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**(6.V.1924 - 13.IV.1999)**

**Boundary Value Problems with Shift for Generalized Analytic Vectors**

**( In honor of Professor Giorgi Manjavidze)**

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 The present work is devoted to boundary value problems of linear conjugation with shift for generalized analytic vectors.

A great place in the works of Giorgi Manjavidze takes the investigation of boun-dary value problems of the theory of functions with shifts. In such problems the boundary values of the desired functions are conjugating in the points which are displaced to each other. The model problem is to find a function holomorphic on the complex plane , cut along some simple closed curves, the boundary values of which and are satisfying the condition

 (\*)

from both sides of , where are given continuous functions on , is continuous function mapping onto in one-to-one manner.

Later on result obtained by him in this direction became known as the theory of conformal sewing. He also studied the problem with shift depended on parameter and proved the invariance of partial indices for conformal mappings. A special cycle of G. Manjavidze’s works is devoted to the investigation of boundary value problems of the function theory by method of successive approximations by means of which solution of the problems with shifts in case of several unknown functions was considerably simplified.

G.Manjavidze obtained marked results for discontinuous boundary value problems with shift for generalized analytic vectors. He studied the Riemann-Hilbert prob-lem in domains with non-smooth boundaries, the Riemann-Hilbert-Poincare prob-lem for generalized analytic functions, and the differential boundary value problem of linear conjugation. In particular, he established both the solvability conditions of these problems and the index formulas and discovered the connection between the problems with shift for analytic and generalized analytic functions.

**1**. **Generalized analytic vectors (** Definitions and Notations).

A vector is called generalized analytic in domain D if it is a solution of the elliptic system

 (1.1)

where are given square matrices of order n of the class

and where is a matrix of the special form: it is quasi-diagonal and every block is lower (upper) triangular matrix satisfying the conditions:

Moreover, and outside of some circle. Under the solution of (1.1) we mean so-called regular solution , i.e. , the gene-ralized derivatives of which where is arbitrary closed subset of .

If then

 (1.2)

and the solutions of (1.2) are called − holomorphic vectors.

The equation (1.2) has a solution of the form

 , (1.3)

where is unit matrix and is a solution of the equation

belonging to are well-known integral operators. The solution (1.3) is analogous of the fundamental homeomorphism of Beltrami equation. The matrix

 (1.4)

is called generalized Cauchy kernel for the equation (1.2), and consider Cauchy type generalized integral

 (1.5)

where is closed simple smooth curve, and

If the density in (1.5) is Holder-continuous on then (1.5) is Holder-conti-nuous in and ; the boundary values of on are given by

 (1.6)

If , then (1.6) are fulfilled almost everywhere on , provided that are angular boundary values of the vector . Here very important role play the analogous integral operators

 (1.7)

If then is completely continuous operator from , onto , moreover the operator is linear bounded operator from to , and

 (1.8)

Using holomorphic vectors the generalized analytic vectors can be represented as follows (Bojarski B. Theory of Generalized Analytic Vector.)

 (1.9)

where is holomorphic vector, , ( is a complete sys­tem of linearly independent solutions of Fredholm equation

 , (1.10)

 turn to be continuous vectors in whole plane vanishing at infinity, are arbitrary real constants, the kernels satisfy integral equations in turn.

Finally the vector has to satisfy the following conditions

 (1.11)

where form system of linearly independent solutions of Fredholm integral equation

It should be mentioned that, generally speaking, the Liouville theorem is not true for the solutions of (1.1). This explains the appearance of the constants in (1.9) and that the condition (1.11) has to be satisfied.

**2. Relation between BVP of Linear Conjugation and Generalized Analytic Functions**

Now we show the connection of linear conjugation problem with shift and the theory of generalized analytic functions. This gives the possibility to consider the problem of linear conjugation in somewhat different formulation.

Let and be Liapunov curves, is function mapping onto in one-to-one manner preserving orientation, is absolutely continuous function, are constants), are given matrices of the class is nonsingular square matrix of order , is matrix.

Find piecewise-holomorphic matrix having finite order at infinity, and satisfying the boundary condition

 (2.1)

We call the piecewise-holomorphic matrix with finite order at infinity the canonical matrix of the problem (2.1) if everywhere except perhaps at the point has a normal form at infinity with respect to columns and

Mapping conformally and into interior and exterior parts of the unit circle respectively we get the same problem (2.1), where maps onto ; the matri-ces have the same properties. Consider the problem in case

After proving some useful propositions we get the following

**Theorem 1.** All solutions of the problem (2.1) are given by the formulas

 (2.2)

where is arbitrary polynomial vector and the vector , is a solution (unique) of the equation

The solutions vanishing at infinity are given by the formulas (2.2) where

are the partial indices of the problem (2.1), is arbitrary polynomial of order if ; if then the vector has to satisfy the following conditions

where

**3. BVP of Linear Conjugation with Shift for Generalized Analytic Vectors.**

Define the classes for -holomorphic vectors. Let be finite (infinite) domain which is bounded by a simple closed Liapunov smooth curve .

Denote by , (D is one of the domains , is a Sobolev space) the class of − holomorphic vectors in the domain satisfying the following conditions

 (3.1)

where is a constant, is an image of the circle , while quasicon-formal mapping of the unit circle onto , is an analytic function in the domain, is a fundamental homeomorphism of the Beltra­mi equation

If is infinite domain , then for the simplicity of notation we suppose that (remind that − holomorphic vectors are the analytic functions in vici-nity of the point z = ∞, because at infinity). By denote the class of the vectors, belonging to the class for some , for which the angular boundary values are belonging to

 (3.2)

Le and are two given matrices, satisfying the conditions when By we denote the class of the vectors defined on cut along plane, belonging to the class (in the domain (), Introduce the classes of the generalized analytic vectors, satisfying the equation of the form

 (3.3)

in case of infinite domain we suppose, that are equal to zero at infinity.

By , , denote the class of the solutions of the equation (3.3) satisfying the conditions

the curve is defined above, is a constant; if is infinite domain, then . Here we consider, that when .

By, , (t) is a function of the form (3.2), denote the class of the vectors , belonging to the class for some λ > 1, for which the angular boundary values of the vector.

 denotes the class of the vectors defined on the plane cut along and belonging to the class ( ] in ().

Consider the following boundary value problem:

Find a vector of the class satisfying boun-dary condition

 (3.4)

 is simple closed Liapunov curve, is function mapping onto in one-to-one manner, preserving the orientation , are given piecewise-continuous matrices on is given vector of the class

The following proposition holds.

**Lemma.** If the vector then it is uniquely representable in the form

 (3.5)

where is a solution of Fredholm integral equation

 (3.6)

 is inverse of .

Substituting the representation (3.5) in the boundary condition (3.4) for the desired vector we obtain the singular integral equation

 (3.7)

The following result holds.

**Theorem 2.** If the equation (3.7**)** is Noetherian in the space then the boundary problem (3.4) is Noetherian in ; the necessary and sufficient solvability conditions have the form

 (3.8)

where is a complete system of linearly independent solutions of conjugate homogeneous equation of the class the index of the problem (3.4) of the class is equal to the index of the equation (3.7) of the class

Consider now the problem (3.4) for generalized analytic vectors satisfying the equations of the form (3.3):

Find the vector satisfying the boundary condition

 (3.9)

We seek the solution of (3.9) by the formula (1.9) in the following form

 (3.10)

where are desired real numbers, - solutions of the corresponding integral equations. The vectors have to satisfy the conditions

where are the complete systems of the homogeneous conjugate equations. With respect to the vector we obtain the following boundary problem

 (3.12)

the operators and are defined as follows

 ],

 (].

Substituting the representations (3.5) first in these formulas we obtain that the ope­rators and are completely continuous operators in with respect to the angular boundary values and then in (3.12) we get singular integral equation with respect to the vector

 (3.13)

where is completely continuous operator, and are singular integral ope­rators

 are linearly independent continuous vectors, represented by are desired real constants.

Besides (3.13) the vector has to satisfy the conditions

 (3.14)

where are linearly independent vectors, represented by .

The necessary and sufficient solvability conditions of the problem (3.9) in the class have the form (Prossdorf S. Some Classes of Singular Equations.)

 (3.15)

where the linearly independent vectors ) belonging to the class , are representable by and by the vectors composing the basis of subspace of the solutions of adjoint homogeneous equation The index of the problem (3.9) is equal to

 (3.16)

where is the index of the operator of the class . Actually in the formula (3.16).

Let be sets of the vectors defined in the domains representable in the form

 (3.17)

This pair of sets coincides with the class . Introduce the norms

 (3.18)

 are Banach spaces. Let be new Banach space with the norm It is evident, that this norm in is independent of Consider the set of the operators

where In order to calculate the index of the problem (3.9) we may take the differential operators of the form and for such operators the numbers are equal to zero and hence in (3.16).

**Theorem 3.**The necessary and sufficient solvability conditions of the problem (3.9) in the class are the conditions (3.15); the index of this problem is equal to the index of the operator .

Note that if the matrices and are continuous then the index of any class is given by the formula

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