ik}$ are defined by $\{v, w\}_{ik} = v_i w_k, \ \ i, k = 1, \cdots, n.$

Notice that in formula (1.11) $\Phi(z)$ is not an arbitrary $Q-$ holomorphic vector. It has to satisfy the conditions

$$\mathrm{Re} \iint_D \Phi(z)v_k(z)d\sigma_z = 0, \ \ k = 1, \cdots, N. \quad (1.13)$$

Finally it should be mentioned that, generally speaking, the Liouville theorem is not true for solutions of (1.1) . This explains the appearance of the constants c_k in the representation formula (1.11) and the fact that the condition (1.13)has to be satisfied (cf [23] and [49]).

19.2 Relation between linear conjugation with displacement and generalized analytic functions

In the present section we shall set the relation between the problem of linear conjugation with displacement and the theory of generalized analytic functions, this will give us the possibility to consider the problem of linear conjugation in somehow different formulation.

Let Γ_1 and Γ_2 be the Liapunov curves, $\alpha(t)$ is a function mapping Γ_1 onto Γ_2 in one-to-one manner preserving the orientation, $\alpha(t(s))$ is an absolutely continuous function, $M \geqslant |\alpha'(t)| \geqslant m > 0$ $(M, m$ are constants$)$ $a(t), b(t)$ are given matrices of the class $H^\mu(\Gamma_1)$ $\left(\mu > \dfrac{1}{2}\right)$, $a(t)$ is a nonsingular quadratic matrix of order n, $b(t)$ is a $(n\times l)-$ matrix; we have to find a piecewise-holomorphic matrix $\varphi(z)$, having the finite order at infinity, $\varphi^+(t), \varphi^-(t) \in H(\Gamma)$ and satisfying the boundary condition

$$\varphi^+[\alpha(t)] = a(t)\varphi^-(t) + b(t), \quad t \in \Gamma_1. \tag{2.1}$$

We call the piecewise-holomorphic matrix $\chi(z)$ with a finite order at infinity the canonical matrix of the problem (2.1) if det $\chi(z) \neq 0$ everywhere except perhaps at the point $z = \infty; \chi(z)$ has a normal form at infinity with respect to columns and

$$\chi^+[\alpha(t)] = a(t)\chi^-(t), \quad t \in \Gamma_1.$$

Mapping conformally D_2^+ and D_1^- into interior and exterior parts of the unit circle Γ respectively we get the same problem as (2.1), where $\alpha(t)$ has to map Γ onto Γ; the matrices $a(t)$, $b(t)$ and the function have the same properties. We shall consider the problem in the case $\Gamma_1 = \Gamma_2 = \Gamma$.

First prove the following lemmas.

Lemma 19.2.1 *Let $\alpha(t)$ be a function satisfying the same conditions as mentioned above and $\omega(z)$ is a piecewise-holomorphic function (bounded at infinity) $\omega^+[\alpha(t)] = \omega^-(t)$ on $\Gamma, \omega^-(t) \in H^*(\Gamma)$. Then $\omega(z)$ is a constant function.*

Consider the following function which is continuous on the whole plane

$$\Omega(z) = \begin{cases} \omega(\alpha(z)), & z \in \overline{D}^+, \\ \omega(z), & z \in D^-, \end{cases}$$

where

$$\alpha(z) = |z|\alpha\left(\frac{z}{|z|}\right). \tag{2.2}$$

On the basis of one Hardy-Littlewood theorem (see [24], [39]) we have

$$|\omega'(z)| \leqslant A(1-|z|)^{\mu-1}, \ \ z \in D^+,$$

$$|\omega'(z)| \leqslant A(|z|-1)^{\mu-1}, \ \ z \in D^-,$$

A is a constant.

Therefore,

$$\partial_z \Omega, \ \ \partial_{\bar{z}} \Omega \in L_p(\mathbb{C}), \ \ 1 < p < (1-\mu)^{-1}.$$

Denoting by $w_0(z)$ the fundamental homeomorphism of Beltrami equation

$$\partial_{\bar{z}} w - q(z) \partial_z w = 0,$$

$$q(z) = \partial_{\bar{z}} \alpha / \partial_z \alpha, \ \ z \in D^+, \ \ q = 0, \ \ z \in D^-, \tag{2.3}$$

we obtain (see [137])

$$\Omega(z) = \Phi(w_0(z)),$$

where $\Phi(z)$ is a holomorphic function on the whole finite plane. $\Omega(z)$ is a bounded function, that's why $\Phi(z) = \text{const}$, $\Omega(z) = \text{const}$ and the lemma is proved[1].

Lemma 19.2.2 *Let Γ be a simple closed smooth curve, $a(t)$ is nonsingular quadratic matrix of order $n, a(t) \in H^\mu(\Gamma)$, $\mu < 1$. If $a(t)$ is sufficiently close to the unit matrix I, i.e. if*

$$\|a_k\|_{H^\mu} \leqslant \varepsilon < \frac{1}{n(1+s_\mu)}, \ \ k = 1, 2, \ \ a_1 = \frac{1}{2}(a - I), \ \ a_2 = \frac{1}{2}({a'}^{-1} - I),$$

s_μ is a norm of the operator $\dfrac{1}{\pi i} \displaystyle\int_\gamma \varphi(t)(t-t_0)^{-1} dt$ in the space $H^\mu(\Gamma)$, then for $a(t)$ there exists the canonical matrix $\chi(z)$ close to the unit matrix:

$$\chi^+(t) = a(t) \chi^-(t), \ \ \chi(z) = I + \zeta_1(z), \ \ \chi^{-1}(z) = I + \zeta_2(z),$$

$$\zeta_1(\infty) = \zeta_2(\infty) = 0, \ \ |\zeta_k^+(t)|_\mu \leqslant C\varepsilon,$$

where the constant C depends only on n and μ and on the curve Γ.

Proof Consider singular integral equations in $H^\mu(\Gamma)$:

$$(I + a_k)\varphi_k - a_k S \varphi_k = I + 2a_k, \ \ k = 1, 2.$$

[1] If we replace the boundedness condition at infinity by the following condition

$$\omega(z) = z + O(z^{-1}),$$

then we get the piecewise-holomorphic function univalent in the domains D^+ and D^- (cf. 20.2).

It is easy to see, that these equations are solvable and also

$$\varphi_k = I + \varphi_k, \quad \|\varphi_k\|_{H^\mu} \leqslant \varepsilon + \frac{n\varepsilon(1+\varepsilon)(1+s_\mu)}{2 - n\varepsilon(1+s_\mu)} = \eta.$$

Introducing the piecewise-holomorphic matrices

$$\chi_k(z) = \begin{cases} \rho_k(z), & z \in D^+, \\ \rho_k(z) + I, & z \in D^-, \end{cases}$$

$$\rho_k = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi_k(t)dt}{t - z}, \quad k = 1, 2,$$

where D^+, D^- are finite and infinite domains, bounded by Γ.

We have

$$\chi_1^+(t) = a(t)\chi_1^-(t), \quad \chi_2^+(t) = [a'(t)]^{-1}\chi_2^-(t),$$

$$\det \chi_1^+\chi_2^+ = \det \chi_1^-\chi_2^-, \quad \det [\chi_1\chi_2] \equiv 1, \quad \chi_1^{-1} = \chi_2^1.$$

Hence $\chi_1(z)$is a canonical matrix for $a(t)$, assuming

$$\chi_1 - I = \zeta_1, \quad \chi_1^{-1} - I = \zeta_2,$$

we obtain

$$|\zeta_k^\pm(t)| \leqslant \frac{1}{2}(1 + s_\mu)$$

Corollary In particular case when γ is a unit circle on the bases of Hardy-Littlewood theorem under the conditions of lemma 19.2.1 for the canonical matrix $\chi(z) = \chi_{ik}(z)$, constructed above we'll have

$$\left|\frac{d\chi_{ik}(z)}{dz}\right| \leqslant \frac{M_1\varepsilon}{(1-|z|)^{1-\mu}}, \quad z \in D^+, \quad \iint\limits_{D^+} \left|\frac{d\chi_{ik}(z)}{dz}\right| dxdy \leqslant M_2\varepsilon^p,$$

$$1 < p < (1-\mu)^{-1}, \quad i, k = 1, \cdots, n,$$

where the constants M_k depend only on n and μ.

Lemma 19.2.3 *If the matrix $a(t)$ is sufficiently close to the unit matrix, then there exists a canonical matrix for the problem* (2.1).

Proof First show that one of the canonical matrices $\chi_\alpha(z)$ we may construct by the formulas

$$\chi_\alpha[\alpha(z)] = \chi(z)[If + I], \quad z \in D^+, \quad \chi_\alpha(z) = \chi(z)[If + I], \quad z \in D^-,$$

where $\chi(z)$ is a canonical matrix when $\alpha(t) = t$, $\chi(\infty) = I$ and f is a solution (unique) of two-dimensional singular integral equation

$$f(z) - q(z)\Pi f - ATf = A, \quad f \in L_p(\overline{D^+}); \tag{2.4}$$

$A = q\chi^{-1}\dfrac{\partial \chi}{\partial z}$, $\alpha(z)$ and $q(z)$ are defined by the formulas (2.2) and (2.3)

If $\|a - I\|_{H^\mu} = \varepsilon$ is a small quantity, then there exists the matrix $\chi(z)$ with the properties from lemma 19.2.1. Since $\mu > \dfrac{1}{2}$ we may take p from the interval $(2, (1-\mu)^{-1})$.

The operator ATf is a linear bounded operator transferring $L_p(D)$ into itself and also it's norm is not more then $M\varepsilon$, the constant M depends only on n and μ.

If we take ε sufficiently small then the equation (2.4) has the unique solution $f \in L_p(D)$.

The matrix $w(z) = Tf$ is Hölder-continuous on the whole plane, vanishes at infinity and satisfies the following equation

$$\partial_{\bar z} w - q(z)\partial_z w - A(z)w = A(z).$$

Assuming $w_1(z) = \chi(z)[w(z) + I]$, $z \in D^+$ we obtain that $w_1(z)$ satisfies the equation

$$\partial_{\bar z} w_1 - q(z)\partial_z w_1 = 0,$$

in D^+ and therefore

$$w_1(z) = \varphi_1[\alpha(z)], \quad z \in D^+,$$

where $\varphi_1(z)$ is a holomorphic matrix in D^+.

If we define the holomorphic matrix in D^- by the formula

$$\varphi_1(z) = \chi(z)[w(z) + I],$$

then we have

$$\varphi_1^+[\alpha(t)] = a(t)\varphi_1^-(t), \quad t \in \Gamma, \quad \varphi_1(\infty) = I.$$

We are able to construct the solution of the boundary problem

$$\varphi_2^+[\alpha(t)] = a^{-1}(t)\varphi_2^-(t), \quad t \in \Gamma, \quad \varphi_2(\infty) = I,$$

analogously as we have

$$\det\,[\varphi_1^+(\alpha(t))\varphi_2^+(\alpha(t))] = \det\,[\varphi_1^-(t)\varphi_2^-(t)], \quad t \in \Gamma$$
$$\det\,[\varphi_1(z)\varphi_2(z)] \equiv 1.$$

and $\varphi_1(z)$ is a canonical matrix for the problem (2.1).

Lemma 19.2.4 *There exists a canonical matrix of the problem* (2.1) *for the arbitrary matrix* $a(t)$ (*satisfying the above indicated conditions*); *it is possible to construct one of them by the formulas*

$$\chi_\alpha[\alpha(z)] = \chi_\alpha^0[\alpha(z)]R(w_0(z)), \quad z \in D^+, \tag{2.5}$$
$$\chi_\alpha(z) = r(z)\chi_\alpha^0(z)R(w_0(z)), \quad z \in D^-,$$

where $r(z)$ and $R(z)$ are respectively chosen matrices, $\chi^0(z)$ is a canonical matrix of the boundary condition

$$\varphi^+[\alpha(t)] = a_0(t)\varphi^-(t), \quad a_0 = ar,$$

$w_0(z)$ is the fundamental homeomorphism of the Beltrami equation

$$\partial_{\bar{z}} w - q(z)\partial_z w = 0.$$

Proof Let us choose the rational matrix $r(z)$ such that the matrix $a_0(t) = a(t)r(t)$ will be close to the unit matrix; denote by $\chi_0(z)$ a canonical matrix of the problem

$$\varphi^+[\alpha(t)] = a_0(t)\varphi^-(t), \quad t \in \Gamma.$$

Consider the piecewise-meromorphic matrix defined in the form

$$\begin{aligned} &\chi_\alpha[\alpha(z)] = \chi_\alpha^0[\alpha(z)]R(w_0(z)), \quad z \in D^+, \\ &\chi_\alpha(z) = r(z)\chi_\alpha^0(z)R(w_0(z)), \qquad z \in D^-, \end{aligned} \tag{2.6}$$

where $R(z)$ is a rational matrix.

The boundary values of this matrix are satisfying the homogeneous boundary condition; it is possible to choose the matrix R such that the matrix defined by (2.6) has to be a canonical matrix of the problem (2.1).

The following theorem holds from these propositions:

Theorem 19.2.1 *All solutions of the problem* (2.1) *are given by the formulas*

$$\begin{aligned} &\varphi[\alpha(z)] = \chi_\alpha[\alpha(z)][Tf + h(z) + P(w_0(z))], \quad z \in D^+, \\ &\varphi(z) = \chi_\alpha(z)[Tf + h(z) + P(w_0(z))], \qquad\qquad z \in D^-, \end{aligned} \tag{2.7}$$

where $P(z)$ is an arbitrary polynomial vector and the vector $f \in L_p(\bar{D^+})$, $(p > 2)$ is a solution (*unique*) *of the equation*

$$Kf =: f(z) - q(z)\Pi f = g(z);$$

$$h(z) = \frac{1}{2\pi i}\int_\gamma \frac{[\chi_\alpha^+(\alpha(t))]^{-1}b(t)}{t-z}dt, \quad g(z) = (g_1, \cdots, g_n) = q(z)h'(z) \in L_p(\bar{D^+}).$$

The solutions vanishing at infinity are given by the formulas (2.7) *in which*

$$P(z) = (P_{\varkappa_1 - 1}, \cdots, P_{\varkappa_n - 1})$$

$$\varkappa_1 \geqslant \cdots \geqslant \varkappa_n$$

are the partial indices of the problem (2.1), $P_j(z)$ *is an arbitrary polynomial of order* j ($P_j(z) = 0$) *if* $j < 0$)*; if* $0 \geqslant \varkappa_{s+1} \geqslant \cdots \geqslant \varkappa_n$*, then the vector* $b(t)$ *has to satisfy the following conditions:*

$$2i \iint\limits_D g_j(\zeta) L(\zeta^k) d\zeta d\eta = \int_\gamma t^k \{[\chi_\alpha^+(\alpha(t))]^{-1} b(t)\}_j dt,$$

$$j = s+1, \cdots, n; \quad k = 0, \cdots, |\varkappa_j| - 1,$$

where L *is operator adjoint to the* K^{-1}*;*

$$Lf = f(z) - \Pi(qf).$$

Consider the set of problems along with the problem (2.1):

$$\begin{aligned} &\varphi^+[\alpha_\lambda(t)] = a(t)\varphi^-(t) + b(t), && t \in \Gamma. \\ &\alpha_\lambda(t) = exp[iV_\lambda(\theta)], \ \ V_\lambda(\theta) = (1-\lambda)\theta + \lambda V(\theta) && 0 \leqslant \lambda < 1. \end{aligned} \tag{2.8}$$

$a(t), b(t)$ are satisfying the conditions of the problem (2.1), $V(\theta)$ is a continuous strongly increasing function on $[0, 2\pi]$, satisfying the conditions mentioned above.

Denote the partial indices of the problem (2.8) by

$$\varkappa_1(\lambda) \geqslant \cdots \geqslant \varkappa_\alpha(\lambda),$$

the sum of non-negative (non-positive) partial indices by $n^+(\lambda)(-n^-(\lambda))$ and also by

$$\delta_1 \geqslant \cdots \geqslant \delta_s \geqslant 0 > \delta_{s+1} \geqslant \cdots \geqslant \delta_n,$$

the partial indices of the problem (2.8) in case when $\lambda = 0$.

We obtain

$$n^+(\lambda) - n^-(\lambda) = \frac{1}{2\pi}[\arg\, \delta(t)]_\Gamma.$$

Introduce the following vector

$$W(z) = \begin{cases} \chi^{-1}(z)\varphi[\alpha_\lambda(z)] - h(z), & z \in D^+, \\ \chi^{-1}(z)\varphi(z) - h(z), & z \in D^-, \end{cases}$$

where $\chi(z)$ denotes a canonical matrix of the problem (2.8) when $\lambda = 0$,

$$h(z) = (h_1, \cdots, h_n) = \frac{1}{2\pi i}\int_\gamma \frac{[\chi^+(t)]^{-1} b(t)}{t - z} dt,$$

$$\alpha_\lambda(z) = \alpha_\lambda(e^{i\theta}).$$

The vector $W(z)$ is continuous on the whole plane is (holomorphic in D^- and may have a pole at infinity); $W(z)$ satisfies the equation

$$\partial_{(\overline{z})} w - q(z,\lambda)\partial_z w + A(z,\lambda)w = B(z,\lambda),$$

$$q(z,\lambda) = \lambda e^{2\pi\theta}\frac{1-V'(\theta)}{2-\lambda+\lambda V(\theta)}, \quad z \in D^+, \quad q(z,\lambda)=0, \quad z \in D^-,$$

$$A(z,\lambda) = -q(z,\lambda)\chi^{-1}(z)\frac{d\chi}{dz}, \quad B(z,\lambda) = q(z,\lambda)\left[h'(z) + \chi^{-1}(z)\frac{d\chi}{dz}h(z)\right]. \tag{2.9}$$

We have to find a solution of the problem (2.9) vanishing at infinity; according to this suppose the solution of the problem (2.9) in the form:

$$W(z) = P(z) + Tf,$$

$f = (f_1, \cdots, f_n) \in L_p((\overline{D})), p > 2$, $P(z) = (P_1, \cdots, P_n)$, $P_j(z)$ is an arbitrary polynomial of order n ($P_j(z) = 0$, if $j < 0$).

With respect to f we obtain the equation

$$K_\lambda f \equiv f(z) - q(z,\lambda)\Pi f + A(z,\lambda)Tf = B(z,\lambda) + q(z,\lambda)P(z) - A(z,\lambda)P(z), \tag{2.10}$$

and the following conditions

$$\iint_{D^+} \xi^k f_j(\xi)d\xi d\eta + \pi a_{jk} = 0, \quad j = s+1, \cdots, ; \quad k = 0, \cdots, |\delta_j| - 1, \tag{2.11}$$

where a_{jk} are the coefficients of the expansion of $h_j(z)$ in the neighborhood of the point $z = \infty$

$$h_j(z) = \sum_{k=0}^{\infty} a_{jk} z^{-k-1}.$$

In case when the partial indices δ_j are non-negative the conditions (2.11) are eliminated.

If for given λ the operator K_λ has the inverse operator K_λ^{-1} then the conditions (2.11) one may rewrite in the following form:

$$\iint_{D^+} \xi^k g_j(\xi,\lambda)d\xi d\eta + \pi a_{jk} = 0, \tag{2.12}$$

where $g_j(\zeta,\lambda)$ denotes j-th component of the vector $K_\lambda^{-1}[B + qP' - AP]$.

The equality (2.12) is a linear algebraic system with respect to the coefficients of the polynomials $P_j(z)$.

It is easy to see that there exists the domain D_λ of the plane λ containing the segment $[0,1]$ of the real axis in which $q(z,\lambda)$ is a holomorphic function with respect to λ and in which the inequality

$$|q(z,\lambda)| \leqslant q_0 < 1$$

is fulfilled. The operator K_λ is analytically depended on λ in D_λ. As Tf is a completely continuous operator, therefore K_λ has an inverse operator for an arbitrary $\lambda \in D_\lambda$ except may be the points of some isolated set D_λ^1 [50], [113]. For $\lambda \in D_\lambda \backslash D_\lambda^1$ the coefficients of the linear system (2.12) are the analytic functions of λ; consequently the corresponding homogeneous system has the same number of linear independent solutions for all $\lambda \in D_\lambda \backslash D_\lambda^1$, except possibly the points of some isolated set.

Hence, the following result takes place.

Theorem 19.2.2 *$n^+(\lambda)$ and $n^-(\lambda)$ have the same values for all $\lambda \in [0,1]$ except possibly the points of some finite set.*

Corollary The partial indices $\varkappa_i(\lambda)$ are admitting constant values for all $\lambda \in [0,1]$ and

$$\delta_1 \geqslant \varkappa_1(\lambda) \geqslant \cdots \geqslant \varkappa_n(\lambda) \geqslant \delta_n.$$

If $\delta_1 - \delta_n \leqslant 1$, then for all $\lambda \in [0,1]$ except possibly the points of some finite set $\varkappa_i(\lambda) = \delta_i$, $i = 1, \cdots, n$.

Remark *As the following example shows there exists the exceptional set.*

Suppose $a(t)$ has the form

$$a(t) = \begin{pmatrix} 1+2t^2+4t & 4t^2 \\ -2t & 1-2t \end{pmatrix},$$

$\alpha_1(t) = e^{i\nu(\theta)}$ is defined by the equality

$$\omega[\alpha_1(t)] = t + 1/4t,$$

where $\omega(z)$ conformally maps the circle $|z| < 1$ onto the interior of the ellipse

$$\frac{x^2}{25} + \frac{y^2}{9} = \frac{1}{16}.$$

It is easy to verify that

$$\delta_1 = \delta_2 = 0, \ \ \varkappa_1(1) = 1, \ \ \varkappa_2(1) = -1.$$

Remark 2 *Let the partial indices of the problem $\varphi^+(t) = a(t)\varphi^-(t), t \in \gamma$ (γ is a simple closed smooth curve) are equal to zero and the function $z = \omega(\zeta)$ is mapping* *conformally the domain, exterior to* γ *onto the exterior of simple closed smooth curve* γ_1. *Then generally speaking as the mentioned results show the partial indices are not equal to zero; but it is possible to find the analytic function* $\omega_0(\zeta)$ *arbitrarily close to* $\omega(\zeta)$ *such that the partial indices of the problem* $\varphi_0^+(\tau) = a[\omega_0(\tau)]\varphi_0^-(\tau)$ *have to be equal to zero* (*cf.* [22], *p.*71).

19.3 Boundary value problem of linear conjugation with displacement for generalized analytic vectors

First let us define the classes for the generalized analytic vectors.

Let $D^+,(D^-)$ be finite (infinite) domain which is bounded by a simple closed smooth Liapunov curve Γ.

Denote by $E_{s,p}(D,Q), s \geqslant 0, , p \geqslant 1, Q(z) = (q_{ik}) \in W_{p_0}^s(\mathbb{C}), p > 2$ (D is one of the domains $D^+, D^-, W_{p_0}^s$ is a Sobolev space) the class of $Q-$ holomorphic vectors $\Phi(z) = (\Phi_1, \cdots, \Phi_n)$ in the domain D satisfying the following conditions

$$\int_{\delta_{kr}} \left| \frac{\partial^s \Phi_k}{\partial z^s} \right|^p |dz| \leqslant C, \quad k = 1, \cdots, n, \tag{3.1}$$

where C is a constant, δ_{kr} is an image of the circle $|\xi| = r, r < 1$ while quasi-conformal mapping $\xi = \omega_k(S_k(z))$ of the unit circle $|\xi| < 1$ onto D, ω_k is an analytic function in the domain $S_k(D)$, S_k is a fundamental homeomorphism of the Beltrami equation

$$\partial_{\bar{z}} S - q_{kk}(z)\partial_z S = 0.$$

If D is infinite domain , then for the simplicity of notation we suppose that $W(\infty) = 0$ (remind that $Q-$ holomorphic vectors are the analytic functions in vicinity of the point $z = \infty$, because $Q = 0$ in this vicinity). By $E_{s,p}(D,Q,S)$ denote the class of the vectors Φ, belonging to the class $E_{s,\lambda}(D,Q)$ for some $\lambda > 1$, for which the angular boundary values are belonging to $L_p(\Gamma, \rho)$,

$$\rho(t) = \prod_{k=1}^{r} |t - t_k|^{\nu_k}, \quad t_k \in \Gamma, \quad -1 < \nu_k < p - 1, \quad p > 1. \tag{3.2}$$

Let $Q^+(z)$ and $Q^-(z)$ are two given matrices, satisfying the conditions mentioned above in §1, $Q^+ \in W_{p_0}^l(\mathbb{C})$, $Q^- \in W_{p_0}^m(\mathbb{C})$, $l, m \geqslant 0, p_0 > 2$. ($Q^+, Q^- \in W_{p_0}^1(\mathbb{C})$ when $l = m = 0$). Let $\rho^+(t), \rho^-(t)$ are the functions of the form (2.12). By $E_{l,m,p}^{\pm}(\Gamma, Q^{\pm}, \rho^{\pm})$ we denote the class of the vectors defined on cut along Γ plane, belonging to the class $(E_{l,p}(D^+, Q^+, \rho^+))$ $(E_{m,p}^-(D^-, Q^-, \rho^-))$, in the domain $D^+(D^-), E_{0,p}(D,Q) \equiv E_p(D,Q)$.

Now introduce the classes of the generalized analytic vectors, satisfying the equation of the form

$$Mw \equiv \partial_{\bar{z}} w - Q\partial_z w + Aw + B\bar{w} = 0; \tag{3.3}$$

in case of infinite domain we suppose that Q, A, B are equal to zero in certain vicinity of $z=\infty$.

By $E_{l,p}(D,M)$, $l \geqslant 0$, $p \geqslant 1$, denote a class of the solutions of the equation (3.3) satisfying the conditions

$$\int_{\delta_{kr}} \left|\frac{\partial^l w_k}{\partial z^l}\right|^p |dz| \leqslant C, \quad \left|\frac{\partial^s w_k}{\partial z^s}\right| \leqslant C, \quad k=1,\cdots,n, \quad s=0,\cdots,l-1,$$

the curve δ_{kr} is defined above, C is a constant; if D is an infinite domain, then $w(\infty)=0$. When we define this case we consider, that $\Gamma \in H_\alpha^m$, $Q(\alpha) \in W_{p_0}^l(\mathbb{C})$, $p_0 > 2$ (in case when $l=0$ $Q \in W_{p_0}^1(\mathbb{C})$, $A, B \in L_\infty(\bar{D})$). $A(z), B(z) \subset H_\alpha^{l-1}(\bar{D})$.

By $E_{l,p}(D,M,\rho)$, $l \geqslant 0$, $p > 1, \rho(t)$ is a function of the form (2.12), denote the class of the vectors $w(z)$, belonging to the class $E_{l,\lambda}(D,M)$ for some $\lambda > 1$, for which the angular boundary values of the vector $\dfrac{\partial^l w}{\partial z^l} \in L_p(\Gamma,\rho)$.

$E^{\pm}_{l,m,p}(\Gamma, M^{\pm}, \rho^{\pm})$ denotes the class of the vectors defined on the plane cut along the Γ and belonging to the class $E_{l,p}(D^+, M^+, \rho^+)[E_{m,p}(D^-, M^-, \rho^-)]$ in $D^+(D^-)$. The vectors of these classes have the following basic properties (see [103], [111]).

A vector of the class $E_p(D,Q)(p>1)$ has the angular boundary values $\Phi(t) \in L_p(\Gamma)$ almost everywhere on Γ and admits the representation by the generalized Cauchy integral

$$\Phi(z) = \frac{1}{2\pi i}\int_\Gamma V(t,z) d_Q t \Phi(t).$$

A vector of the form

$$\Phi(z) = \frac{1}{2\pi i}\int_\Gamma V(t,z) d_Q t f(t), \quad f(t) = L_p(\Gamma), \quad p > 1,$$

belongs to the class $E_p^{\pm}(D^{\pm}, Q^{\pm})$.

The vector $\Phi(z) \in E_{m,p}(D,Q), p > 1$ belongs to the class $H_\alpha^{m-1}(D)$ for some α, $\partial^m \Phi / \partial z^m$ has the angular boundary values of the class $L_p(\Gamma)$.

The vector $\Phi(z)$ of the class $E_{m,p}(D,M), p > 1$ admits the representation of the form

$$w(z) = \Phi(z) + R(z),$$
$$\Phi(z) \in E_{m,p}(D,Q), \quad R(z) \in H_\alpha(D).$$

Consider the following boundary problem.

Find a vector $\Phi(z) = (\Phi_1, \cdots, \Phi_n)$ of the class $E_p^{\pm}(\Gamma, Q^{\pm}, \rho^{\pm})$ satisfying the boundary condition

$$\Phi^+[\alpha(t)] = a(t)\Phi^-(t) + b(t)\overline{\Phi^-(t)} + f(t), \quad t \in \Gamma; \tag{3.4}$$

Γ is a simple closed Liapunov curve, $\alpha(t)$ is a function mapping Γ onto Γ in one-to-one manner keeping the orientation , $a(t), b(t)$ are given piecewise-continuous

$(n \times n)-$ matrices on Γ, $\;inf\ |\det\ a(t)| > 0, f(t)$ is a given vector of the class $L_p(\Gamma, \rho)$, $\rho = \rho^-$ is a function (19.1.1), $\rho^+(t) = \prod_{k=1}^r |t - \alpha(t_k)|^{\nu_k}$.

The following proposition holds (cf. 20.5).

Lemma 19.3.1 *The arbitrary vector $\Phi(z)$ of the class $E_p^{\pm}(\Gamma^{\pm}, Q^{\pm}, \rho^{\pm})$ is uniquely representable in the form*

$$\Phi(z) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_\Gamma V_+(\tau, z) d_{Q^+}\tau\mu[\beta(\tau)], & z \in D^+, \\ \dfrac{1}{2\pi i}\displaystyle\int_\Gamma V_-(\tau, z) d_{Q^-}\tau\mu(\tau), & z \in D^-, \end{cases} \tag{3.5}$$

where $\mu(t) \in L_p(\Gamma, \rho)$ is a solution of the Fredholm integral equation

$$\begin{aligned} N_\mu \equiv & \,\mu(t) + \frac{1}{2\pi i}\int_\Gamma \left[V_+(\alpha(\tau), \alpha(t)) d_{Q^+}\alpha(\tau) - V_-(\tau, t) d_{Q^+}\tau\mu(\tau)\right] \\ = & \,\Phi^+[\alpha(t)] - \Phi^-(t), \end{aligned} \tag{3.6}$$

$\beta(t)$ is inverse function to $\alpha(t)$.

Proof Let $\mu_0 \in L_p(\Gamma, \rho)$ be a solution of the equation $N\mu = 0$. Composing the vector $\Phi_0(z)$ by the formulas (3.5), and assuming $\mu = \mu_0$ we obtain

$$\Phi_0^+[\alpha(t)] = \Phi_0^-(t), \;\; t \in \Gamma,$$

from which it follows (cf. 19.2) that $\Phi_0(z) \equiv 0$.

Then we get

$$\mu_0(t) = F^+(t), \;\; \mu_0[\beta(t)] = F^-(t), \tag{3.7}$$

where the vector $F(z) \in E_\lambda^{\pm}(\Gamma, Q^{\pm})$ for some $\lambda > 1$.

From (3.7) it follows the inequality

$$F^+[\beta(t)] = F^-(t),$$

therefore $F(z) \equiv 0$, $\mu_0(z) \equiv 0$, the equation (3.6) is solvable for every right - hand side value and lemma is proved.

Now begin to solve the problem (3.4) . Substituting the representation (3.5) into the boundary condition (3.4) for the desired vector $\mu(t)$, we obtain the following singular integral equation

$$\begin{aligned} \mathcal{L}_\mu \equiv & \,[I + a(t_1)]\mu(t) + b(t)\overline{\mu(t)} + \frac{1}{\pi i}\int_\Gamma M_1(\tau, t)\mu(\tau) d\tau \\ & + \frac{1}{\pi i}\int_\Gamma M_2(\tau, t)\overline{\mu(\tau)} d\tau = 2f(t), \end{aligned} \tag{3.8}$$

$$\begin{aligned} M_1(\tau, t)\mu(\tau) d\tau = & \left[V_+(\alpha(\tau), \alpha(t)) d_{Q^+}\alpha(\tau) - a(t) V_-(\tau, t) d_{Q^-}\tau\right] \mu(\tau), \\ M_2(\tau, t)\overline{\mu(\tau)} d\tau = & \, b(t)\overline{V_-(\tau, t)} d_{Q^-}\overline{\tau\mu(\tau)}. \end{aligned}$$

The Noetherity of the equation (3.8) is determined by the same matrix G, introduced in the chapter 18,

$$G = S^{-1}D, \quad S = \begin{pmatrix} I & b \\ 0 & \bar{a} \end{pmatrix}, \quad D = \begin{pmatrix} a & 0 \\ \bar{b} & I \end{pmatrix}.$$

Suppose the Noetherity conditions of (3.8) are fulfilled. In order to solve the equation (3.8) in $L_p(\Gamma, \rho)$ it is necessary and sufficient, that

$$Re \int_\Gamma f(t)\psi_k(t)dt = 0, \quad k = 1, \cdots, \ell', \tag{3.9}$$

where $\psi_k(t)(k = 1, \cdots, \ell')$ is a complete system of linearly independent solutions of conjugate homogeneous equation $\mathcal{L}'\psi = 0$ of the class $L_q(\Gamma, \rho^{1-q})$.

And so as the representation (3.5) is unique we obtain the following result.

Theorem 19.3.1 *If the equation* (3.8) *is Noetherian in the space* $L_p(\Gamma, \rho)$, *then the boundary problem* (3.4) *is Noetherian in* $E_p^{\pm}(\Gamma, Q^{\pm}, \rho^{\pm})$; *the necessary and sufficient solvability conditions have the form* (3.4); *the index of the problem* (3.4) *of the class* $E_p^{\pm}(\Gamma, Q^{\pm}, \rho^{\pm})$ *is equal to the index of the equation* (3.8) *of the class* $L_p(\Gamma, \rho)$.

Consider now the problem (3.4) for generalized analytic vectors satisfying the equations of the form (3.3): we have to find the vector $w(z) \in E_p^{\pm}(\Gamma, M^{\pm}, \rho^{\pm})$, satisfying the boundary condition

$$w^+[\alpha(t)] = a(t)w^-(t) + b(t)\overline{w^-(t)} + f(t), \quad t \in \Gamma. \tag{3.10}$$

The solution of (3.10) will be found by the formula (1.9) in the following form

$$w^{\pm}(z) = \Phi^{\pm}(z) + \iint\limits_{D^{\pm}} \left[\Gamma_1^{\pm}(z,t)\Phi^{\pm}(t) + \Gamma_2^{\pm}(z,t)\Phi^{\pm}(t)\right] d\sigma_t + \sum_{k=1}^{N^{\pm}} c_k^{\pm} w_k^{\pm}(z), \tag{3.11}$$

where $\Phi^{\pm}(z) \in E_p^{\pm}(\Gamma, Q^{\pm}, \rho^{\pm})$, $c_k^{\pm}(k = 1, \cdots, N^{\pm})$ are desired real numbers, $w_k^{\pm}$ is a solution of the corresponding integral equations.

The vectors $\Phi^{\pm}(t)$ have to satisfy the conditions

$$Im \int_\Gamma \Phi^{\pm}(t) d_{Q^{\pm}} t \psi_k^{\pm}(t) = 0, \quad k = 1, \cdots, N^{\pm},$$

where Ψ_k^+, Ψ_k^- are the complete systems of the homogeneous conjugate equations.

With respect to the vector $\Phi^{\pm}(z)$ we obtain the following boundary problem

$$\begin{aligned} &\Phi^+[\alpha(t_0)] = a(t_0)\Phi^-(t_0) + b(t_0)\overline{\Phi^-(t_0)} + \mathcal{L}_+\Phi^+ + \mathcal{L}_-\Phi^- + f_0(t_0), \\ &f_0(t) = f(t) + \sum_{k=1}^{N^-} c_k^- \left[a_k(t)w_k^-(t) + b_k(t)\overline{w_k^-(t)}\right] - \sum_{k=1}^{N^+} c_k^+ w_k[\alpha(t)]; \end{aligned} \tag{3.12}$$

the operators $\mathcal{L}_+$ and $\mathcal{L}_-$ are defined by the formulas

$$\mathcal{L}_+\Phi^+ = -\iint\limits_{D^+} \left[\Gamma_1^+(\alpha(t_0,t))\Phi^+(t) + \Gamma_2(\alpha(t_0),t)\overline{\Phi^+(t)}\right] d\sigma_t,$$

$$\mathcal{L}_-\Phi^+ = \alpha(t_0)F(t_0) + b(t_0)\overline{F(t_0)},$$

$$F(t_0) = \iint\limits_{D^-} \left[\Gamma_1(t_0,t)\Phi^-(t) + \Gamma_2^-(t_0,t)\overline{\Phi^-(t)}\right] d\sigma_t.$$

Substituting in these formulas the following representations

$$\Phi^+(t) = \frac{1}{2\pi i}\int_\Gamma V_+(\tau,t)d_{Q^+}\tau\Phi^+(\tau), \quad t \in D^+,$$

$$\Phi^-(t) = -\frac{1}{2\pi i}\int_\Gamma V_-(\tau,t)d_{Q^-}\tau\Phi^-(\tau), \quad t \in D^-,$$

we obtain that the operators $\mathcal{L}_+$ and $\mathcal{L}_-$ are the completely continuous operators in the spaces $L_p(\Gamma,\rho^+)$, $L_p(\Gamma,\rho^-)$ with respect to the angular boundary values $\Phi^+(\tau)$, $\Phi^-(\tau)$.

Searching the solution of the problem (3.12) again in the form (3.5) we get the singular integral equation with respect to the vector $\mu(t)$

$$\Omega_\mu \equiv (\Omega_0 + \Omega_1)\mu + \Omega_2\bar{\mu} = 2\overline{f(t)} + \sum_{k=1}^{N} d_k\eta_k(t), \tag{3.13}$$

where Ω_0 is a completely continuous operator, Ω_1 and Ω_2 are the singular integral operators

$$\Omega_k\mu \equiv a_k(t)\mu(t) + \frac{b_k(t)}{\pi i}\int_\Gamma \frac{\mu(\tau)d\tau}{\tau - t},$$
$$a_1 = I + a, \quad b_1 = I - a, \quad a_2 = b_2 = b,$$

$\eta_k(t)$ are continuous linearly independent vectors, represented by $w_k^\pm(t)$, $d_k (k = 1,\cdots,N, N \leqslant N^+ + N^-)$ are desired real constants.

Besides the equation (3.13) the vector μ has to satisfy the conditions

$$\operatorname{Im}\int_\Gamma \mu(t)\omega_k(t)dt = 0, \quad k = 1,\cdots,P, \tag{3.14}$$

where $\omega_k(t) (k = 1,\cdots,P)$ are the linearly independent vectors, represented by the vectors $\Psi_k^\pm(t)$.

Using the theorem indicated in [119], we imply that necessary and sufficient solvability conditions of the problem (3.10) in the class $E_p^\pm(\Gamma, M^\pm, \rho^\pm)$ have the form

$$\operatorname{Re}\int_\Gamma f(t)\gamma_k(t)dt = 0, \quad k = 1,\cdots,R, \tag{3.15}$$

where the linearly independent vectors $\gamma_k(t)(k = 1, \cdots, R)$ belong to the class $L_p(\Gamma, \rho^{1-q})$, are representable by the vectors Ψ_k^+, Ψ_k^- and by the vectors, composing the basis of subspace of the solutions of the adjoint homogeneous equation $\Omega' v = 0$; the index of the problem (3.10) is equal to

$$\varkappa + N - R, \tag{3.16}$$

where $\varkappa$ is the index of the operator Ω of the class $L_p(\Gamma, \rho)$. Now show that actually in the formula (3.16) $N = R$.

Let $X^{\pm}$ are the sets of defined vectors $w^{\pm}(z)$ in the domains $D^{\pm}$ representable in the form

$$\begin{gathered} w^{\pm}(z) = \Phi^{\pm}(z) + h^{\pm}(z), \\ \Phi^{\pm}(z) \in E_p^{\pm}(\Gamma, Q^{\pm}, \rho^{\pm}), \quad h^{\pm}(z) \in H^{\alpha}(D^{\pm}). \end{gathered} \tag{3.17}$$

A pair of sets $X^{\pm}$, because of the properties indicated at the beginning of the chapter, coincides with the class $E_p^{\pm}(\Gamma, M^{\pm}, \rho^{\pm})$.

Introduce the norms

$$|w^{\pm}|_{X^{\pm}} = inf \ \left\{ |\Phi^{\pm}|_{L_p(\Gamma, \rho^{\pm})}, |h^{\pm}|_{H^{\alpha}(D^{\pm})} \right\}, \tag{3.18}$$

where the infimum runs over all possible representations (3.17). Then the sets $X^{\pm}$ are Banach spaces. Let $X = (X^+, X^-)$ be a new Banach space with the norm $|w|_{\chi} = max[|w^+|_{X^+}, |w^-|_{X^-}]$, Consequently we have introduced the norm in $E_p^{\pm}(\Gamma, M^{\pm}, \rho^{\pm})$ which evidently doesn't depend on $A^{\pm}, B^{\pm}$.

Consider the set of the operators

$$M_{\lambda}^{\pm} w^{\pm} = \partial_{\bar{z}} w^{\pm} - Q^{\pm} \partial_z w^{\pm} + \lambda \left[A^{\pm} w^{\pm} + B^{\pm} \overline{w^{\pm}} \right],$$

where $\lambda \in [0, 1]$; we have to come to the conclusion, that in order to calculate the index of the problem (3.10) we may take the differential operators of the form $\partial_{\bar{z}} w^{\pm} - Q^{\pm} \partial_z w^{\pm}$ and for such operators the numbers N^+, N^- are equal to zero and hence $N = R$ in the formula (3.16).

Therefore we obtain the following result

Theorem 19.3.2 *The necessary and sufficient solvability conditions of the problem* (3.10) *in the class* $E_p^{\pm}(\Gamma, M^{\pm}, \rho^{\pm})$ *are the conditions* (3.15); *the index of the problem* (3.10) *is equal to the index* $\varkappa$ *of the operator* Ω.

Note that if the matrices $a(t)$ and $b(t)$ are continuous, then the index of any class is given by the formula

$$\varkappa = \frac{1}{\pi} \left[\arg \det \, a(t) \right]_{\Gamma}.$$

19.4 The problem of linear conjugation with displacement for an elliptic system of differential equations

Consider the following equation

$$\partial_{\bar{z}}w - Q_1(z)\partial_z w - Q_2(z)\partial_z \overline{w} + A(z)w + B(z)\overline{w} = 0, \quad z \in D, \tag{4.1}$$

D is finite or infinite domain bounded by the Liapunov curve Γ. $w(z) = (w_1, \cdots, w_n)$ is a desired vector, $A(z), B(z)$ are given quadratic matrices of order n, belonging to the class $L_r(\bar{D})$, $r > 2$, $Q_k = (q_{ij}^k)$ are given lower triangular matrices where q_{ij}^k are bounded measurable functions in D; moreover the elliptic conditions

$$|q_{kk}^1| + |q_{kk}^2| \leqslant q_0 < 1, \quad k = 1, \cdots, n$$

are fulfilled. We assume that the matrices A, B, Q_1, Q_2 are equal to zero outside of some circle with the sufficiently large radius in case of infinite domain.

Consider the boundary problem: let $\Gamma_k (k = 1, 2)$ be simple Liapunov curves, bounding the domains $D^{\pm}, \alpha(t)$ is a function, mapping Γ_1 onto Γ_2 in one-to-one manner keeping the orientation, $0 \neq \alpha'(t) \in H(\Gamma)$; find the vectors $w^+ \in W_p^2[\bar{D}_2^+]$, $w^- \in W_p^1(\bar{D}_2^-)$, ($p > 2, W_p^l$ are the Sobolev spaces), $w(\infty) = 0$, satisfying the equations

$$\partial_{\bar{z}}w^{\pm} - Q_1(z)\partial_z w^{\pm} - Q_2(z)\overline{\partial_z w^{\pm}} + A^{\pm}(z)w^{\pm} + B^{\pm}(z)\overline{w^{\pm}} = 0 \tag{4.2}$$

in the domains D_2^+ and D_1^- respectively and the boundary condition

$$w^+[\alpha(t)] = a(t)w^-(t) + f(t), \quad t \in \Gamma_1, \tag{4.3}$$

where $f(t)$ is a given vector, $a(t)$ is given nonsingular lower triangular matrix, $a(t), f(t) \in H^{\mu}(\Gamma_1)$, $\mu > 1/2$.

Since the equation (4.2) is not changing it's form while conformal mapping, using the function $\omega(z)$, constructed in chapter 21, it is possible to reduce the problem (4.3) to the case, when $\alpha(t) = t$, $\Gamma_1 = \Gamma_2 = \Gamma$, $D_1^{\pm} = D_2^{\pm} = D^{\pm}$ and so we consider the boundary condition

$$w^+(t) = a(t)w^-(t) + f(t), \quad t \in \Gamma. \tag{4.4}$$

Let $\chi(z)$ be a canonical matrix for the matrix $a(t)$; we assume that $\chi(z)$ has a lower triangular form

$$\chi^{\pm}(t) \in H(\Gamma), \quad \chi'(z) \in L_{\delta}(\bar{D}^+), \quad \chi'(z) - P(z) \in L_{\delta}(\bar{D}^-), \quad \delta > 2,$$

where $P(z)$ is some polynomial vector.

Introducing the notations

$$w^{\pm}(z) = \chi^{\pm}(z)w_0^{\pm}(z),$$

for the vectors $w_0^{\pm}(z)$ we obtain boundary condition

$$w_0^+(t) = w_0^-(t) + h(t), \quad h(t) = [\chi^+(t)]^{-1} f(t), \quad t \in \Gamma. \tag{4.5}$$

The equations for the vectors $w_0^{\pm}$ have the form (4.2), where the matrices $Q_1^{\pm}, Q_2^{\pm}, A^{\pm}, B^{\pm}$ are changed by the matrices

$$R_1^{\pm} = [\chi^{\pm}]^{-1} Q_1^{\pm} \chi^{\pm}, \quad R_2^{\pm} = [\chi^{\pm}]^{-1} Q_2^{\pm} \chi^{\pm},$$
$$A_0^{\pm} = [\chi^{\pm}]^{-1} A^{\pm} \chi^{\pm} - [\chi^{\pm}]^{-1} Q_1 \chi'^{\pm}, \quad B_0^{\pm} = -[\chi^{\pm}]^{-1} Q_2^{\pm} \chi'^{\pm} + [\chi^{\pm}]^{-1} B \overline{\chi^{\pm}}.$$

Note that the matrices R_1 and R_2 are satisfying the elliptic conditions. Represent the vectors $w_0^{\pm}$ in the form

$$w_0^{\pm}(z) = \Phi^{\pm}(z) + T_{D^{\pm}} \omega^{\pm}, \quad \omega^{\pm}(z) \in L_{\gamma}(\bar{D}^{\pm}), \quad \gamma > 2; \tag{4.6}$$

$\Phi^{\pm}(z)$ is a piecewise-holomorphic vector, having finite order at infinity, principal part of $\Phi^-(z)$ is the polynomial vector $P(z) = (P_1, \cdots, P_n)$; $P_k(z)$ is a polynomial of order $\delta_k, \delta_1 \geqslant \cdots \geqslant \delta_n$ are the partial indices of the matrix $a(t)$ ($P_k(z) = 0$ when $\delta_k < 0$).

Substituting (4.7) in the boundary condition (4.6) we obtain

$$\begin{gathered} \Phi^+(t) - \Phi^-(t) = h_1(t), \quad t \in \Gamma, \\ h_1 = h + T\omega^- - T\omega^+. \end{gathered} \tag{4.7}$$

From the boundary condition (4.7) we have

$$\Phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{h(t)dt}{t - z} + \Phi_1(z) + P(z), \tag{4.8}$$

where $\Phi(z)$ is a piecewise-holomorphic vector, defined by the formula

$$\Phi_1(z) = -\frac{1}{\pi} \iint\limits_{D^-} \frac{\omega^-(\xi) d\sigma_{\xi}}{\xi - z}, \; z \in D^+, \quad \Phi_1(z) = -\frac{1}{\pi} \iint\limits_{D^+} \frac{\omega^-(\xi) d\sigma_{\xi}}{\xi - z}, \; z \in D^-.$$

In order to define $w_0^{\pm}(z)$, substituting the formulas (4.7), (4.8) in the differential equation we get the two dimensional singular integral equation

$$E\Omega \equiv \Omega(z) - R_1(z)\Pi\Omega - R_2(z)\overline{\Pi\Omega} + V\Omega = F(z), \tag{4.9}$$

where Ω is desired vector

$$\Omega(z) = \begin{cases} \omega^+(z), & z \in D^+, \\ \omega^-(z), & z \in D^-, \end{cases}$$

the matrices R_1, R_2 and the vector F are defined by the formulas

$$R_k(z) = \begin{cases} R_k^+(z), & z \in D^+, \\ R_k^-(z), & z \in D^-, \end{cases}$$

$$F^{\pm}(z) = -R_1^{\pm}(z)H'(z) - R_2^{\pm}(z)\overline{H'(z)} + A^{\pm}(z)H(z) + B^{\pm}(z)\overline{H(z)},$$
$$H(z) = P(z) + \frac{1}{2\pi i}\int_{\Gamma}\frac{h(t)dt}{t-z};$$

and the operator V is determined by the formulas

$$V\Omega = \begin{cases} A^{+}(z)T\Omega + B^{+}(z)\overline{T\Omega}, & z \in D^{+}, \\ A^{-}(z)T\Omega + B^{-}(z)\overline{T\Omega}, & z \in D^{-}, \end{cases}$$

V is a completely continuous operator in the space $L_{\delta}(\mathbb{C})$, $\delta > 2$.

The equation (4.9) is the Fredholm equation in $L_{2+\varepsilon}(\mathbb{C})$ for sufficiently small $\varepsilon > 0$ (see [20], [137]) and for it's solvability it is necessary and sufficient the validity of the conditions

$$\mathrm{Re}\iint_{\mathbb{C}} F(z)\Psi_k(z)d\sigma_z = 0, \quad k = 1, \cdots, \ell',$$

where $\{\Psi_k(z)\}$ is a complete system of linearly independent solutions in $L_q(\mathbb{C})$ $(q = (2+\varepsilon)/(1+\varepsilon))$ of the adjoint equation $E'\Psi = 0$.

Considering the set of equations

$$\partial_{\bar{z}}w_0^{\pm} - \lambda[R_1^{\pm}\partial_z w_0 + R_2^{\pm}\partial_z\overline{w}_0 - A_0^{\pm}w_0^{\pm} - B_0^{\pm}\overline{w}_0^{\pm}] = 0,$$

where the real parameter $\lambda \in [0,1]$, we are convinced, that the index of the formulated problem (4.3), (4.6) is equal to the index of the same problem when the equation (4.3) is replaced by the equation

$$\partial_{\bar{z}}w = 0.$$

Consequently the index is defined by the formula

$$\varkappa = \frac{1}{\pi}[\arg\det a(t)]_{\Gamma}.$$

19.5 Differential boundary value problems for generalized analytic vectors

We begin with some auxiliary propositions:

Lemma 19.5.1 *Let $\Phi(z)$ be a vector of the class $E^{\pm}_{\ell,m,p}(\Gamma, Q^{\pm}, \rho^{\pm})$, $p > 1$, $m = 0$ or $1, \Phi(0) = 0$, $\Gamma \in H^1_{\alpha}$ is a simple closed curve ($O \in D^{+} = \mathrm{int}\Gamma$), $\rho^{-}(t) = \rho(t) = \prod_{k=1}^{p}|t-t_k|^{\nu_k}, \rho^{+}(t) = \prod_{k=1}^{p}|t-\alpha(t_k)|^{\nu_k}, t_k \in \Gamma, -1 < \nu_k < p-1$, $\alpha(t)$ is mapping Γ onto itself in one-to-one manner keeping the orientation, $0 \neq \alpha'(t) \in C(\Gamma)$. Then we may represent the vector $\Phi(z)$ by the formula*

$$\Phi(z) = \begin{cases} \dfrac{\zeta_{+}(z)}{2\pi i}\displaystyle\int_{\Gamma} S_{+}(z,\tau)d_{Q^{+}}\tau\mu[\beta(\tau)], & z \in D^{+}, \\[2ex] \dfrac{(-1)^{m-1}}{2\pi i}\displaystyle\int_{\Gamma} S_{-}(z,\tau,m)d_{Q^{-}}\tau\mu(\tau), & z \in D^{-}, \end{cases} \tag{5.1}$$

where $\mu(t) \in L_p(\Gamma, \rho)$ *is a solution of the Fredholm integral equation*

$$N_\mu \equiv \mu(t) + \frac{1}{2\pi i}\int_\Gamma [V_+(\alpha(\tau), \alpha(t))d_{Q^+}\alpha(\tau) - V_-(\tau, t)d_{Q^-}\tau]\mu(t)$$

$$= [\zeta_+(\xi)\Phi'_+(\xi) + \zeta'_+(\xi)\Phi'_+(\xi)]_{\zeta=\alpha(t)} - f_m(t), \tag{5.2}$$

$$f_0(t) = \Phi_-(t), \quad f_1(t) = \zeta_-(t)\Phi'_-(t), \quad \Phi'(t) = \partial\Phi/\partial t,$$

$$S_+(z, t) = -\zeta_+^{-1}(z)\ln[I - \zeta_+(z)\zeta_+^{-1}(t)], \tag{5.3}$$

$$S_-(z, t, 1) = -\zeta_-^{-1}(t)\ln[I - \zeta_-(\tau)\zeta_-^{-1}(z)], \quad S_-(z, t, 0) = V_-(t, z).$$

$\beta(t)$ *is an inverse function to* $\alpha(t), V_\pm(t, z)$ *and* $\zeta_\pm(t)$ *is the generalized Cauchy kernel and the principal solution with respect to the coefficient* $Q^\pm(z)$*; in the formulas* (5.3) $\ln[I - \zeta_+(z)\zeta_+^{-1}(t)]\{\ln[I - \zeta_-(t)\zeta_-^{-1}(t)]\}$ *means the branch which is univalent in the plane cut along the curve* $\ell_t^+\{\ell_t^-\}$ *connecting the point* $t \in \Gamma$ *with the point* $z = \infty$ *and lying in the domain* D^- *(connecting the point* $t \in \Gamma$ *with the point* $z = 0$ *and lying in the domain* D^+ *and is equal to zero at infinity).*

Proof Let $\mu_0 \in L_p(\Gamma, \rho)$ be a solution of the homogeneous equation $N_\mu = 0$.

Assume

$$\varphi(z) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_\Gamma V_+(\tau, z)d_{Q^+}\tau\mu_0(\beta(\tau)), & z \in D^+, \\ \dfrac{1}{2\pi i}\displaystyle\int_\Gamma V_+(\tau, z)d_{Q^-}\tau\mu_0(\tau), & z \in D^-. \end{cases} \tag{5.4}$$

It is easy to see that $\varphi(z)$ is a vector of the class $E_p^\pm(\Gamma, Q^\pm, \rho^\pm)$. From (5.4) it follows that

$$\varphi^+[\alpha(t)] - \varphi^-(t) = N\mu_0 = 0. \tag{5.5}$$

The first components $\varphi^+(z)$, $\varphi^-(z)$ are satisfying the Beltrami equation

$$\partial_{\overline{z}}\varphi^\pm - q_{11}\partial_z\varphi^\pm = 0$$

and therefore

$$\varphi_1^\pm(z) = \Phi_1^\pm(s_1^\pm(z)),$$

where $s_1^\pm(z)$ are the fundamental homeomorphisms, $\Phi_1^\pm(s)$ are the holomorphic functions in the corresponding domains. From (5.5) we obtain

$$\Phi_1^+[s_1^+(\alpha(t))] = \Phi_1^-[s_1^-(t)], \quad t \in \Gamma,$$

from which $\Phi_1 = 0$, $\varphi_1 = 0$. Continuing this argument we get, that

$$\varphi_2 = 0, \cdots, \varphi_n = 0, \quad \varphi = 0.$$

It follows from the equalities (5.4), that

$$\mu_0[\beta(t)] = F_-(t), \quad \mu_0(t) = F_+(t),$$

where $F_\pm$ are the angular boundary values of the vector $F(z) \in E_p^\pm(\Gamma, Q^\pm, \rho^\pm)$. By virtue of the last formulas

$$F_+[\beta(t)] = F_-(t), \quad t \in \Gamma.$$

Therefore $F(z) = 0$ and the homogeneous equation $N\mu = 0$ has the only trivial solution in $L_p(\Gamma, \rho)$. That's why for the given vector $\Phi(z) \in E^\pm_{\ell,m,p}(\Gamma, Q^\pm, \rho^\pm)$ the equation (5.2) has the unique solution. Using the solution $\mu(t)$ construct the vector $\Psi(z)$ by the formula (5.1); let us show, that $\Psi(z) = \Phi(z)$.

We have

$$\begin{aligned} [\partial_z \zeta_+(z)]^{-1}\, \partial_z \Psi(z) &= \frac{1}{2\pi i}\int_\Gamma V_+(\tau, z) d_{Q^+}\tau \mu[\beta(\tau)], \quad z \in D^+, \\ \zeta_-^m(z)\frac{\partial^m \Psi}{\partial z^m} &= \frac{1}{2\pi i}\int_\Gamma V_-(\tau, z) d_{Q^-}\tau \mu(\tau), \qquad\qquad z \in D^-. \end{aligned} \tag{5.6}$$

From (5.6) we obtain

$$[\partial_\xi \zeta_+(\xi)]^{-1}\partial_\xi \Psi_+(\xi)]_{\xi=\alpha(t)} - \zeta_-^m(t)\frac{\partial^m \Psi_-(t)}{\partial t^m} = N\mu.$$

Hence

$$\left\{[\partial_\xi \zeta_+(\xi)]^{-1}\partial_\xi \omega_+(\xi)\right\}_{\xi=\alpha(t)} = \zeta_-^m(t)\frac{\partial^m \omega_-(t)}{\partial t^m}, \quad \omega = \Phi - \Psi \tag{5.7}$$

If follows from (5.7), that $\omega = 0$, $\Phi(z) = \Psi(z)$.

Corollary Let $\varphi(z)$ is a vector of the class $E_{e,m,p}(\Gamma, Q^\pm, \rho^\pm)$, $m = 0$ or 1. Then $\Phi(z)$ is representable by the formula

$$\Phi(z) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_\Gamma S_+(\tau, z) d_{Q^+}\tau \mu[\beta(\tau)], & z \in D^+, \\ \dfrac{1}{2\pi i}\displaystyle\int_\Gamma S_+(\tau, z) m) d_{Q^-}\tau \mu(\tau), & z \in D^-, \end{cases} \tag{5.8}$$

where $\mu(t) \in L_p(\Gamma, \rho)$ is a solution of the equation

$$N\mu = [\zeta_+(\xi)\Phi'_+(\xi) + \zeta'_+(\xi)\Phi_+(\xi)]_{\xi=\alpha(t)} - f_m(t).$$

Compose the vector

$$\Phi_0(z) = \begin{cases} \zeta_+(z)\Phi(z), & z \in D^+, \\ \Phi(z), & z \in D^-. \end{cases}$$

By the lemma 19.5.1 it is possible to represent $\Phi_0(z)$ in the form (5.1), consequently $\Phi(z)$ admits the representation (5.8).

Consider the following boundary problem:

Find a vector $\Phi(z)$ of the class $E^{\pm}_{e,m,p}(\Gamma, Q^{\pm}, \rho^{\pm})$ ($m = 0$ or $1, \Gamma, \rho^{\pm}, \alpha(t)$ are defined in lemma 19.5.1 satisfying the boundary condition:

$$H\Phi \equiv \sum_{k=0}^{m}\{a_k(t)\Phi_+^{(k)}[\alpha(t)] + b_k(t)\overline{\Phi_+^{(k)}[\alpha(t)]} + N_k^+\Phi_+^{(k)}\} \\ + \sum_{k=0}^{m}\{c_k(t)\Phi_-^{(k)}(t) + d_k(t)\overline{\Phi_-^{(k)}(t)} + N_k^-\Phi_-^{(k)}\} = f(t), \tag{5.9}$$

where $a_k(t), b_k(t), c_k(t), d_k(t)$ are given piecewise-continuous matrices, $N_k^{\pm}$ are the operators of the form

$$N_k^{\pm}\Phi = \int_\Gamma [H_{1k}^{\pm}(t,\tau)\Phi(\tau) + H_{2k}^{\pm}(t,\tau)\overline{\Phi(\tau)}]d\tau,$$

where the kernels $H_{ik}^{\pm}$ have the form

$$H_{ik}^{\pm}(t,\tau) = h_{ik}^{\pm}(t,\tau)|t-\tau|^{\eta}, \quad -1, < \eta \leqslant 0;$$

$h_{ik}^{\pm}(t,\tau)$ is a measurable bounded matrix, $f(t)$ is a given vector of the class $L_p(\Gamma,\rho)$, $\Phi_+^{(k)}(t), \Phi_-^{(k)}(t)$ are the singular boundary values of the vectors $\partial^k\varphi/\partial z^k$ from the domains D^+ and D^-.

Searching the solution of the problem (5.9) in the form (5.8) in order to determine the vector $\mu(t)$ we get the following system of singular integral equations

$$K\mu \equiv K_1\mu + \overline{K_2\mu} = 2g(t), \tag{5.10}$$

$$K_s\mu \equiv A_s(t)\mu(t) + \frac{B_s(t)}{\pi}\int_\Gamma \frac{\mu(\tau)d\tau}{\tau - t} + \int_\Gamma k_s(t,\tau)\mu(\tau)d\tau,$$

$$A_1(t) = a_1(t)q(t) - c_m(t)q_m(t), \quad A_2(t) = \overline{b_1(t)}q(t) - \overline{d_m(t)}q_m(t),$$

$$B_1(t) = a_1(t)q(t) + c_m(t)q_m(t), \quad B_2(t) = \overline{b_1(t)}q(t) + \overline{d_m(t)}q_m(t),$$

$$q(t) = \zeta_+^{-1}[\alpha(t)], \quad q_m(t) = \zeta_-^{-m}(t)$$

$k_s(t,\tau)$ are the certain matrices with weak singularities.

Note, that the problem (5.9) in the class $E^{\pm}_{l,m,p}(\Gamma, Q^{\pm}, \rho^{\pm})$ and the equation (5.10) in the class $L_p(\Gamma,\rho)$ are equivalent. They are either simultaneously solvable or not and their indices coincide (it is clear, that the equation (5.10) is Noetherian in $L_p(\Gamma,\rho)$; then the problem (5.9) is Noetherian in the class $E^{\pm}_{l,m,p}(\Gamma, Q^{\pm}, \rho^{\pm})$).

In order to clarify the Noetherity problem for the equation (5.10) we have to compose the following block matrices

$$A = \begin{pmatrix} A_1, & \overline{A}_2 \\ A_2, & \overline{A}_1 \end{pmatrix}, \quad B = \begin{pmatrix} B_1, & -\overline{B}_2 \\ B_2, & -\overline{B}_1 \end{pmatrix},$$

and have to require that

$$\inf |\det S(t)| > 0, \quad \inf |\det D(t)| > 0, \;\; t \in \Gamma, \tag{5.11}$$

$$S = A + B, \;\; D = A - B.$$

It is easy to see, that

$$\det S(t) = \det \overline{D(t)},$$

and therefore it is sufficient to require the fulfillment of one of the (5.11) inequalities.

It is easy to establish the following formula (see [46])

$$\det D(t) = (-1)^n \det \Omega(t),$$

where

$$\Omega = \begin{pmatrix} c_m & b_1 \\ \bar{d}_m & a_1 \end{pmatrix}.$$

As $\det \zeta(t) \neq 0, \quad \det [\partial_t \zeta(t)] \neq 0, \; t \in \Gamma_1$ then the necessary and sufficient conditions for the inequalities (5.11) to be valid is the following condition

$$\inf_{t\in\Gamma} |\det \Omega(t)| > 0. \tag{5.12}$$

Suppose that (5.12) holds, denote $S^{-1}D \equiv G$ and consider the equation

$$\det [G(t_k - 0) - \lambda G(t_k + 0)] = 0, \;\; k = 1, \cdots, \tau. \tag{5.13}$$

Let λ_{kj} $(k = 1, \cdots, \tau, \;\; j = 1, \cdots, 2n)$ are the roots of the equation (5.13) and

$$\mu_{kj} = \frac{1}{2\pi} \arg \lambda_{kj}, \;\; 0 \leqslant \arg \lambda_{kj} < 2\pi;$$

If the inequalities

$$\frac{1 + \nu_k}{p} \neq \mu_{kj} \;\; (k = 1, \cdots, r, \;\; j = 1, \cdots, 2n) \tag{5.14}$$

are fulfilled, then the equation (5.10) is Noetherian in $L_p(\Gamma, \rho)$.

Consequently we obtain the following result.

Theorem 19.5.1 *The boundary value problem* (5.9) *under the consideration of inequalities* (5.12) *and* (5.14) *is solvable in the class* $E^{\pm}_{l,m,p}(\Gamma, Q^{\pm}, \rho^{\pm})$ *if and only if*

$$\mathrm{Re} \int_{\Gamma} f(t) \Psi_k(t) dt = 0, \;\; k = 1, \cdots, l',$$

where $\Psi_k(t)$ $(k = 1, \cdots, l')$ *is a complete system of linear independent solutions of the adjoint homogeneous equation* $K'\Psi = 0$ *of the class* $L_q(\Gamma, \rho^{1-q})$, *the index of* (5.9) *of the class* $E^{\pm}_{l,m,p}(\Gamma, Q^{\pm}, \rho^{\pm})$ *is calculated by the formula*

$$\varkappa = \frac{1}{2\pi}\left[\arg\frac{\det G(t)}{\prod\limits_{k=1}^{r}(t - z_0)^{\sigma_k}}\right]_{\Gamma}, \tag{5.15}$$

the numbers σ_k *are defined by the formulas* (3.22), *(chapter 19).*

Remark *If* a_k, b_k, c_k, d_k *are continuous matrices then the conditions* (5.14) *are automatically fulfilled and when the condition* $\det\,\Omega(t) \neq 0$ $(t \in \Gamma)$ *holds, then the problem* (5.9) *is Noetherian in certain class* $E^{\pm}_{l,m,p}[\Gamma, Q^{\pm}, \rho^{\pm}]$ *and the index formula takes the following form*:

$$\varkappa = \frac{1}{2\pi}[\arg\det G(t)]_{\Gamma}.$$

If all coefficients a_k, b_k, c_k, d_k of the matrix $h_{ik}(t, \tau)$ and the right hand side function $f(t)$ are Hölder-continuous then the solutions of the problem (5.9) of an arbitrary class will belong to the class $H^1(H^m)$ in the closed domain $\overline{D^+}(\overline{D^-})$.

In this section we consider the boundary problem of the form (5.9) for generalized analytic vectors.

We have to find the generalized analytic vector $w(z)$ belonging to the class $E^{\pm}_{l,m,p}(\Gamma, \mathcal{L}^{\pm}, \rho^{\pm})$ and satisfying the condition

$$Hw = f(t), \quad t \in \Gamma, \tag{5.16}$$

where H is the operator of the form (5.9).

Let us represent the vector $w(t)$ by the Q-holomorphic vectors

$$w^{\pm}(z) = \Phi^{\pm}(z) + \iint\limits_{D^{\pm}} \Gamma_1(z,\tau)\Phi(\tau) + \Gamma_2(z,\tau)\overline{\Phi(\tau)}]d\sigma_\tau + \sum_{k=1}^{N^{\pm}} c_k^{\pm} w_k^{\pm}(z), \tag{5.17}$$

$c_k^{\pm}$ $(k = 1, \cdots, N^{\pm})$ are the real constants, $\Phi(z)$ is a vector of the class $E^{\pm}_{l,m,p}(\Gamma, Q^{\pm}, \rho^{\pm})$, satisfying the conditions

$$I_m \int_{\Gamma} \Phi^{\pm}(t) d_{Q'_{\pm}} t \Psi_k(t) = 0, \quad k = 1, \cdots, N_0^{\pm}, \tag{5.18}$$

where $\Psi_k^{\pm}$ $(k = 1, \cdots, N_0^{\pm})$ is a complete system of linearly independent solutions of the equation, conjugate to the equation $M^{\pm}w = 0$, which are continuous on the whole plane and are equal to zero at infinity.

Substituting the representations (5.16) into the boundary condition (5.9) for the desired vector $\Phi(z)$ we obtain the following boundary condition

$$H_1\Phi = f(t) + \sum_{i=1}^{N} c_i v_i(t), \quad t \in \Gamma; \tag{5.19}$$

Moreover the vector $\Phi(z)$ has to satisfy also the conditions (5.17).

The linear independent vectors $v_i(t)$ are representable by w_k^+, w_k^-, c_i are unknown real constants.

The operator H_1 in the boundary condition (5.16) has the form of the operator H, the difference between H_1 and H_2 may be only the completely continuous operators.

The problem (5.17),(5.18) is Noetherian. When the operator K is Noetherian and the indices are connected by the following formula

$$\tilde{\varkappa} = \varkappa + N - N_0, \quad N_0 = N_0^+ + N_0^-,$$

where $\varkappa$ is the index of the operator K of the class $L_p(\Gamma, \rho)$.

Using the method of homotopy one can prove that $N = N_0 (cf.19.3)$.

It is possible to study the boundary problems containing the derivatives of higher order analogously [112].

In this chapter we apply the author's articles written together with Ngo V.L. [103].

Differential boundary problems for analytic and generalized analytic functions were investigated in [7], [13], [14], [134], [136], [34], [62], [91], [92], [130], [122] and also in several (other) monographs. The representation of generalized analytic vectors indicated in §5 is the generalization of the representation constructed in [62].

The boundary problems of linear conjugation with displacement for generalized analytic functions were investigated in [4] and (see also [89]).

The boundary problems for elliptic system of the general form was studied in the article [138] and in other works.

In Chapter 19 we often apply the results and terminology from [23]. Various aspects of the theory of generalized analytic vectors are illuminated in [55], [49], [37]. The references concerning this problem one can found in the monograph [49] in detail.

Investigation of the theory of differential equation of elliptic type using the complex analysis methods has old history. This problem is studied in the monographs of [1], [11], [17], [18], [19], [134], [135], [139], [49], [42], [43], [107], [117] and in many other monographs.

Chapter 20

On Boundary Value Problems for Non-Linear Systems of Partial Differential Equations in the Plane

by Giorgi F. Manjavidze and Wolfgang Tutschke

20.1 Introduction

Let G be a bounded domain on the plane of the complex variable z, the boundary Γ of which consists of one or finite number of simple closed Liapunov curves (i.e. the angle, between the tangent towards them and a constant direction is Hölder-continuous). Consider the following system of differential equations in G:

$$\frac{\partial w_k}{\partial \bar{z}} = F_n\left(z, w_1, \cdots, w_n, \frac{\partial w}{\partial z}, \cdots, \frac{\partial w_n}{\partial z}\right), \quad k = 1, \cdots, n.$$

We shall write this system in the short form as the equation

$$\frac{\partial w}{\partial \bar{z}} = F\left(z, w, \frac{\partial w}{\partial z}\right). \tag{$*$}$$

In this paper some boundary value problems for the system (*) are studied. These problems have unique solution in holomorphic case ($F \equiv 0$). The desired solution is constructed as a solution of the system of nonlinear integral equations

$$\begin{aligned} w(z) &= \psi(z) + \Phi_{(w,h)}(z) - \frac{1}{\pi}\iint\limits_G \frac{F(\zeta, w(\zeta), h(\zeta))}{\zeta - z} d\xi d\eta, \\ h(z) &= \psi'(z) + \Phi'_{(w,h)}(z) - \frac{1}{\pi}\iint\limits_G \frac{F(\zeta, w(\zeta), h(\zeta))}{(\zeta - z)^2} d\xi d\eta, \end{aligned} \tag{$**$}$$

$$\zeta = \xi + i\eta.$$

The vector ψ is a holomorphic solution of the considered boundary problem,

$\Phi_{(w,h)}$ is the holomorphic vector, such that

$$\Phi_{(w,h)}(z) - \frac{1}{\pi}\iint\limits_G \frac{F(\zeta, w(\zeta), h(\zeta))}{\zeta - z} d\xi d\eta$$

satisfies the corresponding homogeneous boundary condition[1]. Due to the notations of the book Vekua I. [134], denote the integral operators on the right-hand sides of the equations (**) by T_G and Π_G correspondingly.

In this paper all considered boundary functions are supposed to belong to the space $C_\alpha(\Gamma)$. Therefore, it is provided, that the holomorphic solution ψ (in case of $F \equiv 0$) belongs to the space $C_\alpha(\overline{G})$. It is well-known that from $\psi \in C_\alpha(\overline{G})$ it follows that $\psi' \in L_p(G)$ if

$$p < \frac{1}{1-\alpha}. \tag{1.1}$$

In order to ensure the existence of the number $p > 2$ such that $\psi' \in L_p(G)$, in the sequel we shall suppose that $\frac{1}{2} < \alpha \leqslant 1$.

On the right-hand side we assume, that $F(z, w, h)$ is defined when $z \in G$, $|w| \leqslant R$ and for all h. For all considered boundary problems it is required that F satisfies the following two conditions:

$$F(z, 0, 0) \in L_p(G), \tag{I}$$

$$|F(z, w, h) - F(z, \tilde{w}, \tilde{h})| \leqslant L_1|w - \tilde{w}| + L_2|h - \tilde{h}|, \tag{II}$$

where L_1, L_2 are non-negative constants, $|w| = \max\limits_k |w_k|$. Moreover, sometimes we will need the following assumptions

$$|F(z_2, w, h) - F(z_1, w, h)| \leqslant l|z_2 - z_1|^\gamma$$

for all $z_1, z_2 \in G$, $\frac{p-2}{2} < \gamma < 1$ and

$$|F(z, 0, 0)| \leqslant m$$

for all $z \in G$; m, l, γ - are constants.

By virtue of the assumptions (I), (II) the composite vector-function (when it is measurable)

$$f(z) = F(z, w(z), h(z)) \tag{1.2}$$

belongs to the space $L_p(G)$, if $h \in L_p(G)$ and $w = w(z)$ is continuous in $\overline{G}$; on the other hand the operator T_G is mapping the space $L_p(G)$ into $C_\beta(\overline{G})$, $\beta = \frac{p-2}{p} < \alpha$.

[1]cf. [131]. In case F is not depending on $\partial w/\partial z$, the system (**) consists only of the first line (see [12]). More references may be found for example in [139].

Hence, naturally we look for the solution of the equation (*) with the property

$$w \in C_\beta(\bar{G}), \quad \frac{\partial w}{\partial z} \in L_p(G).$$

Consider a Banach space consisting of the pairs of vectors (w, h), in which the norm is defined in the following way:

$$||(w,h)|| = \max(\|w\|_{C_\beta(\overline{G})}, \|h\|_{L_p(G)})$$

From $\Phi_{(w,h)} \in C_\beta(\overline{G})$ it does not follow that $\Phi'_{(w,h)}$ belongs to each space $L_p(G)$ $(p > 2)$. In order to avoid this difficulty the assumptions are intensified usually: we shall take the given boundary values from the class C^1_μ, and the right-hand side of the equation will satisfy the additional condition, such assumptions provide the existence of the solution in C^1_μ (see [131], [132]). The second possibility is to seek the solution in the Sobolev Space $W^1_p(C)$; this will allow us to construct the solution if only the Lipshitz condition (II) is fulfilled; with respect to the given boundary functions it is sufficient to suppose, that they belong to the Slobodetski Space $W_{1-\frac{1}{p}}(\Gamma)$ (see [121]).

In this paper using one property of the operator T_G we will prove that $\Phi'_{(w,h)} \in L_p(G)$. This will permit the corresponding a-priori estimate, when the following assumptions hold:

a) the given boundary functions are Hölder-continuous (with an exponent larger than 1/2);

b) the right-hand side of the equation (*) satisfies the conditions (I), (II). The constructed solution will turn out to be Hölder-continuous in $\overline{G}$.

20.2 Dirichlet problem in simply connected domains

20.2.1 Formulation of the problem

Let G be a simply connected domain and $g = (g_1, \cdots, g_n)$ is a given real vector on Γ, $g \in C_\alpha(\Gamma)$, $\alpha > 1/2$. Let z_0 be a fixed point in $\overline{G}$ and $c = (c_1, \cdots, c_n)$ is a given vector with constant real components.

We have to find a solution w continuous in $\overline{G}$ of the differential equation (*) satisfying the conditions

$$\begin{aligned} &\text{Re } w = g \quad \text{on } \Gamma, \\ &\text{Im } w(z_0) = c. \end{aligned} \tag{2.1}$$

Without loss of generality we may assume that G is the unit circle since it is possible to map conformally every simply connected domain (bounded by a Liapunov curve) onto the disk, and also the mapping functions $z = z(\zeta)$, $\zeta = \zeta(z)$

belong to C^1_μ (see for example [108]). After changing the variables the differential equation (*) takes the form

$$\frac{\partial w}{\partial \bar{\zeta}} = \left(\overline{\frac{dz}{d\zeta}}\right) F\left(z(\zeta), w, \left(\frac{dz}{d\zeta}\right)^{-1} \frac{\partial w}{\partial \zeta}\right);$$

and consequently the form of differential equation remains.

It is easy to see that the right-hand side is also satisfying the condition (II) where L_1 and L_2 are to be replaced by

$$L_1 \sup\left|\frac{dz}{d\zeta}\right| \quad \text{and} \quad L_2 \sup\left|\frac{dz}{d\zeta}\right| \left(\inf\left|\frac{dz}{d\zeta}\right| \right)^{-1}$$

correspondingly. Therefore, we may suppose from the very beginning, that the given domain G is the unit disk.

20.2.2 About one property of the T_G-operator in case of the disk

Let G denote an arbitrary circle $\{z :| z - a |< r\}$ and let $t \in \Gamma$, Γ is the boundary of the domain G, as above.Let the function $f \in L_p(G)$, $p > 1$. Let us show, that the boundary values of the function $\overline{T_G f}$ are the boundary values of a function holomorphic in G. We have

$$\overline{(T_G f)(t)} = -\frac{1}{\pi} \iint\limits_G \frac{\overline{f(\zeta)} d\xi d\eta}{\bar{\zeta} - \bar{a} - \frac{r^2}{t-a}} = \frac{t-a}{\pi} \iint\limits_G \frac{\overline{f(\zeta)} d\xi d\eta}{r^2 - (\bar{\zeta} - \bar{a})(t-a)} \quad . \tag{2.2}$$

One can see immediately, that $\overline{(T_G f)(t)}$ are the boundary values of the holomorphic in G function φ_0:

$$\varphi_0(z) = \frac{z-a}{\pi} \iint\limits_G \frac{\overline{f(\zeta)} d\xi d\eta}{r^2 - (\bar{\zeta} - \bar{a})(z-a)} \quad . \tag{2.3}$$

Then

$$\varphi_0'(z) = \frac{r^2}{\pi} \iint\limits_G \frac{\overline{f(\zeta)} d\xi d\eta}{(r^2 - (\bar{\zeta} - \bar{a})(z-a))^2} \quad . \tag{2.4}$$

The operators defined by the right-hand sides of the formulas(2.2),(2.3) have the analogous properties of the operators T_G and Π_G (see [20]):

$$\begin{aligned} &\|\varphi_0\|_{C_\beta(\overline{G})} \leqslant c_1 \|f\|_{L_p(G)}, \\ &\|\varphi_0'\|_{L_p(G)} \leqslant c_2 \|f\|_{L_p(G)}, \end{aligned} \tag{2.5}$$

where $c_1,\ c_2$ are constants depending only on p and G.

Later on it will be necessary to consider the case when G is an infinite domain $\{z : |z-a| > r\}$, $f \in L_p(G)$, where $f \equiv 0$ in some neighborhood of the point $z = \infty$. It is easy to see that in this case the formulas (2.2)-(2.5) will take place. The functions φ_0 and φ_0' will be holomorphic in the infinite doman G, the function φ_0' vanishes at the point $z = \infty$ and

$$\lim_{z\to\infty} \varphi_0(z) = -\frac{1}{\pi} \iint\limits_G \frac{\overline{f(\zeta)} d\zeta d\eta}{\bar{\zeta} - \bar{a}}.$$

20.2.3 An a-priori estimate of the holomorphic solution of the boundary value problem

Let now G be the unit disk. As in Section 20.1, we denote by ψ the holomorphic solution of the considered boundary problem, i.e. the problem (2.1). By virtue of the Privalov theorem we have the following estimate

$$\|\psi\|_{C_\alpha(\overline{G})} \leqslant c_3 \|g\|_{C_\alpha(\Gamma)} + |c|,$$

where the constant c_3 depends only on α, $o < \alpha < 1$. In case $o < \beta < \alpha < 1$ we obtain

$$\|\psi\|_{C_\beta(\overline{G})} \leqslant 2\|\psi\|_{C_\alpha(\overline{G})} \leqslant 2(c_3\|g\|_{C_\alpha(\Gamma)} + |c|). \tag{2.6}$$

The statement of the Hardy-Littlewood theorem (see for example [48]) can be written in the form

$$|\psi'(z)| \leqslant c_4 \|\psi\|_{C_\alpha(\overline{G})} \frac{1}{(1-|z|)^{1-\alpha}},$$

where c_4 depends only on α. From the last inequality it follows that

$$\|\psi'\|_{L_p(G)} \leqslant c_5 \|\psi\|_{C_\alpha(\overline{G})},$$

c_5 depends on α and p. It is possible to estimate $\|\psi'\|_{L_p(G)}$ by $\|g\|_{C_\alpha(\Gamma)}$ and $|c|$, namely the following a-priori estimate

$$\|\psi'\|_{L_p(G)} \leqslant c_5(\|g\|_{C_\alpha(\Gamma)} + |c|) \tag{2.7}$$

holds. We need also the following a-priori estimate of the holomorphic function $\Phi(w, h)$ defined in Section 20.1. For this purpose let us consider the boundary value problem

$$\begin{aligned} &\text{Re } \varphi = \text{Re } T_G\sigma \quad \text{ on } \Gamma \\ &\text{Im } \varphi(z_0) = \text{Im } [(T_G\sigma)(z_0)]; \end{aligned} \tag{2.8}$$

$\varphi(z)$ is the desired holomorphic function in G which is continuous in $\overline{G}$, σ is a given function in G, $\sigma \in L_p(G)$, $p > 2$.

Rewrite the first condition of (2.8) in the form

$$\text{Re } \varphi = \text{Re } \overline{T_G\sigma} = \text{Re } \varphi_0 \quad \text{on } \Gamma.$$

Taking the second condition of (2.8) into account, we have

$$\varphi(z) = \varphi_0(z) - 2iIm\varphi_0(z_0)$$

From the formulas (2.5) we get

$$\|\varphi\|_{C_\beta(\overline{G})} \leqslant 3c_1\|\sigma\|_{L_p(G)},$$

$$\|\varphi'\|_{L_p(G)} \leqslant c_2\|\sigma\|_{L_p(G)}.$$

The arguments show that for the vector $\Phi(w,h)$ defined above the same estimates

$$\begin{aligned} &\|\Phi_{(w,h)}\|_{C_\beta(\overline{G})} \leqslant 2c_1\|f\|_{L_p(G)}, \\ &\|\Phi'_{(w,h)}\|_{L_p(G)} \leqslant c_2\|f\|_{L_p(G)} \end{aligned} \tag{2.9}$$

hold where the vector f is defined by (1.2).

20.2.4 Estimation of the operators on the right-hand side of the system (**)

Let (w,h) be an element of the space $(C_\beta(\overline{G}), L_p(G))$. It is supposed that $|w(z)| \leqslant R$ everywhere in $\overline{G}$ (concerning the number R see Section 20.1); with the help of the right-hand sides of the system (**) define the following operator: to each pair (w,h) corresponds the pair (W,H):

$$\begin{aligned} W &= \psi + \Phi_{(w,h)} + T_G F(\cdot, w, h) \\ H &= \psi' + \Phi'_{(w,h)} + \Pi_G F(\cdot, w, h). \end{aligned} \tag{2.10}$$

Let (w,h), $(\tilde{w},\tilde{h})$ be two pairs from $(C_\beta(\overline{G}), L_p(G))$ and (W,H), $(\widetilde{W},\widetilde{H})$ are their images. Let then

$$\tilde{f}(z) = F(z,w,h) - F(z,\tilde{w},\tilde{h}).$$

By virtue of the condition (II), we have

$$\begin{aligned} \|\tilde{f}\|_{L_p(G)} &\leqslant L_1\|w-\tilde{w}\|_{L_p(G)} + L_2\|h-\tilde{h}\|_{L_p(G)} \\ &\leqslant L_1\pi^{1/p}\|w-\tilde{w}\|_{C_\beta\overline{(G)}} + L_2\|h-\tilde{h}\|_{L_p(G)}. \end{aligned} \tag{2.11}$$

On the other hand, $\Phi_{(w,h)} - \Phi_{(\tilde{w},\tilde{h})}$ turns out to be a holomorphic solution of the boundary problem

$$\operatorname{Re}\ [\Phi_{(w,h)} - \Phi_{(\tilde{w},\tilde{h})} + T_G\tilde{f}] = 0 \quad on\ \ \Gamma,$$

$$\operatorname{Im}\ [\Phi_{(w,h)} - \Phi_{(\tilde{w},\tilde{h})} + T_G\tilde{f}](z_0) = 0.$$

If we apply the estimate (11), then we obtain

$$\|\Phi_{(w,h)} - \Phi_{(\tilde{w},\tilde{h})}\|_{C_\beta(\overline{G})} \leqslant 3c_1\|\tilde{f}\|_{L_p(G)},$$

$$\|\Phi'(w,h) - \Phi'(\tilde{w},\tilde{h})\|_{C_p(G)} \leqslant c_2\|\tilde{f}\|_{L_p(G)}.$$

Therefore, we may estimate the distance of the images (W,H) and $(\widetilde{W},\tilde{H})$ of the pairs (w,h) and $(\tilde{w},\tilde{h})$:

$$\|(W,H) - \widetilde{W},\widetilde{H})\| \leqslant \max(3c_1 + \|T_G\|_{L_p(G),C_\beta(\overline{G})},\ c_2 + \|\Pi_G\|_{L_p(G),L_p(G)})$$

$$(L_1\pi^{1/p} + L_2)\|(w,h) - (\tilde{w},\tilde{h})\|. \tag{2.12}$$

From the last inequality immediately follows the following lemma.

Lemma 20.2.1 *The operator defined by the formulas* (2.10) *is continuous.*

Now let us consider in the space $(C_\beta(\overline{G}), L_p(G))$ a polycylinder

$$D = \{(w,h) : \|w\|_{C_\beta(\overline{G})} \leqslant R_1,\ \|h\|_{L_p(G)} \leqslant R_2\},$$

where $R_1 \leqslant R$. By virtue of the assumptions (II), we have

$$\begin{aligned}|F(z,w,h)| &\leqslant |F(z,0,0)| + |F(z,w,h) - F(z,0,0)| \\ &\leqslant |F(z,0,0)| + L_1|w| + L_2|h|.\end{aligned} \tag{2.13}$$

Taking into account (I), from this inequality follows that as we have already noted in §1, the composed function f, defined by the equality (1.2) belongs to the space $L_p(G)$ (if it is measurable). From (2.13) it follows that

$$\|f\|_{L_p(G)} \leqslant M + L_1\|w\|_{L_p(G)} + L_2\|h\|_{L_p(G)} \leqslant M + L_1\pi^{1/p}\|w\|_{C_\beta(\overline{G})} + L_2\|h\|_{L_p(G)},$$

$$M = \|F(z,0,0)\|_{L_p(G)}.$$

Consequently

$$\|f\|_{L_p(G)} \leqslant M + L_1\pi^{1/p}R_1 + L_2R_2, \tag{2.14}$$

for all $(w,h) \in D$. Taking into account (2.7) and (2.14), we obtain finally the following estimate

$$\|W\|_{C_\beta(\overline{G})} \leqslant \|\psi\|_{C_\beta(\overline{G})} + (3c_1 + \|T_G\|_{L_p(G),C_\beta(\overline{G})})(M + L_1\pi^{1/p}R_1 + L_2R_2),$$

$$\|H\|_{L_p(G)} \leqslant \|\psi'\|_{L_p(G)} + (c_2 + \|\Pi_G\|_{L_p(G),L_p(G)})(M + L_1\pi^{1/p}R_1 + L_2R_2),$$

for all $(w,h) \in D$.

20.2.5 Solution of the boundary problem (2.1)

Applying the Banach fixed-point theorem, from the inequalities (2.11) and (2.15) it follows immediately:

Theorem 20.2.1 *Let $g \in C_\alpha(\Gamma), \alpha > 1/2$ and $2 < p < \dfrac{1}{1-\alpha}$. It is assumed, that the right-hand side $F(z, w, h)$ of the differential equation $(*)$ satisfies the conditions (I) and (II). Suppose, that there exist numbers $R_1 (\leqslant R)$ and R_2 such that the following inequalities*[2]

$$\begin{aligned} &\|\psi\|_{C_\beta(\overline{G})} + (3c_1 + \|T_G\|_{L_p(G), C_\beta(\overline{G})})(M + L_1 \pi^{1/p} R_1 + L_2 R_2) \leqslant R_1, \\ &\|\psi'\|_{L_p(\overline{G})} + (c_2 + \|\Pi_G\|_{L_p(G), L_p(G)})(M + L_1 \pi^{1/p} R_1 + L_2 R_2) \leqslant R_2 \end{aligned} \tag{III}$$

are fulfilled and

$$\max\left(3c_1 + \|T_G\|_{L_p(G), C_\beta(\overline{G})},\ c_2 + \|\Pi_G\|_{L_p(G), L_p(G)}\right)(L_1 \pi^{1/p} + L_2) < 1. \tag{2.15}$$

Then there exists one and only one solution w of the boundary problem (2.1) for which the pair $\left(w, \dfrac{\partial w}{\partial z}\right)$ belongs to the polycylinder D.

20.3 Dirichlet problem in multiply connected domains

20.3.1 Formulation of the problem

Let now G be a bounded $(m+1)$–connected domain, $m \geqslant 1$, with the boundary $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_m$. It is assumed, that Γ_0 contains all other boundary curves. Let $g = (g_1, \cdots, g_n)$ be a given real vector on the boundary, $g \in C_\alpha(\Gamma)$, $\alpha > 1/2$. Let also $c = (c_1, \cdots, c_n)$ be a given vector with the constant real components. Let, finally, z_0 be *a* fixed point in $\overline{G}$.

We have to find *a* solution w of the differential equation $(*)$ in G continuous in $\overline{G}$ and satisfying the boundary conditions

$$\begin{aligned} &\text{Re}\ w = g + k^{(j)} \quad \text{on}\ \Gamma_j, j = 0, 1, \cdots, m, \\ &\text{Im}\ w(z_0) = c, \end{aligned} \tag{3.1}$$

where $k^{(j)}$ are real constant vectors, which are not given beforehand, one of them can be fixed arbitrarily; assume, that $k^{(0)} = (0, \cdots, 0)$; then all remaining $k^{(j)}$ are defined uniquely by g, c and F. As for the solution of the modified Dirichlet

[2] These conditions provide, that the operator (2.10) maps D into itself. In these inequalities we may replace $\|\psi\|_{C_\beta(\overline{G})}$ and $\|\psi'\|_{L_p(G)}$ by the right-hand sides of the inequalities (2.6) and (2.7).

problem for the holomorphic function ($F \equiv 0$) (see [121]), it is shown that there exist a unique solution[3].

It follows from the choice $k^{(0)} = (0, \cdots, 0)$ that in case when $m = 0$ the problem (3.1) coincides with the problem (2.1).

Without loss of generality we may assume that the curves Γ_j are circles, because it is possible to map each $(m+1)$-connected domain onto a domain of such form. From above formulated assumptions about the boundary curves it follows that the conformally mapping function $z = z(\zeta) \in C^1_\mu$. Everything that was said in Subsection 20.2.1 about the form of the differential equation (*) is also valid after the mapping.

20.3.2 An a-priori estimate of the holomorphic solution

Let φ be a holomorphic solution[4] of the problem (3.1). Rewrite (3.1) in the following form.

$$\varphi(t) + \overline{\varphi}(t) = \widetilde{g}(t), \widetilde{g}(t) = 2(g(t) + k^{(j)}) \quad \text{on } \Gamma_j. \tag{3.2}$$

Multiplying both sides of (3.2) by $\dfrac{1}{2\pi i}\dfrac{dt}{t-z}$, $z \in G$ and integrating along Γ, we obtain

$$\varphi(z) + \frac{1}{2\pi i}\int_\Gamma \frac{\overline{\varphi(t)}dt}{t-z} = h(z)\ , \tag{3.3}$$

where

$$h(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\widetilde{g}(t)dt}{t-z}\ .$$

The limiting process $z \to t_0$, $t_0 \in \Gamma$, in (3.3) yields

$$\varphi(t_0) + \frac{1}{2}\overline{\varphi(t_0)} + \frac{1}{2\pi i}\int_\Gamma \frac{\overline{\varphi}(t)dt}{t-t_0} = h(t_0), \tag{3.4}$$

$$h(t_0) = \frac{1}{2}\widetilde{g}(t_0) + \frac{1}{2\pi i}\int_\Gamma \frac{\widetilde{g}(t)dt}{t-t_0}\ .$$

Then we have

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma \frac{\overline{\varphi(t)}dt}{t-t_0} &= \frac{1}{2\pi i}\int_\Gamma \frac{\overline{\varphi(t)}\overline{dt}}{\overline{t}-\overline{t_0}} + \frac{1}{2\pi i}\int_\Gamma \overline{\varphi(t)}\left[\frac{dt}{t-t_0} - \frac{\overline{dt}}{\overline{t}-\overline{t_0}}\right]\\ &= -\frac{1}{2}\overline{\varphi(t_0)} + \textstyle\int_\Gamma H(t_0,t)\overline{\varphi(t)}dt\end{aligned}$$

---

[3]The constant vectors $k^{(j)} = (k_1^{(j)}, \cdots, k_n^{(j)})$ are uniquely defined by the vector g , they have the form $k_s^{(j)} = \displaystyle\int_{\Gamma_j} \rho_j^{(s)}(t) g_{|s|}(t)dt$, $(s = 1, \cdots, n)$, where $\rho_j^{(s)}$ are some real functions depending only on the contours $\Gamma = \Gamma_0, \Gamma_1, \cdots, \Gamma_m$.

[4]In the holomorphic case the components of the desired vector do not depend on each other. Consequently, it is sufficient to consider the case $n = 1$.

where $H(t_0, t)$ is an infinitely differentiable function of the variables t_0 and t which depends only on the circles Γ_k. The equation (3.4) takes the form

$$\varphi(t_0) + \int_\Gamma H(t_0, t)\overline{\varphi(t)}dt = h(t_0) \ , \tag{3.5}$$

which is a Fredholm equation; its generalized resolvent will have the form

$$Rh = h(t_0) + \int_\Gamma R_1(t_0, t)h(t)dt + \int_\Gamma R_2(t_0, t)\overline{h(t)}dt; \tag{3.6}$$

we may consider that $R_1(t_0, t)$, $Re(t_0, t)$ are infinitely differentiable functions depending only on the circles Γ_k. The equation (3.5) is solvable and its general solution has the form

$$\varphi(t_0) = h(t_0) + h_1(t_0) + id, \tag{3.7}$$

where

$$h_1(t_0) = \int_\Gamma R_1(t_0, t)h(t)dt + \int_\Gamma R_2(t_0, t)\overline{h_(t)}dt$$

and d is a real constant. Taking into account that

$$h(t) = \frac{1}{2}\widetilde{g}(t) + \frac{1}{2\pi i}\int_\Gamma \frac{\widetilde{g}(\tau)d\tau}{\tau - t} = g(t) + \frac{1}{\pi i}\int_\Gamma \frac{g(\tau)d\tau}{\tau - t},$$

we obtain

$$h_1(t_0) = \int_\Gamma R_3(t_0, t)g(t)dt \equiv Kg, \tag{3.8}$$

where $R_3(t_0, t)$ is infinitely differentiable function depending only on the circles Γ_k. From (3.7) we have

$$\varphi(z) = \frac{1}{\pi i}\int_\Gamma \frac{g(t)dt}{t - z} + \frac{1}{2\pi i}\int_\Gamma \frac{h_1(t)dt}{t - z} + id, \tag{3.9}$$

where $h_1(t) = Kg$ is given by the formula (3.8). Finally the desired solution φ has the form

$$\varphi(z) = \frac{1}{\pi i}\int_\Gamma \frac{g(t)dt}{t - z} + \frac{1}{2\pi i}\int_\Gamma \frac{Kg(t)dt}{t - z} + id, \tag{3.10}$$

where one may choose the real constant d from the condition

$$d = c - Im\widetilde{\varphi}(z_0), \tag{3.11}$$

$$\widetilde{\varphi}(z) = \frac{1}{\pi i}\int_\Gamma \frac{g(t)dt}{t - z} + \frac{1}{2\pi i}\int_\Gamma \frac{Kg(t)dt}{t - z} \ .$$

The second term in (3.10),

$$\frac{1}{2\pi i}\int_\Gamma \frac{Kg(t)dt}{t - z} = \frac{1}{2\pi i}\int_\Gamma \frac{h_1(t)dt}{t - z} = \varphi_2(z)$$

is by virtue of (3.8) together with its derivatives of any order a continuous function in the closure of the domain, where

$$\|\varphi_2^{(n)}(z)\|_{C_\alpha(\overline{G})} \leqslant M_n^\alpha \int_\Gamma |g(t)| \cdot |dt| \,, \tag{3.12}$$

M_n^α are some constants, depending only on α, n and the circles Γ_k. As for the term

$$\frac{1}{\pi i}\int_\Gamma \frac{g(t)dt}{t-z} = \varphi_1(z)$$

if $g \in C_\alpha(\Gamma), \alpha > \dfrac{1}{2}$, then

$$\begin{aligned} &\|\varphi_1\|_{C_\alpha\overline{(G)}} \leqslant M_\alpha \|g\|_{C_\alpha(\Gamma)}, \\ &\|\varphi_1'\|_{L_p(G)} \leqslant N_p \|g\|_{C_\alpha(\Gamma)}, \end{aligned} \tag{3.13}$$

where $2 < p < \dfrac{1}{1-\alpha}$, and M_α and N_p are constants depending only on the circles Γ_k and on α and p.

Let now ψ be a holomorphic solution of the boundary problem (3.1). Taking into account, that $\widetilde{\varphi} = \varphi_1 + \varphi_2$ from the formulas (3.11) and from the estimations (3.12) and (3.13) we get immediatly

$$\|\psi\|_{C_\beta(\overline{G})} \leqslant c_5 \|g\|_{C_\alpha(\Gamma)} + |c|,$$

$$\|\psi'\|_{L_p(G)} \leqslant c_6 \|g\|_{C_\alpha(\Gamma)};$$

c_5, c_6 are constants, they do not depend on g and c (they depend only on Γ_k, α and p).

Let then g has the form

$$2g = T_G\sigma + \overline{T_G\sigma},$$

where $\sigma \in L_p(G), p > 2$. As the boundary values of the function $T_G\sigma$ are the boundary values of a function, which is holomorphic outside $\overline{G}$ and vanishes at infinity, we have

$$\varphi_1(z) = \frac{1}{\pi i}\sum_{k=0}^{m}\int_{\Gamma_k}\frac{g(t)dt}{t-z} = \frac{1}{2\pi i}\sum_{k=0}^{m}\int_{\Gamma_k}\frac{\overline{(T\sigma)(t)}dt}{t-z}.$$

According to an above mentioned statement (see §2, section 2), one has $\varphi \in C_\beta(\overline{G}),\ \beta = \dfrac{p-2}{p},\ \varphi_1' \in L_p(G)$, and also

$$\|\varphi_1\|_{C_\beta(\overline{G})} \leqslant A_p \|\sigma\|_{L_p(G)}, \tag{3.14}$$

$$\|\varphi_1'\|_{L_p(G)} \leqslant B_p \|\sigma\|_{L_p(G)},$$

where the constants A_p, B_p depend only on Γ_k and p. Applying these estimates in case of the vector $\sigma = -f$, where f is defined by (1.2), we obtain for the vector $\Phi(w,h)$ the following estimates

$$\|\Phi_{(w,h)}\|_{C_\beta(\overline{G})} \leqslant c_7\|f\|_{L_p(G)}, \tag{3.15}$$
$$\|\Phi'_{(w,h)}\|_{L_p(G)} \leqslant c_8\|f\|_{L_p(G)},$$

where c_7, c_8 are constants not depending on (w,h) (they depend only on Γ_k and p).

20.3.3 Estimates of the operators on the right-hand sides of the system (**)

As in Subsection 20.2.4, consider the operator (2.10). Let

$$\sigma(z) = -(f(z,w(z),h(z)) - f(z,\tilde{w}(z),\tilde{h}(z))).$$

Since $\Phi(w,h) - \Phi(\tilde{w},\tilde{h})$ is a solution of the boundary value problem

$$\text{Re }[\Phi(w,h) - \Phi(\tilde{w},\tilde{h})] = \text{Re }\sigma \quad \text{on } \Gamma,$$
$$\text{Im }[\Phi(w,h) - \Phi(\tilde{w},\tilde{h})](z_0) = 0,$$

one obtains from (3.11), (3.12) and (3.14) the following estimates analogous to (3.15):

$$\|\Phi(w,h) - \Phi(\tilde{w},\tilde{h})\|_{C_\beta(\tilde{G})} \leqslant c_7\|F(\cdot,w,h) - F(\cdot,\tilde{w},\tilde{h})\|_{L_p(G)},$$
$$\|\Phi'(w,h) - \Phi'(\tilde{w},\tilde{h})\|_{L_p(G)} \leqslant c_8\|F(\cdot,w,h) - F(\cdot,\tilde{w},\tilde{h})\|_{L_p(G)}.$$

Due to the estimate (2.12), we have

$$\begin{aligned}\|(W,H) - (\widetilde{W},\widetilde{H})\| \leqslant &\max(c_7 + \|T_G\|_{L_p(G),C_\beta(\overline{G})}, c_8 + \|\Pi_G\|_{L_p(G),L_p(G)}) \\ &\times \left(L_1(mG)^{1/p} + L_2\right)\|(w,h) - (\tilde{w},\tilde{h})\| \ .\end{aligned}$$

Hence, the following lemma is valid (cf. §2, section 4).

Lemma 20.3.1 *In case of the modified Dirichlet problem the operator* (2.10) *is continuous.*

Consider as in Subsection 20.2.4 the polycylinder D. In this case the estimates (2.15) hold if $3c_1, c_2$ are replaced by c_7 and c_8 correspondingly.

20.3.4 Solution of the modified Dirichlet problem

Taking (3.14), (2.15) (where $3c_1, c_2$ are replaced by c_7 and c_8) and (3.15) into account, we may prove the following theorem.

Theorem 20.3.1 *Assume that the right-hand side $F(z,w,h)$ satisfies the conditions $(I),(II)$. Let us suppose, that there exist numbers $R_1(\leqslant R)$ and R_2 such that the following inequalities*[1].

$$\begin{aligned} &\|\varphi\|_{C_\beta(\overline{G})} + (c_7 + \|T_G\|_{L_p(G),C_\beta(\overline{G})})(M + L_1(mG)^{1/p}R_1 + L_2R_2) \leqslant R_1, \\ &\|\varphi'\|_{L_p(G)} + (c_8 + \|\Pi_G\|_{L_p(G),L_p(G)})(M + L_1(mG)^{1/p}R_1 + L_2R_2) \leqslant R_2 \end{aligned} \tag{V}$$

are fulfilled.

Let

$$\max(c_7 + \|T_G\|_{L_p(G),C_\beta(\overline{G})}, c_8 + \|\Pi_G\|_{L_p(G),L_p(G)})(L_1(mG)^{1/p}R_1 + L_2) < 1. \tag{VI}$$

Then there exists the unique solution w of the modified Dirichlet problem $(*)$, (3.1) *for which the pair $\left(w, \dfrac{\partial w}{\partial z}\right)$ belongs to the polycylinder D.*

20.4 Riemann-Hilbert problem for simply connected domains

20.4.1 Formulation of the problem

Let G be a simply connected domain on the plane z. Without loss of generality (cf. Subsection 20.2.1), we may assume that G is the unit disk. We have to consider the differential equation (*) for the vector $w = (w_1, \cdots, w_n)$. Let $A(t)$ be a nonsingular quadratic matrix of order n given on Γ, $A(t) \in C_\alpha(\Gamma)$, $\alpha > 1/2$. We assume $\varkappa_k \geqslant -1$ $(k = 1, \cdots, n)$ for the partial indices of the matrix $A^{-1}(t)\overline{A(t)}$. Let, further, $g \in C_\alpha(\Gamma)$ $\alpha > 1/2$, be a given vector on Γ.

Find a solution satisfying the boundary condition

$$\operatorname{Re}\ [A(t)w(t)] = g(t) \quad on \ \Gamma \tag{4.1}$$

and also some normalization condition.

First of all we shall establish the normalization for the Riemann-Hilbert problem in case of holomorphic vectors (see the next section) and then the same normalization will be applied for the problem (4.1) for the equation (*).

20.4.2 Normalization of Riemann-Hilbert problem in the holomorphic case

Assume, that the partial indices of the matrix $a(t) = -A^{-1}(t)\overline{A(t)}$ are satisfying the condition

$$\varkappa_1 \geqslant \cdots \geqslant \varkappa_n \geqslant -1.$$

[1] In these inequalities $\|\psi\|_{C_\beta(\overline{G})}$ and $\|\psi'\|_{L_p(G)}$ may be replaced by the right-hand sides of (30)

Then the homogeneous Riemann-Hilbert problem (4.1) has $l = \sum_{k=1}^{n} \varkappa_n + n$ linear independent solutions over the real number field (see [136]).

The general solution of Riemann-Hilbert problem (4.1) is given by the formula

$$\Phi(z) = \Phi^o(z) + \chi(z)P(z), \tag{4.2}$$

where $\Phi^o(z)$ is a partial solution of the problem of the following form

$$\Phi^o(z) = 1/2[w(z) + w_*(z)], \tag{4.3}$$

$$w(z) = \frac{\chi(z)}{2\pi i} \int_\Gamma \frac{[\chi^+(t)]^{-1}A^{-1}(t)g(t)}{t-z} dt, \quad w_*(z) = \overline{w(\bar{z}-1)}$$

and $\chi(z)$ is the canonical matrix of the boundary value problem $\varphi^+(t) = a(t)\varphi^-(t)$ satisfying the condition $\chi_*(z) = \chi(z) diag[z^{\varkappa_1}, \cdots, z^{\varkappa_n}]$ and the polynomial vector $P(z)$ has the form

$$P(z) = (p^{(1)}, \cdots, p^{(n)}), \quad p^{(s)}(z) = 0, \quad when \ \ \varkappa_s = -1,$$

$p^{(s)}(z) = c_0^s z^{\varkappa_s} + \cdots + c_{\varkappa_s}^s$ is an arbitrary polynomial whose coefficients are connected by the relations

$$\overline{c_{\varkappa_s - k}^s} = c_k^s, \quad k = 0, \cdots, \varkappa_s, \ s = 1, \cdots, n. \tag{4.4}$$

The relations (4.4) contain $2\varkappa_s + 2$ real constants Re c_k^s, Im c_k^s; using these relations it is possible to express $\varkappa_s + 1$ of them by the remainings; denoting them by $d_0^s, \cdots, d_{\varkappa_s}^s$, the polynomial $P^{(s)}(z)$ has the form

$$P^{(s)}(z) = \sum_{k=0}^{\varkappa_s} d_k^s \lambda_k^s(z),$$

where $\lambda_k^s(z)$ are linearly independent functions.

For simplicity, let us suppose that

$$\varkappa_1 > \varkappa_2 > \cdots > \varkappa_{n-1} > \varkappa_n > -1.$$

Consider the matrix

$$B(t) = A(t)\chi^+(t).$$

This matrix is nonsingular for all $t \in \Gamma$. Therefore, it is possible to find different points $t_k^{n-1} \in \Gamma$ $(k = 1, \cdots, \varkappa_{n-1} - \varkappa_n)$ and quadratic matrices of order $(m-1)$ composed from the first $n-1$ columns of the matrix $B(t)$ such that the determinant of this matrix is not equal to zero at the points t_k^{n-1}. Take one of such matrices and denote it by $B_{\nu_1}^{n-1}$ if the ν_1-th row of the $B(t)$ matrix is not contained in it.

Then choose distinct points $t_k^{n-2} \in \Gamma$ $(k = 1, \cdots, \varkappa_{n-2} - \varkappa_{n-1})$ (different from the points t_k^{n-1}) and the quadratic matrix B_{ν_1,ν_2}^{n-2} of order $n-2$ composed of the first $n-2$ columns of the matrix $B(t)$ such that det $B_{\nu_1,\nu_2}^{n-2}(t_k^{n-2}) \neq 0$; the lower indices ν_1, ν_2 denote that the ν_1- and ν_2-th rows of the matrix $B(t)$ are not contained in B_{ν_1,ν_2}^{n-2}.

Continuing in such a way, we shall choose distinct points $t_k^1,\ k = 1, \cdots, \varkappa_1 - \varkappa_2)$ (different from the points chosen earlier) and the elements $b_{\nu_1,\cdots,\nu_{n-1}}(t)$ from the first column of the matrix $B(t)$, satisfying the conditions

$$B_{\nu_1,\cdots,\nu_{n-1}}(t_k^1) \neq 0.$$

Besides, take also the distinct points $t_k^n \in \Gamma,\ k = 1, \cdots, \varkappa_{n+1}$, different from the points chosen earlier.

Prescribe the vector Im $[A(t)\Phi(t)]$ at the points t_k^n:

$$\text{Im}\ [A(t_k^n)\Phi(t_k^n)] = c_k^n, \quad k = 1, \cdots, \varkappa_n + 1, \tag{4.5}$$

where c_k^n are arbitrary fixed real n-dimensional vectors.

Take now in the general solution (4.2) the last component $P^n(z) \equiv 0$; we get the vector, which we denote by $\Phi^{n-1}(z)$ and put the condition

$$\text{Im}\ [A(t_k^{n-1})\Phi^{n-1}(t_k^{n-1})]\}_{(\nu_1)} = c_k^{n-1}, \quad k = 1, \cdots, \varkappa_{n-1} - \varkappa_n, \tag{4.6}$$

c_k^{n-1} is an arbitrary fixed real $(n-1)$-dimensional vector.

Here we use the following notation: if $W = (W_1, \cdots, W_n)$ is some n-dimensional vector, then

$$\{W\}_{(\nu_1,\cdots,\nu_s)}$$

denotes the $(n-s)$-dimensional vector, which is obtained from the vector W if we omit the components with the numbers $\nu_1, \cdots, \nu_s$.

Then take in the general solution (4.2) $P^{(n-1)}(z) \equiv 0,\ P^{(n)}(z) \equiv 0$; the obtained vector is denoted by $\Phi^{n-2}(z)$, and put the condition

$$\{\text{Im}\ [A(t_k^{n-2})\Phi^{n-2}(t_k^{n-2})]\}_{(\nu_1,\nu_2)} = c_k^{n-2}, \quad k = 1, \cdots, \varkappa_{n-2} - \varkappa_{n-1}, \tag{4.7}$$

c_k^{n-2} are fixed real $(n-2)$-dimensional vectors.

If we continue further, in the last step we have to put the condition

$$\{\text{Im}\ [A(t_k^1)\Phi(t_k^1)]\}_{(\nu_1,\cdots,\nu_{n-1})} = c_k^1, \quad k = 1, \cdots, \varkappa_1 - \varkappa_2, \tag{4.8}$$

c_k^1 are fixed real constants. The relations (4.5)-(4.8) are a linear algebraic system for the real unknown $d_k^s,\ k = 0, \cdots, \varkappa_s,\ s = 1, \cdots, n$; the number of unknowns coincides with the number of the equations of the system and is equal to $l = \sum_{k=1}^n \varkappa_k + n$.

Let us prove that this system is solvable. For this purpose consider the homogeneous system, which we get if we suppose that $g(t) \equiv 0$, all $c_k^s = 0$. Let d_{k0}^s $(k = 0, \cdots, \varkappa_s,\ s = 1, \cdots, n)$ be some solution of the homogeneous system. We shall prove, that all $d_{k0}^s = 0$.

Indeed, using the equalities

$$\text{Re}\,[A(t_k^n)\Phi(t_k^n)] = 0,\ \ k = 0, \cdots, \varkappa_n + 1,$$

in the homogeneous case we may rewrite the equation (4.5) as follows

$$A(t_k^n)\chi(t_k^n)P(t_k^n) = 0;$$

from here we have

$$P(t_k^n) = 0,\ \ k = 0, \cdots, \varkappa_n + 1$$

and in particular $P^{(n)}(t_k^n) = 0,\ \ P^{(n)}(z) \equiv 0,\ \ d_{k0}^n = 0.$

In the second step we get, analogously,

$$P^{(n-1)}(t_k^{n-1}) = 0,\ \ k = 0, \cdots, \varkappa_{n-1} - \varkappa_n,$$

from which we obtain $P^{(n-1)}(z) \equiv 0,\ \ d_{k0}^n = 0$ and etc.

Hence, the homogeneous system has only the trivial solution.

We may carry out an analogous normalization if among the partial indices $\varkappa_k$ $(k = 1, \cdots, n)$ some are equal to each other and some are equal to -1.

As it is easy to see, for the solution of the system (4.5)-(4.8) we shall have the estimate

$$|d| \leqslant H_1 \|g\|_{C_\alpha(\Gamma)} + H_2 \max_{i,j} |c_j^i|. \tag{4.9}$$

where the constants H_1 and H_2 depend on the matrix $A(t)$.

20.4.3 An a-priori estimate for the Riemann-Hilbert problem in the holomorphic case

First we prove an estimate, which we will need further.

Let G be a domain bounded by a simple closed smooth curve Γ and let $\varphi(t) \in C_\mu(\Gamma)$ and let $\rho(t)$ be the boundary values of some function $\rho(z)$, analytic in G and continuously extendable to Γ.

Consider the Cauchy-type integral

$$\Phi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(t)\rho(t)}{t - z} dt$$

and its derivative

$$\Phi'(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(t)\rho(t)}{(t - z)^2} dt.$$

Denote by $R_0 = R_0(\alpha_0)$ the standard radius for the curve Γ, corresponding to some arbitrarily fixed acute angle α_0 [108].

Let t_0 be an arbitrarily fixed point on Γ and δ_0 be a positive constant, $\delta_0 > R_0$.

In the following we shall always assume, that the distance δ of the point z and t_0 is not larger than δ_0:

$$\delta = |z - t_0| \leqslant \delta_0,$$

and that the non-obtuse angle between the interval $\overline{t_0 z}$ and the tangent Γ at t_0 is not less than some fixed $\beta_0 > \alpha_0$.

Consider the circle γ_0 with the radius R_0, and denote by $l = ab$ the part of Γ inside γ_0. Represent $\Phi'(z)$ in the form

$$\Phi'(z) = \frac{1}{2\pi i}\int_{\Gamma - l} \frac{\varphi(t)-\varphi(t_0)}{(t-z)^2}\rho(t)dt + \frac{1}{2\pi i}\int_{l} \frac{\varphi(t)-\varphi(t_0)}{(t-z)^2}\rho(t)dt + \varphi(t_0)\rho'(z).$$

From the last equation we have:

$$|\Phi'(z)| \leqslant M \max|\rho(t)| \max|\varphi(t)| \qquad (4.10)$$

$$+N \sup\frac{|\varphi(t_1)-\varphi(t_2)|}{|t_1-t_2|^{\mu}} \max|\rho(t)|\delta^{\mu-1} + \max|\varphi(t)||\rho'(z)|.$$

In the estimate (4.10) the constant M depends only on the curve Γ, and the constant N depends on Γ and the exponent μ.

From this estimate one can see that if $\mu > 1/2$ and $\rho'(z) \in L_s(G)$, $s > 2$, then $\Phi'(z) \in L_p(G)$ for some $p > 2$, namely p has to satisfy the inequalities

$$2 < p < \frac{1}{1-\mu}, \quad p \geqslant s.$$

If G is the unit disk, then the estimate (4.10) will have the form

$$|\Phi'(z)| \leqslant M \max|\rho(t)| \max|\varphi(t)| +$$

$$N \sup\frac{|\varphi(t_1)-\varphi(t_2)|}{|t_1-t_2|^{\mu}} \max|\rho(t)|(1-r)^{\mu-1} + \max|\varphi(t)||\rho'(z)|. \quad (r = |z|). \qquad (4.11)$$

When $|z| < \varepsilon < 1$, it takes place the estimate

$$|\Phi'(z)| \leqslant \frac{1}{(1-\varepsilon)^2} \max|\varphi(t)| \max|\rho(t)|,$$

hence an estimate of the form (4.10) is fulfilled.

The analogous estimates are valid in case when $\varphi(t)$ and $\rho(t)$ are matrices.

As it was mentioned above, one of the solutions of the Riemann-Hilbert problem (when the partial indices of the problem $\varkappa_k \geqslant -1$) is given by the formulas (4.3). From these formulas we have

$$\Phi^+(t_0) = A^{-1}(t_0)g(t_0) + \frac{A^{-1}(t_0)}{\pi i}\int_\Gamma \frac{g(t)dt}{t-t_0} + R(t_0) + \overline{\chi^-(t_0)}B, \tag{4.12}$$

where

$$B = \frac{1}{2\pi i}\int_\Gamma [\overline{\chi^+(t)}]^{-1}\overline{A^{-1}(t)}g(t)\bar{t}dt,$$

$$R(t_0) = \frac{\chi^+(t_0)}{2\pi i}\int_\Gamma \frac{[\chi^+(t)]^{-1}A^{-1}(t) - [\chi^+(t_0)]^{-1}A^{-1}(t_0)}{t-t_0}g(t)dt$$
$$-\frac{\overline{\chi^-(t_0)}}{2\pi i}\int_\Gamma \frac{[\overline{\chi^+(t)}]^{-1}\overline{A^{-1}(t)} - [\overline{\chi^+(t_0)}]^{-1}\overline{A^{-1}(t_0)}}{t-t_0}g(t)dt$$

Consider the representation

$$\varphi(t_0) = \int_\Gamma \frac{H(t)-H(t_0)}{t-t_0}\psi(t)dt,$$

where the matrix $H(t) \in C_\mu(\Gamma)$, $\psi(t)$ is a measurable bounded vector, Γ_0 here denotes a simple close smooth curve.

We obtain

$$||\varphi||_{C_{\mu-\varepsilon}(\Gamma)} \leqslant E||H||_{C_{\mu(\Gamma)}} \max(\sup|\psi_1(t)|, \cdots, \sup|\psi_n(t)|),$$

where the constant E depends only from Γ_0, ε, μ (see [108], §5).

Therefore (if $A \in C_\alpha(\Gamma)$),

$$\|R\|_{C_{\alpha-\varepsilon}(\Gamma)} \leqslant Q\max(\max|g_1(t)|, \cdots, \max|g_n(t)|), \tag{4.13}$$

where the constant Q depends only from the matrix A, α and ε; if $\alpha > 1/2$, then we may suppose that in the estimate (4.13) $\alpha - \varepsilon > 1/2$.

If now we are solving the Riemann-Hilbert problem for the holomorphic vector, then the vector g has the form

$$g(t) = T_G(\sigma) + \overline{T_G(\sigma)},$$

where $\sigma(z) \in L_p(G)$ $(p > 2)$ is some given vector. Hence in this case

$$g(t_0) \equiv A^{-1}(t_0)g(t_0) + \frac{A^{-1}(t_0)}{\pi i}\int_\Gamma \frac{g(t)d(t)}{t-t_0} = 2A^{-1}(t_0)\overline{T_G(\sigma)}.$$

Define by

$$S(z) = \frac{1}{2\pi i}\int_\Gamma \frac{g(t)dt}{t-z}.$$

Note that $\overline{T_G(\sigma)}$ are the boundary values of a vector holomorphic in G. Using (4.11), we get

$$\|S'\|_{L_p(G)} \leqslant K_1 \|\sigma\|_{L_p(G)}, \tag{4.14}$$

where the constant K_1 depends only on p and the matrix $A(t)$.

Besides, it is evident that

$$\|S\|_{C_\beta(G)} \leqslant K_2 \|\sigma\|_{L_p(G)}, \tag{4.15}$$

where the constant K_2 depends on p and the matrix $A(t)$.

Now we consider the case that the given vector has the form

$$g = g_0 + T_G\sigma + \overline{T_G\sigma},$$

$g_0 \in C_\alpha(\Gamma),\ \sigma \in L_p(G)$. Let Φ be a normed holomorphic solution of the problem (4.1).

As it is easy to see from the formulas (4.9), (4.13)-(4.15), this holomorphic solution admits an estimate of the form

$$\|\Phi\|_{C_\beta(G)} \leqslant c_9 \|g\|_{C_\alpha(\Gamma)} + c_{10} \|\sigma\|_{L_p(G)} + c_{11} \max\nolimits_{k,\nu} \|c_k^\nu\|,$$

$$\|\Phi'\|_{L_p(G)} \leqslant c_{12} \|g_0\|_{C_\alpha(\Gamma)} + c_{13} \|\sigma\|_{L_p(G)} + c_{14} \max\nolimits_{k,\nu} \|c_k^\nu\|.$$

In the sequel the estimates (4.13) will be applied in the following two cases: First they give us the possibility to estimate the norm $\|\psi\|_{C_\beta(\overline{G})}$ and $\|\psi'\|_{L_p(G)}$ of the holomorphic solution ψ of the considered problem (4.1) by the given datas g and C_k^ν $(\sigma \equiv 0)$:

$$\|\psi\|_{C_\beta(G)} \leqslant c_9 \|g\|_{C_\alpha(\Gamma)} + c_{11} \max\nolimits_{k,\nu} |c_k^\nu|,$$

$$\|\psi'\|_{L_p(G)} \leqslant c_{12} \|g\|_{C_\alpha(\Gamma)} + c_{14} \max\nolimits_{k,\nu} |c_k^\nu|.$$

Second, for the vector $\Phi(w,h)$ defined in §1 we have $g_0 = 0,\ c_k = 0,\ \sigma = -f$ and, consequently, by virtue of (4.13)

$$\|\Phi_{(w,h)}\| \leqslant c_{10} \|f\|_{L_p(G)},$$

$$\|\Phi'_{(w,h)}\| \leqslant c_{13} \|f\|_{L_p(G)}.$$

20.4.4 An a-priori estimate of the operators on the right-hand sides of the system (**)

The holomorphic vector $\Phi(w,h) - \Phi(\tilde{w},\tilde{h})$ satisfies the normed boundary condition (4.1), and

$$\sigma = -(F(\cdot, w, h) - F(\cdot, \tilde{w}, \tilde{h})).$$

By virtue of (2.12), we have

$$\|\sigma\|_{L_p(G)} \leqslant (L_1\pi^{1/p} + L_2)\|(w,h) - (\tilde{w},\tilde{h})\|.$$

Taking into account (49), we get

$$\|\Phi_{(w,h)} - \Phi_{(\tilde{w},\tilde{h})}\|_{C_\beta(G)} \leqslant c_{10}(L_1\pi^{1/p} + L_2)\|(w,h) - (\tilde{w},\tilde{h})\|,$$

$$\|\Phi'_{(w,h)} - \Phi'_{(\tilde{w},\tilde{h})}\|_{L_p(G)} \leqslant c_{13}(L_1\pi^{1/p} + L_2)\|(w,h) - (\tilde{w},\tilde{h})\|.$$

Finally we obtain

$$\begin{aligned}\|(W,H) - (\tilde{W},\tilde{H})\| \leqslant &\max(c_{10} + \|T_G\|_{L_p(G),C_\beta(G)}, c_{13} + \|\Pi_G\|_{L_p(G),L_p(G)}) \\ &\times(L_1\pi^{1/p} + L_2)\|(w,h) - (\tilde{w},\tilde{h})\|.\end{aligned}$$

From here it follows immediately that analogously to the lemmas 4.1, 4.2 the following lemma holds.

Lemma 20.4.1 *The operator* (2.10) *corresponding to the Riemann-Hilbert boundary value problem* (4.1) *is continuous.*

Analogous to the arguments in the Subsections 20.2.4 and 20.3.3, the operator (2.10) can be estimated in the polycylinder D. The estimates (2.15) are also valid if $3c_1,\ c_2$ are replaced by $c_{10},\ c_{13}$.

20.4.5 Solution of the Riemann-Hilbert problem (4.1)

On the basis of the estimate (4.16) and the inequalities (2.15) (in which $3c_1,\ c_2$ are replaced by $c_{10},\ c_{13}$) one may prove the following theorem:

Theorem 20.4.1 *Assume that the rigth-hand side $F(z,w,h)$ of the equation $(*)$ satisfies the conditions (I) and (II). Suppose, further, that there exist non-negative numbers $R_1(\leqslant R)$ and R_2 such, that the following inequalities are fulfilled*

$$\begin{aligned}&\|\psi\|_{C_\beta(G)} + (c_{10} + \|T_G\|_{L_p(G),C_\beta(G)})(M + L_1\pi^{1/p}R_1 + L_2R_2) \leqslant R_1,\\ &\|\psi'\|_{L_p(G)} + (c_{13} + \|\Pi_G\|_{L_p(G),L_p(G)})(M + L_1\pi^{1/p}R_1 + L_2R_2) \leqslant R_2.\end{aligned} \tag{4.16}$$

Finally, let the following condition be fulfilled

$$\max(c_{10} + \|T_G\|_{L_p(G),C_\beta(G)}, c_{13} + \|\Pi_G\|_{L_p(G),L_p(G)})(L_1\pi^{1/p} + L_2) < 1. \tag{VIII}$$

Then the normalized Riemann-Hilbert problem (4.1) *for the equation* $(*)$ *is* (*uniquely*) *solvable in the polycylinder D.*

20.5 Application of the Schauder principle

20.5.1 Proof of the existence of solutions of the boundary value problems (2.1), (3.1) and (4.1) on the basis of the Schauder principle

In the paragraphs 2, 3 and 4 the solution of above mentioned boundary value problems was constructed in the space $(C_\beta(\overline{G}), L_p(G))$ using the succesive approximations method. For this purpose we impose on the Lipshitz constants $L_1,\ L_2$ the restrictions (IV), (VI) and (VIII) correspondingly. On the other hand, from the lemmas 4.1-4.3 it follows that the operator (2.10) is continuous with respect of the metric of the space $(C_\beta(\overline{G}), L_p(G))$. The last assertion is valid also without any restrictions on the constants $L_1,\ L_2$. But in the definition of the operator (2.11) the operator Π_G is contained, which is not compact. Therefore the second version of the Schauder principle is not applicable in this case. Consequently, the application of the Schauder principle requires to consider the operator (12) on a compact subset of the space $(C_\beta(\overline{G}), L_p(G))$.

As mentioned above, the polycylinder D is not compact. In order to construct a compact set, first of all note that the ball

$$\{w \in C_\beta(\overline{G}) : \|w\| \leqslant R_1'\}$$

is compact in $C_\beta'(\overline{G})$ if $\beta' < \beta$. Let γ and r be given numbers, $0 < \gamma < 1$ and $r > 1$. The norm in the space $L_r^\gamma(G)$ is usually defined by

$$\|h\|_{L_r^\gamma(G)} = \|h\|_{L_r(G)} + \sup_{z,\Delta z} \frac{\|f(z+\Delta z) - h(z)\|_{L_r(G)}}{|\Delta z|^\gamma}.$$

It is quite enough to consider increments Δz for which $|\Delta z| < 1$. Besides $L_r^\gamma(G)$ the space $L_r^\gamma(G) \cap L_{2r}(G)$ is considered, which is normed by the formula

$$\| \cdot \|_{L_r^\gamma(G)\cap L_{2r}(G)} = \| \cdot \|_{L_r^\gamma(G)} + \| \cdot \| L_{L_{2r}(G)}.$$

For a given $r > 1$ we choose a number p such that $r < p < 2r$. Note secondly, that the ball

$$\{h \in L_r^\gamma(G) \cap L_{2r}(G) : \|h\|_{L_r^\gamma(G)\cap L_{2r}(G)} \leqslant R_2'\}$$

is compact in $L_p(G)$. Therefore we get that the polycylinder

$$D' = \{(w,h) : w\|)_{C_\beta(\overline{G})} \leqslant R_1',\ \|h\|_{L_r^\gamma(G)\cap L_{2r}(G)} \leqslant R_2'\}$$

is compact in $(C_\beta(\overline{G}), L_p(G))$ if $\beta' < \beta$ and $2 < p < 2r$; it is supposed that $R_1' \leqslant R$. The Schauder Principle requires that the operator (2.10) maps the polycylinder D into itself. Thus for the solution ψ of the boundary problems (2.1), (3.1) or (4.1) in the holomorphic case ($F \equiv 0$) one has to demand that $\psi' \in L_r^\gamma(G)$. In order to fulfill this, let us suppose additionally that $dg/ds \in L_r^\alpha(G)$, s is the arc length, $0 < \alpha < 1$, (see the next section).

20.5.2 Solution of the boundary problems (2.1), (3.1) and (4.1) in the holomorphic case with derivatives in $L_r^\gamma(G)$

Let r and α be given numbers, $r > 1$, $0 < \alpha < 1$. As in the above section, the number p is chosen from the interval $2 < p < 2r$. The numbers β, μ and γ are defined by $\beta = 1 - 2/p$, $\mu = 1 - 1/r$, $\gamma = \min(\alpha, 1/r)$. We have to find the solution ψ of the boundary problem (33) in the holomorphic case ($F \equiv 0$). About the vector-function g (more precisely its components) in the considered boundary conditions, it is assumed that

a) g_k is absolutely continuous,

b) $dg_k/ds \in L_r^\alpha(\Gamma)$, s is the arc length [5]

From these assumptions it follows, in particular, that $g \in C_\mu(\Gamma)$ and hence $\psi \in C_\mu(\overline{G})$. Since $\mu > \beta$ the function ψ belongs to the space $C_\beta(\overline{G})$ too. The other properties of the holomorphic function ψ are proved in the Subsections 20.2.3, 20.3.2 and 20.4.3 correspondingly. Additionally we need estimates of the derivative ψ' in the spaces $L_r^\gamma(G)$ and $L_{2r}(G)$. Next we outline these estimates.

In order to solve the Dirichlet type problem (2.1) we have (for our purpose it is sufficient to consider the case $n = 1$):

$$\psi'(z) = \frac{1}{\pi i}\int_\Gamma \frac{dg/dt}{t-z}dt,$$

$$\psi'(e^{is_0}) = -ie^{-is_0}dg/ds_0 - \frac{1}{2\pi}\int_0^{2\pi} e^{-is_0}dg/ds\; ctg\frac{s-1}{2}ds + \frac{1}{2\pi i}\int_0^{2\pi} e^{-is}dg(s);$$

$\psi'(e^{is_0})$ denotes the boundary value of the function $\psi'(z)$ at the point $t_0 = e^{is_0}$.

From these formulas one can see that $\psi'(z)$ belongs to the Hardy class $H_r(G)$, $\psi'(t) \in L_r^\alpha(\Gamma)$ ([48], [24], [140]). On the other hand, from $\psi'(z) \in H_r(G)$ it follows that $\psi'(z) \in L_{2r}(G)$, $\|\psi'\|_{L_{2r}(G)} \leqslant \|\psi'\|_{H_r(G)}$ [55]. Using the arguments mentioned in the book [24] (pp. 78-79) it is possible to show that

$$\psi'(z) \in L_r^\gamma(\Gamma),\quad \gamma = \min(\alpha, 1/r),\quad \|\psi'\|_{L_r^\gamma(\Gamma)} \leqslant const\|g\|_{L_r^\alpha(\Gamma)}.$$

Turn now to the Riemann-Hilbert Problem. The canonical matrix of the correspoding problem of linear conjugation may be constructed in the following way ([96], [108]).

Consider the sequence of the matrices

$$\varphi_m(z) = \tfrac{1}{2\pi i}\int_\Gamma \frac{b_0(t)\varphi_{m-1}^-(t)}{t-z}dt + \frac{1}{2\pi i}\int_\Gamma \frac{a_0(t)}{t-z}dt, \qquad m = 1, 2, \cdots, \varphi_0^- = 0, \tag{5.1}$$

[5] In case of the Riemann-Hilbert problem an analogous condition will be imposed on the matrix $A(t)$.

where $a_0(t) = I + b_0(t) = a(t)R(t),\ R(z)$ is a rational matrix, such that the norm of the matrix $b_0(t)$ will be sufficiently small.

The matrix defined by the equalities

$$\chi_0(z) \lim_{m\to\infty} \varphi_m(z),\ \ z \in G, \quad \chi_0(z) = I + \lim_{m\to\infty} \varphi_m(z),\ \ z \notin G \cap \Gamma,$$

will be a canonical matrix for the matrix $a_0(t)$.

The canonical matrix for the initial matrix $a(t)$ is constructed by the formula

$$\chi(z) = \begin{cases} \chi_0(z)Q(z), & z \in G, \\ R(z)\chi_0(z)Q(z), & z \notin G \cup \Gamma, \end{cases}$$

where $Q(z)$ is the correspondingly chosen rational matrix.

If $a'(t) \in L_r^\alpha(\Gamma),\ 0 < \alpha < 1$, then $b_0'(t) \in L_r^\alpha(\Gamma)$ and we have

$$\varphi_m'(z) = \frac{1}{2\pi i}\int_\Gamma \frac{b_0(t)\varphi_{m-1}'^-(t)}{t-z}dt + \frac{1}{2\pi i}\int_\Gamma \frac{b_0'(t)\varphi_{m-1}^-(t)}{t-z}dt + \frac{1}{2\pi i}\int_\Gamma \frac{a_0'(t)}{t-z}dt,$$
$$m = 1, 2, \cdots, \varphi_0^- = 0, \tag{5.2}$$

From the formulas (5.1), (5.2) we get $\chi_0^{'+}(t),\ \chi_0^{'-}(t) \in L_r^\alpha(\Gamma)$ and hence $\chi^{'+}(t)$, $\chi^{'-}(t) \in L_r^\alpha(\Gamma)$; then from the formulas (4.2), (4.3) we obtain that all solutions of the Riemann-Hilbert problem have the corresponding properties.

The mentioned arguments (and also the analogous arguments, which we may carry out with respect to the modified Dirichlet problem) show that the following proposition is valid.

Lemma 20.5.1 *The solution of the boundary value problems* (2.1) *and* (3.1) *and of the normalized boundary value problem* (4.1) *in the holomorphic case* ($F \equiv 0$) *admits the a-priori estimates*

$$\|\psi\|_{C_\mu(\overline{G})} \leqslant c_{15}\|g\|_{C_\mu(\Gamma)} + c_{16}N(c),$$
$$\|\psi'\|_{L_r^\gamma(G)} \leqslant c_{17}\|g\|_{C_r^\alpha(\Gamma)} + c_{18}N(c),$$
$$\|\psi'\|_{L_{2r}(G)} \leqslant c_{19}\|g\|_{L_r(\Gamma)} + c_{20}N(c).$$

In case of the boundary problems (2.1) and (3.1) $c_{16} = 1$, $c_{18} = c_{20} = 0$, for all these problems $N(c) = |c|$, and for the problem (4.1) $N(c) = \max\limits_{|k,\nu|} |c_k^\nu|$.

20.5.3 Behaviour of the operator (2.10) in the space $L_r^\gamma(G)$

Let $|\Delta z| \leqslant 1$. We have

$$(\Pi_G h)(z + \Delta z) - (\Pi_G h)(z) = -\frac{1}{\pi}\iint\limits_{G_0} \frac{h(\xi + \Delta z) - h(\xi)}{(\xi - z)^2}d\xi d\eta,$$

where G_0 is a disk with the center at the point $z = 0$ and with the correspondingly chosen fixed radius ($G_0 \supset G,\ h \equiv 0$ outside of G).

Taking into account the definition of the norm in the space $L_r^\gamma(G)$, it follows from the last formula that the operator Π_G is a bounded operator in the space $L_r^\gamma(G)$ with the norm $\|\Pi_{G_0}\|_{L_r(G_0),L_r(G_0)}$.

The same is valid for the operator on the right-hand side of the inequality (2.4), since this operator may be reduced to the operator Π_G (see. [20]).

20.5.4 Existence theorem on the basis of the Schauder principle

Strengthening the formula (I), require that $F(z,0,0) \in L_{2r}(G)$. Denote by M the norm $\|F(z,0,0)\|_{L_{2r}(G)}$. For all $(w,h) \in D'$ we have

$$\|w\|_{L_{2r}(G)} \leqslant \|w\|_{C_\beta(\overline{G})}(mG)^{1/2r} \leqslant R_1'(mG)^{1/2r}, \quad \|h\|_{L_{2r}(G)} \leqslant R_2';$$

it follows from (II) that

$$\|f\|_{L_{2r}(G)} \leqslant M + L_1(mG)^{1/2r}R_1' + L_2R_2',$$

where f is defined by (1.2).

We have analogously

$$\|h\|_{L_r(G)} \leqslant \|h\|_{L_r^\gamma(G)} \leqslant R_2',$$

$$\|F(z,0,0)\|_{L_r(G)} \leqslant M(mG)^{1/2r};$$

from (II) if follows

$$\|f\|_{L_r(G)} \leqslant M(mG)^{1/2r} + L_1(mG)^{1/r}R_1' + L_2R_2'.$$

In order to estimate the $L_r^\gamma(G)$- norm of the element f, we consider the expression

$$\begin{aligned}&|F(z+\Delta z), w(z+\Delta z), h(z+\Delta z) - F(z, w(z), h(z))|\\ &\leqslant L_1|w(z+\Delta z) - w(z)| + L_2|h(z+\Delta z) - h(z)|\\ &+|F(z+\Delta z, w(z), h(z)) - F(z, w(z), h(z))|.\end{aligned}$$

Since the first summand on the right-hand side may be estimated by $L_1\|w\|_{C_\beta(\overline{G})}$ $|\Delta z|^\beta \leqslant L_1R_1'|\Delta z|^\beta$, from this inequality it follows

$$\begin{aligned}&\|F(z+\Delta z, w(z+\Delta z), h(z+\Delta z)) - F(z, w(z), h(z))\|_{L_r(G)}\\ &\leqslant L_1R_1'|\Delta z|^\beta(mG)^{1/r} + L_2\|h(z+\Delta z) - h(z)\|_{L_r(G)}\\ &+\|F(z+\Delta z, w(z), h(z)) - F(z, w(z), h(z))\|_{L_r(G)}.\end{aligned}$$

Now assume additionally that

$$\|F(z_2,w,h)-F(z_1,w,h)\|_{L_r(G)}\leqslant l|z_2-z_1|^{\tau}\quad(0<\tau\leqslant 1).\tag{IX}$$

Dividing the last inequality by $|\Delta z|^{\gamma}$, we get the following result:
If $\gamma\leqslant\min(\beta,\tau)$, then $f\in L_r^{\gamma}(G)$ and

$$\|f\|_{L_r^{\gamma}(G)}\leqslant M+L_1((mG)^{1/2r}+(mG)^{1/r})^{1/r}R_1'+2L_2R_2'+l.$$

Let now

$$c_{21}=3c_1,\quad c_{22}=\tilde{c}_2,\quad c_{23}=\hat{c}_2$$

in case of the boundary problem (2.1),

$$c_{21}=3c_7,\quad c_{22}=\tilde{c}_8,\quad c_{23}=\hat{c}_8$$

in case of the boundary problem (3.1) and

$$c_{21}=3c_{10},\quad c_{22}=\tilde{c}_{13},\quad c_{23}=\hat{c}_{13}$$

in case of the Riemann-Hilbert problem (4.1). The constant c_1,c_7,c_{10} are corresponding to the space $L_p(G)$, the constants $\tilde{c}_2,\tilde{c}_8,\tilde{c}_{13}$ - to the space $L_r(G)$, and the constants $\hat{c}_7,\hat{c}_8,\hat{c}_{13}$ to the space $L_{2r}(G)$.

Note that

$$\|f\|_{L_p(G)}\leqslant\|f\|_{L_{2r}(G)}(mG)^{1/p-1/2r}.$$

Taking into account the arguments of the above section, we get the following estimates

$$\|W\|_{C_\beta(\overline{G})}\leqslant\|\psi\|_{C_\beta(\overline{G})}+\Big(c_{21}+\|T_G\|_{L_p(G),C_\beta(\overline{G})}\Big)$$
$$\times(M+L_1(mG)^{1/2r}R_1'+L_2R_2')(mG)^{1/p-1/2r},$$

$$\|H\|_{L_r^{\gamma}(G)}\leqslant\|\psi'\|_{L_r^{\gamma}(G)}+\big(c_{22}+\|\Pi_{G_0}\|_{L_r(G_0),L_r(G_0)}\big)$$
$$\times(M(mG)^{1/2r}+2L_1(mG)^{1/r}R_1'+2L_2R_2'+l),$$

$$\|H\|_{L_{2r}(G)}\leqslant\|\psi'\|_{L_{2r}(G)}+\big(c_{23}+\|\Pi_G\|_{L_{2r}(G),L_{2r}(G)}\big)$$
$$\times(M+L_1(mG)^{1/2r}R_1'+L_2R_2')$$

instead of (2.15).

On the other hand, due to the lemmas 4.1-4.3, the operator (2.10) is continuous in $(C_\beta(\overline{G}),L_p(G))$ and hence also in $(C_{\beta'}(\overline{G}),L_p(G))$, $\beta'<\beta$. Using the Schauder principle one can prove the following theorem.

Theorem 20.5.1 *Let r and α be given numbers, $r > 1,\ 0 < \alpha < 1$. Suppose, that the right-hand side $F(z, w, h)$ of the equation $(*)$ satisfies the conditions (I) (where p is to be replaced by $2r$), (II) and (IX).*

We consider one of the boundary value problems (2.1), (3.1) or (4.1) where it is assumed that the given boundary functions are absolutely continuous and their derivative $dg_k/ds \in L_r^\alpha(\Gamma)$, s is the arc length[1]

The number p is chosen in the interval $2 < p < 2r$, and $\beta = 1 - 2/p$. Define the number γ, which is contained in the definition of the polycylinder D' by[2]

$$\gamma = \min(1/r, \alpha, \beta, \tau).$$

Assume then that there exist the non-negative numbers $R_1' (\leqslant R)$ and R_2' such that the following inequalities are fulfilled[3]

$$\begin{aligned}
&\|\psi\|_{C_\beta(\overline{G})} + \Big(c_{21} + \|T_G\|_{L_p(G), C_\beta(\overline{G})}\Big) \\
&\times \Big(M + L_1 (mG)^{1/2r} R_1' + L_2 R_2'\Big) (mG)^{1/p - 1/2r} \leqslant R_1, \\
&\|\psi'\|_{L_r^\gamma(G) \cap L_{2r}(G)} + \big(c_{22} + \|\Pi_{G_0}\|_{L_r(G_0), L_r(G_0)}\big) \\
&\times \Big(M (mG)^{1/2r} + 2L_1 (mG)^{1/r} R_1' + 2L_2 R_2' + l\Big) \\
&+ \big(c_{23} + \|\Pi_G\|_{L_{2r}(G), L_{2r}(G)}\big) \Big(M + L_1 (mG)^{1/2r} R_1' + L_2 R_2'\Big) \leqslant R_2.
\end{aligned} \tag{X}$$

Then there exists at least one solution w of the differential equation $(*)$ satisfying the boundary condition (2.1), (3.1) or (4.1) and such that $(w, \partial w/\partial z)$ belongs to the polycylinder D'.

20.5.5 Concluding remarks

a) Using the Schauder principle (instead of the Banach fixed point theorem), the conditions (IV), (VI) and (VIII) are superfluous, and the conditions (III), (V) or (VII) are to be replaced by the modified condition (X). On the other hand, the Schauder principle needs the additional assumptions (IX) with respect to the right-hand side (this requirement together with the condition (II) provides the measurability of the function (1.2)). Moreover, we shall assume that the given boundary values g have derivatives $dg/ds \in L_r^\alpha(\Gamma)$, s is the arc length.

[1] In case of the Riemann-Hilbert problem an analogous condition is to be satisfied by the matrix $A(t)^-$.

[2] $L_r^{\gamma_1} \supset L_r^{\gamma_2}$, if $\gamma_1 < \gamma_2$.

[3] This condition provides that the operator (2.10) maps D' into itself. Using the lemma 4, we may estimate the norms of ψ, ψ' by the norms of the given boundary function g.

b) In case when $r > 2$ we may take $p = r$. Additionally, it is sufficient to consider the space $L_r^\gamma(G)$ intead of $L_r^\gamma(G) \cap L_{2r}(G)$. Hence, the second condition (X) is simplified.

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