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Differential Boundary Value Problem of the Theory of Analytic Functions on a Cut Plane

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The differential boundary value problem in case of several unknown functions is considered. The necessary and sufficient solvability conditions and the index formula are established.

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Let D denote the plane of the complex variable $z = x + iy$, cut along some Liapunov-smooth arcs $a_k b_k, k = 1, \dots, p$. Denote $\Gamma_k = a_k b_k$ and $\Gamma = \bigcup_{k=1}^{p} \Gamma_k$. Let us consider the following boundary value problem: Find an analytic vector

$$\phi(z) = (\phi_1(z), \dots, \phi_n(z))$$

satisfying the boundary condition

$$\operatorname{Re} \sum_{k=0}^{m} \left[a_{\pm}^{(k)}(t) \phi_{\pm}^{(k)}(t) \right] = f_{\pm}(t), \quad t \in \Gamma, \tag{1}$$

where $a_+^{(k)}(t), a_-^{(k)}(t)$, $k = 0, \ldots, m$, are given quadratic matrices of order n on Γ, $f_+(t), = (f_+^1, \ldots, f_+^n)$, $f_-(t) = (f_-^1, \ldots, f_-^n)$ are given real vectors on Γ, $a_+^{(k)}(t)$, $a_-^{(k)}(t)$ are Hölder-continuous matrices, $f_+(t), f_-(t)$ are satisfying the condition, cf. [1–3],

$$f(t)\Pi(t) \in H^*(\Gamma), \quad \Pi(t) = \prod_{k=1}^{p} (t - a_k)^{m-1}(t - b_k)^{m-1}.$$

In the boundary condition (1) $\phi_+^{(k)}(t)$, $\phi_-^{(k)}(t)$ denotes the boundary values of the vector

$$\frac{d^k \phi(z)}{dz^k}$$

from the left and from the right on Γ. The boundary condition (1) will be fulfilled everywhere on Γ except at the end points a_k, b_k. In the neighbourhood of these points the estimation

$$|\phi^{(m)}(z)| \leq \text{const}\, |z - c|^{-\beta}, \quad 0 \leq \beta < m, \quad c = a_k, \text{ or } b_k. \tag{2}$$

holds.

At the point of infinity $\phi(z)$ may have a pole

$$\phi_k(z) = q_k(z) + O\left(\frac{1}{z}\right), \quad (k = 1, \ldots, n),$$

where $q_k(z)$ is a polynomial.

If we introduce a new unknown vector

$$\psi(z) = \phi^{(m)}(z), \tag{3}$$

then

$$\left.\begin{aligned} \phi(z) &= \frac{1}{(m-1)!} \int_{z_0}^{z} (z - \xi)^{m-1} \psi(\xi) d\xi + P(z), \\ \phi^{(k)}(z) &= \frac{1}{(m-k-1)!} \int_{z_0}^{z} (z - \xi)^{m-k-1} \psi(\xi) d\xi + P^{(k)}(z), \\ & k = 1, \ldots, m-1, \end{aligned}\right\} \tag{4}$$

where z_0 is a fixed finite point in $D, P(z)$ is a polynomial vector of order not more then $m - 1$,

$$P(z) = \sum_{k=0}^{m-1} c_k (z - z_0)^k, \tag{5}$$

where $c_k = (c_k^1, \ldots, c_k^n)$ are constant vectors, $c_k = \phi^{(k)}(z_0)/k!$.

In formulas (4) the integration is made along a piecewise smooth arc, which is situated in D and is connecting the point z_0 with the variable point z. Because of the independence of these integrals from the path of integration, the conditions

$$\int_{\lambda_j} t^k \psi(t)dt = 0, \quad k = 0, 1, \ldots, m-1, \quad j = 1, \ldots, p, \tag{6}$$

should be fulfilled, where λ_j denotes a piesewise smooth simple closed curve, which surrounds the arc Γ_j and the remaining arcs Γ_k are outside of it.

Therefore, with respect to the vector $\psi(z)$ we get the following boundary condition on Γ

$$\operatorname{Re}\left[a_{\pm}^{(m)}(t)\psi_{\pm}(t) + \sum_{k=0}^{m-1} a_{\pm}^{(k)}(t)\rho_{\pm}^{(k)}(t)\right] = f_{\pm}(t) - \operatorname{Re}\sum_{k=0}^{m-1} a_{\pm}^{(k)}(t)P^{(k)}(t), \tag{7}$$

where $\rho_{\pm}^{(k)}(t)$ denotes the boundary values of the vector

$$\frac{1}{(m-k-1)!}\int_{z_0}^{z}(z-\xi)^{m-k-1}\psi(\xi)d\xi$$

from both sides on Γ; $\psi(z)$ will satisfy the integral conditions (6) and the estimation:

$$|\psi(z)| \leq \text{const}|z-c|^{-\beta}, \quad c = a_k \text{ or } b_k. \tag{8}$$

In the following we need two lemmas.

LEMMA 1 *Let $\phi(z)$ be a holomorphic function in the domain δ, which represents a circle cut along the smooth arc $ab, a \in \delta$ and $\phi'(z)$ satisfy the condition*

$$|\phi'(z)| \leq A|z-a|^{-\beta}, \quad 0 \leq \beta. \tag{9}$$

When $\beta > 1$ then $\phi(z)$ satisfies the condition

$$|\phi(z)| \leq B|z-a|^{1-\beta}, \tag{10}$$

when $\beta < 1$, then $\phi(z)$ satisfies the Hölder-condition with the exponent $1-\beta$ in δ.

Proof Without loss of generality we can assume, that δ is the unit circle cut along the interval [0,1]. Let us take the point z_0, $\mathrm{Im} z_0 = 0$, $-1 < \mathrm{Re} z_0 < 0$ and write the formula

$$\phi(z) - \phi(z_0) = \int_{z_0}^{z} \phi'(\xi) d\xi, \tag{11}$$

where the integration is made along a curve, which is connecting the points z_0 and z. It consists of the interval $(z_0, -|z|)$ of the real axis and the arc of the circle with center in the point $z = 0$ connecting the points $-|z|$ and z.

From (6) we have

$$\begin{aligned} \phi(z) - \phi(z_0) &= -z_0 \int_0^{\tau_0} \phi'(1-\tau) d\tau + |z| i \int_{\pi}^{\theta_0} \phi'(|z| e^{i\theta}) d\theta, \\ \tau_0 &= 1 - \left|\frac{z}{z_0}\right|, \quad |\theta_0 - \pi| \leq \pi, \\ |\phi(z) - \phi(z_0)| &\leq \mathrm{A} |z_0|^{1-\beta} \int_0^{\tau_0} (1-\tau)^{\beta} d\tau + \mathrm{A} |z|^{1-\beta} |\theta_0 - \pi| \\ &= \mathrm{A}\left\{ |\beta - 1|^{-1} \left| |z|^{1-\beta} - |z_0|^{1-\beta} \right| + |z|^{1-\beta} \pi \right\}. \end{aligned} \tag{12}$$

The inequality (12) proves the lemma in case, when $\beta > 1$.

Now let us consider the case, when $\beta < 1$. We have

$$\int_{z_1}^{z_2} \phi'(\xi) d\xi = \phi(z_2) - \phi(z_1). \tag{13}$$

In the formula (13) the integration is taken along the following path. Consider the following two cases:

I. The points z_1 and z_2 are situated on the radius, which originates from the point $z = 0$, $z_1 = |z_1| e^{i\theta}$, $z_2 = |z_2| e^{i\theta}$, $|z_1| < |z_2|$

$$\phi(z_2) - \phi(z_1) = \int_{|z_1|}^{|z_2|} \phi'(r e^{i\theta}) dr,$$

$$\begin{aligned} |\phi(z_2)| - |\phi(z_1)| &\leq \mathrm{A} \int_{|z_1|}^{|z_2|} r^{-\beta} dr = \frac{\mathrm{A}}{1-\beta} \left[|z_2|^{1-\beta} - |z_1|^{1-\beta} \right] \\ &\leq \mathrm{const} |z_2 - z_1|^{1-\beta}. \end{aligned}$$

II. The modules of z_1 and z_2 are equal,

$$z_1 = re^{i\theta_1}, \quad z_2 = re^{i\theta_2},$$

then

$$\phi(z_2) - \phi(z_1) = \int_{\theta_1}^{\theta_2} \phi'(re^{i\theta}) ri^{i\theta} d\theta,$$

$$|\phi(z_2) - \phi(z_1)| \leq \mathrm{A}\left|\int_{\theta_1}^{\theta_2} r^{1-\beta} d\theta\right| = \mathrm{A}r^{1-\beta}|\theta_2 - \theta_1|.$$

However

$$|\theta_2 - \theta_1| \leq K\left|e^{i\theta_2} - e^{i\theta_1}\right| < K_1\left|e^{i\theta_2} - e^{i\theta_1}\right|^{1-\beta}.$$

therefore

$$|\phi(z_2) - \phi(z_1)| \leq AK_1\left|re^{i\theta_2} - re^{i\theta_1}\right|^{1-\beta}.$$

Consequently, the function $\phi = \phi(re^{i\theta})$ satisfies a Hölder-condition separately as well with respect to the variable r, as with respect to the variable $e^{i\theta}$ and the lemma is proved.

From this lemma the next result follows.

COROLLARY *Let $\phi(z)$ be a holomorphic function in the neighbourhood of a smooth arc ab, and $\phi^{(m)}(z)$ be continuously extendable on ab from both sides, except, possibly, the points a and b and satisfy the estimation*

$$|\phi^{(m)}(z)| \leq \text{const}|z - c|^{-\alpha}, \quad c = a \text{ or } b, \quad 0 \leq \alpha < m;$$

then $\phi(z), \phi'(z), \ldots, \phi^{(m-1)}(z)$ are continuously extendable functions on each point of ab from both sides except, possibly, the points a and b and are satisfying the estimation

$$|\phi(z)| \leq \text{const}, \quad \left|\phi^{(k)}(z)\right| \leq \text{const}|z - c|^{-\alpha_k}, \quad c = a \text{ or } b,$$
$$0 \leq \alpha_k < k, \quad k = 1, 2, \ldots, m-1.$$

Besides, the function $\phi(z)$ is satisfying the Hölder-condition in the domain.

LEMMA 2 *Let $\omega(\xi) = \int_{ab} \mu(t)dt/(t - \xi)$, $\mu(t) \in H^*(a, b)$, the function $\omega(\xi)/((\xi - a)(\xi - b))$ has a single-valued primitive function $F(\xi)$; then* *the function* $H(z) = \Pi(z)F(z) = (z-a)(z-b)F(z)$ *is continuously extendable on the arc ab from both sides and satisfies a Hölder-condition in the neighbourhood of the arc ab.*

Proof Rewrite the function $H(z)$ in the following way:

$$H(z) = \int_{z_0}^{z} \omega(\xi)d\xi + \int_{z_0}^{z} (z-\xi)^2 \frac{\omega(\xi)}{(\xi-a)(\xi-b)} d\xi$$
$$+ \int_{z_0}^{z} (z-\xi)\frac{\omega(\xi)}{\xi-a} d\xi + \int_{z_0}^{z} (z-\xi)\frac{\omega(\xi)}{\xi-b} d\xi.$$

There occurs the following estimations

$$|\omega(z)| \leq \frac{\text{const}}{|z-c|^{\beta}}, \quad 0 \leq \beta < 1, \quad c = a \text{ or } b,$$

$$\left|\frac{\omega(\xi)}{(\xi-a)(\xi-b)}\right| \leq \frac{\text{const}}{|\xi-c|^{\beta+1}}, \quad \left|\frac{\omega(\xi)}{\xi-a}\right| \leq \frac{\text{const}}{|\xi-a|^{\beta+1}}, \quad \left|\frac{\omega(\xi)}{\xi-b}\right| \leq \frac{\text{const}}{|\xi-b|^{\beta+1}}.$$

As a result all four terms are satisfying a Hölder-condition and thus the functions $H^{+}(t)$ and $H^{-}(t)$ are satisfying a Hölder-condition on ab.

Therefore, if the function $\psi(z)$ is the solution of the boundary value problem (7), then the vector

$$\psi_1(z) = \int_{z_0}^{z} \psi(\xi)d\xi$$

is admitting the estimation (when $\beta > 1$)

$$|\psi_1(z)| \leq \text{const}|z-c|^{1-\beta}$$

and the vector

$$\Omega(z) = \Pi(z)\psi_1(z)$$

is continuously extendable everywhere on Γ from both sides, in the points a_k, b_k limits are equal to zero and the boundary values $\Omega^{+}(t)$, $\Omega^{-}(t)$ are satisfying a Hölder-condition.

The same we can say with respect to the vector $\Pi(z)\rho^{(k)}(z)$.

Multiplying the boundary condition (7) by the function $|\Pi(t)|$, we have

$$\operatorname{Re}\left[\frac{a_{\pm}^{(m)}(t)|\Pi(t)|}{\Pi(t)}\Pi(t)\psi_{\pm}(t)+|\Pi(t)|\sum_{k=0}^{m-1}a_{\pm}^{(k)}(t)\rho_{\pm}^{(k)}(t)\right]=F_{\pm}(t) \quad (14)$$

where

$$F_{\pm}(t)=|\Pi(t)|\left[f_{\pm}(t)-\operatorname{Re}\sum_{k=0}^{m-1}a_{\pm}^{(k)}(t)p^{(k)}(t)\right].$$

If we introduce a new unknown vector

$$\omega(z)=\Pi(z)\psi(z), \quad (15)$$

then $\omega(z)$ will satisfy the estimation

$$|\omega(z)|\leq \mathrm{A}|z-c|^{-\alpha}, \quad 0\leq\alpha<1,$$

and that is why we can represent it as

$$\omega(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{\mu(t)dt}{t-z}+Q(z), \quad (16)$$

where $Q(z)$ is a polynomial vector $Q(z)=\Pi(z)q^{(m)}(z)+\gamma(z)$, where $\gamma(z)$ is a polynomial vector of order not more then $s=2p(m-1)-m-1$. The desired vector $\mu(t)$ belongs to $H^{*}(\Gamma)$.

Inserting the representation (16) in the boundary condition (14), we get

$$\operatorname{Re}\left[b_{\pm}(t_0)\left(\pm\frac{1}{2}\mu(t_0)+\frac{1}{2\pi i}\int_{\Gamma}\frac{\mu(t)dt}{t-t_0}\right)\right]+L_{\pm}\mu=g_{\pm}(t_0), \quad (17)$$

where

$$g_{\pm}(t)=F_{\pm}(t)+R_{\pm}(t), \quad b_{\pm}(t)=a_{\pm}^{(m)}(t)\frac{|\Pi(t)|}{\Pi(t)}. \quad (18)$$

In (18) $R_{\pm}(t)$ linearly contains the coefficients of the vector $Q(z)$ and $L_{\pm}\mu$ are completely continuous operators.

We may rewrite equation (16) in the following way

$$\left.\begin{aligned}
&\frac{b_+(t_0)}{2}\left[\mu(t_0)+\frac{1}{\pi i}\int_\Gamma\frac{\mu(t)dt}{t-t_0}\right]+\frac{\overline{b_+(t_0)}}{2}\left[\overline{\mu(t_0)}-\frac{1}{\pi i}\int_\Gamma\frac{\overline{\mu(t)dt}}{t-t_0}\right]\\
&\qquad +M_+\mu=2g_+(t_0),\\
&\frac{b_-(t_0)}{2}\left[-\mu(t_0)+\frac{1}{\pi i}\int_\Gamma\frac{\mu(t)dt}{t-t_0}\right]+\frac{\overline{b_-(t_0)}}{2}\left[\overline{-\mu(t_0)}-\frac{1}{\pi i}\int_\Gamma\frac{\overline{\mu(t)dt}}{t-t_0}\right]\\
&\qquad +M_-\mu=2g_-(t_0),
\end{aligned}\right\} \tag{19}$$

Let us represent the vector $\mu=\mu_1+i\mu_2$, where μ_1 and μ_2 are the real vectors, then we have

$$\left.\begin{aligned}
&\frac{b_+(t_0)+\overline{b_+(t_0)}}{2}\mu_1(t_0)+\frac{1}{2}\left[b_+(t_0)-\overline{b_+(t_0)}\right]\mu_2(t_0)+\frac{b_+(t_0)-\overline{b_+(t_0)}}{2}\\
&\times\frac{1}{\pi i}\int_\Gamma\frac{\mu_1(t)dt}{t-t_0}+i\frac{b_+(t_0)+\overline{b_+(t_0)}}{2}\frac{1}{\pi i}\int_\Gamma\frac{\mu_2(t)dt}{t-t_0}+M_+\mu=2g_+(t_0),\\
&-\frac{b_-(t_0)+\overline{b_-(t_0)}}{2}\mu_1(t_0)-\frac{1}{2}\left[b_-(t_0)-\overline{b_-(t_0)}\right]\mu_2(t_0)+\frac{b_-(t_0)-\overline{b_-(t_0)}}{2}\\
&\times\frac{1}{\pi i}\int_\Gamma\frac{\mu_1(t)dt}{t-t_0}+i\frac{b_-(t_0)+\overline{b_-(t_0)}}{2}\frac{1}{\pi i}\int_\Gamma\frac{\mu_2(t)dt}{t-t_0}+M_-\mu=2g_-(t_0).
\end{aligned}\right\} \tag{20}$$

Considering the vector $\nu=(\operatorname{Re}\mu,\operatorname{Im}\mu)$ with $2n$ components, we will reduce (7) to the following singular integral equation

$$A(t_0)\nu(t_0)+\frac{B(t_0)}{\pi i}\int_\Gamma\frac{\nu(t)dt}{t-t_0}+K\nu=g, \tag{21}$$

where

$$A=\begin{bmatrix}\operatorname{Re}b_+, & -\operatorname{Im}b_-\\ -\operatorname{Re}b_-, & \operatorname{Im}b_-\end{bmatrix},\quad B=i\begin{bmatrix}\operatorname{Im}b_+, & \operatorname{Re}b_+\\ \operatorname{Im}b_-, & \operatorname{Re}b_-\end{bmatrix},\quad g=(2g_+,2g_-), \tag{22}$$

and $K\nu$ is a completely continuous operator.

Therefore

$$A+B=\begin{bmatrix} b_+, & ib_+ \\ -\overline{b}_-, & i\overline{b}_- \end{bmatrix}, \quad A-B=\begin{bmatrix} \overline{b}_+, & -\overline{b}_+ i \\ -\overline{b}_-, & -b_{-i} \end{bmatrix} \tag{23}$$

It is evident that $\det(A+B)\neq 0$, $\det(A-B)\neq 0$, if $\det a_{\pm}^{(m)}(t)\neq 0$. Consider the matrices

$$(A+B)^{-1}=\frac{1}{2}\begin{bmatrix} b_+^{-1}, & -i\overline{b}_-^{-1} \\ -ib_+^{-1}, & -i\overline{b}_-^{-1} \end{bmatrix} \tag{24}$$

$$G(t)=(A+B)^{-1}(A-B) \tag{25}$$

and the equations

$$\det\big[G(b_k-0)-\lambda I\big]=0, \tag{26}$$

$$\det\big[G(a_k+0)+\lambda I\big]=0. \tag{27}$$

Denote by $\lambda_j^k (k=1,\dots,p,\ j=1,\dots,n)$ the roots of equation (26) and by $\lambda_j^{p+k} (k=1,\dots,p,\ j=1,\dots,n)$ the roots of equation (27). Then introduce the notations

$$\mu_j^k=\frac{1}{2\pi i}\ln\lambda_j^k, \quad k=1,2,\dots,2p, \quad -1<\operatorname{Re}\mu_j^k\leq 0. \tag{28}$$

We call the point from a_k or b_k singular if the corresponding

$$\operatorname{Re}\mu_j^k=0, \quad j=1,\dots,n,$$

and we call the other points non-singular [1]. We seek the solution of (1) in $H^*(\Gamma)$. The solution will satisfy the condition (6), which has the form

$$\int_{\lambda_j}\tau^k[\Pi(\tau)]^{-1}\left(\frac{1}{2\pi i}\int_\Gamma\frac{\mu(t)dt}{t-\tau}+Q(\tau)\right)d\tau=0,$$

$$j=1,\dots,p,\ k=0,\dots,m-1, \tag{29}$$

or

$$\frac{1}{2\pi i}\int_{\Gamma}\mu(t)\left(\int_{\lambda_j}\frac{\tau^k[\Pi(\tau)]^{-1}d\tau}{t-\tau}\right)dt+\int_{\lambda_j}\tau^k[\Pi(\tau)]^{-1}Q(\tau)d\tau=0\,. \qquad (30)$$

Therefore, $\mu(t)\in H^*(\Gamma)$ will satisfy (21), the right-hand side of which we can represent as

$$g(t)=g_0(t)+\sum_{k=1}^{N}d_k g_k(t), \qquad (31)$$

where the vector $g_0(t)$ is representable by the vectors $f_+(t)$ and $f_-(t)$ and $g_k(t)$ are linearly independent vectors, d_k are arbitrary real constants, depending on the coefficients of the polynomial vectors $P(t)$ and $Q(t)$.

The vectors $f_+(t)$ and $f_-(t)$ and the parameters d_k will satisfy the solvability conditions

$$\int_{\Gamma}g(t)\lambda_j(t)dt=0, \quad j=1,\ldots,l, \qquad (32)$$

where $\lambda_j(t)$ is a complete system of linearly independent solutions of the adjoint homogeneous equation of (21) of the adjoint class $h(c_1,c_2,\ldots,c_s)$, where $c_1,c_2,\ldots,c_s$ are all non-singular points among the points a_k, b_k [3].

Besides, the vectors $\mu(t)$ and $Q(t)$ will satisfy the conditions (30).

After defining the vectors $\mu(t)$, $P(t)$, $Q(t)$ from (21) and from the conditions (30), (32), we shall find $\omega(z)$ from (16), afterwards we shall find $\psi(z)$ from (15) and define $\phi(z)$ by the formula (4).

Rewrite the conditions (30) as

$$\int_{\Gamma}F_{jk}(t)\mu(t)dt+\int_{\lambda_j}\tau^k[\Pi(\tau)]^{-1}\gamma(\tau)d\tau=0,$$
$$j=1,\ldots,p,\ k=0,\ldots,m-1, \qquad (33)$$

where F_{jk} denotes the function

$$F_{jk}(t)=\frac{1}{2\pi i}\int_{\lambda_j}\frac{\tau^k[\Pi(\tau)]^{-1}d\tau}{t-\tau}.$$

The function $F_{jk}(t)$ is a holomorphic function on Γ and in it's neighbourhood. The conditions (33) can be considered (for a fixed μ) as a linear algebraic system with respect to the coefficients of the polynomial vector $\gamma(\tau)$. The number of equations is $2pmn$ (we consider the real equations), and the number of unknowns is $2[2p(m-1)-m]n$ if $m > 1$ (if $m = 1$, then $\gamma \equiv 0$).

Let us suppose that the rank of the system (33) is r

$$r \leq \min[2mpn, 2n(2p(m-1)-m)],$$

then the necessary and sufficient solvability conditions for this system are the $2mpn - r$ linear independent conditions

$$\int_\Gamma \sum_{k=1}^{n} \left[H_{jk}^{(1)} Re\mu_k(t) + H_{jk}^{(2)} Im\mu_k(t) \right] ds = 0, \quad j = 1, \ldots, 2mnp - r. \quad (34)$$

If the condition (34) is valid, then from the $2n[2p(m-1)-m]$ coefficients r are representable with the help of the vector μ and others will be undefined.

Hence the conditions (33) must be changed into (34), the values of the coefficients of the polynomial vector $\gamma(z)\,(\Omega(z))$ we will put on the right-hand side of the equation. N will be changed to $N - r$ and the obtained finite-dimensional operators we will transfer to the left-hand side. This will not affect the equations' Noetherity and it's index.

Therefore, the index of the problem for finding the vector $\mu(t)$ will be

$$\kappa_\mu = \kappa + N - r - (2mnp - r) = \kappa + N - 2mnp, \quad (35)$$

where κ denotes the index of the class H^* of the operator

$$A(t_0)\nu(t_0) + \frac{B(t_0)}{\pi i} \int_\Gamma \frac{\nu(t)dt}{t - t_0}.$$

It is evident, that the index of the problem for finding the vector $\omega(z)$ is the same as for the vector $\psi(z)$, κ_μ, and for the vector $\phi(z)$ the index will be

$$\kappa_\phi = \kappa_\mu + 2mn \quad (36)$$

if the coefficients of the polynomial vector $p(t)$ remain undefined during this procedure. If we have defined δ coefficients of the polynomial vector $p(t)$, then in the formula (36) instead of $2mn$, will be $2mn - \delta$. In conclusion we have the following result.

THEOREM *The problem* (1) *is normally-solvable in the class* $H^*(\Gamma)$ *when* $\det a_{(\pm)}^{(m)}(t) \neq 0$. *In this case the necessary and sufficient solvability conditions are* (6) *and the index of this class is calculated by the formula* (36), *where* κ_μ *is the index of problem* (17) *of the same class.*

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