THE CAUCHY-LEBESGUE TYPE INTEGRALS FOR THE GENERALIZED BELTRAMI SYSTEMS

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Consider the first order system of partial differential equations in the complex plane C

$$w_{\overline{z}} = Q\, w_z, \tag{1}$$

where Q is given $n \times n$ complex matrix of the class $W_p^1(C)$, $p > 2$ and $Q(z) = 0$ outside of some circle. (The notation $A \in K$, where A is a matrix and K is some class of functions, means that every element $A_{\alpha\beta}$ of A belongs to K. If K is some linear normed space with the norm $\| \cdot \|_K$, then $\|A\|_K = \max\limits_{\alpha,\beta} \|A_{\alpha\beta}\|_K$.)

Bojarski [1] assumed that the variable matrix Q in (1) is lower quasidiagonal matrix of the special form having the eigenvalues less than 1. Hile [2] noticed that what appears to be the essential property of the elliptic systems of the form (1) for which one can obtain a useful extension of the analytic function theory is the sell-commuting property of the matrix Q, which is

$$Q(z_1)\, Q(z_2) = Q(z_2)\, Q(z_1) \tag{2}$$

for any two points $z_1, z_2 \in C$. Following Hile if Q is sell-commuting in C, and if $Q(z)$ has eigenvalues the less than 1, then the system (1) is called generalized Beltrami system. Solutions of such a system will be called Q-holomorphic vectors. Under the solution of the equation (1) in some domain D we understand so-called regular solution (see [4]) i.e. $w(z) \in L_2(\overline{D})$, whose generalized derivatives $w_{\overline{z}}$, w_z belong to $L_r(D')$, $r > 2$, where $D' \subset D$ is an arbitrary closed subset. Equation (1) is to be satisfied almost everywhere in D.

Remark. We say that A and B commute in D if A and B are matrix valued functions defined in D and satisfy the condition

$$A(z_1)\, B(z_2) = B(z_2)\, A(z_1) \quad \text{for all} \quad z_1, z_2 \in D.$$

It is well-known, that if $\{Q_\alpha\}$ is a commutative collection of complex $n \times n$ matrices then there exists an invertible complex $n \times n$ matrix S such that for each Q_α,

$$S\, Q_\alpha\, S^{-1} = \begin{bmatrix} A_{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & A_{\alpha_m} \end{bmatrix}$$

and each λ_{α_j} has the following form:

$$A_{\alpha_j} = \begin{bmatrix} \lambda_{\alpha_j} & & 0 \\ & \ddots & \\ * & & \lambda_{\alpha_j} \end{bmatrix}$$

Each square block matrix A_{α_j} has the same size for all α, but may change with respect to j (the star-denotes possibly nonzero terms). This shows, in particular, that the self-commuting condition is really very strong condition.

The matrix valued function $\Phi(z)$ is a generating solution of the system (1) if it satisfies the following properties ([2]):

(i) $\Phi(z)$ is a C^1-solution of (1) in C;

(ii) $\Phi(z)$ is self-commuting and commutes with Q in C;

(iii) $\Phi(t) - \Phi(z)$ is invertible for all z, t in C, $z \neq t$;

(iv) $\partial_{\overline{z}}\Phi(z)$ is invertible for all z in C.

We call the matrix

$$V(t,z) = \partial_t \, \Phi(t)\big[\Phi(t) - \Phi(z)\big]^{-1}$$

the generalized Cauchy kernel for the system (1).

Let now Γ be a union of simple closed non-intersecting Liapunov-smooth curves, bounding finite or infinite domain. If Γ is one closed curve then D^+ denotes the finite domain; if Γ consists of several curves then by D^+ we denote the connected domain the boundary of which is Γ. On these curves the positive direction is chosen such, that when moving to this direction the domain D^+ remains left; the complement of open set $D^+ \cup \Gamma$ till the whole plane denote by D^-.

Assume the vector $\varphi(t) \in L(\Gamma)$ is given and consider the following integral:

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} V(t,z)\, d_Q\, t\, \varphi(t), \tag{3}$$

where $d_Q t = I\,dt + Q\,d\overline{t}$, I is a identity matrix. It is evident, that $\Phi(z)$ is Q-holomorphic vector everywhere outside of Γ, $\Phi(\infty) = 0$. We call the vector $\Phi(z)$ the generalized Cauchy-Lebesgue type integral for the system (1) with the jump line Γ. The boundary values of $\Phi(z)$ on Γ are given by the formulas:

$$\Phi^{\pm} = \pm\frac{1}{2}\,\varphi(t) + \frac{1}{2\pi i} \int\limits_{\Gamma} V(\tau,t)\, d_Q\, \tau\, \mu(\tau). \tag{4}$$

The formulas (4) are to be fulfilled almost everywhere on Γ, provided that $\Phi^{\pm}$ are angular boundary values of the vector $\Phi(z)$ and the integral in (4) is to be understood in the sense of Cauchy principal value.

Theorem 1. *Let $\Phi(z)$ be a Q-holomorphic vector on the plane cut along the Γ, $\Phi(\infty) = 0$. Let $\Phi(z)$ has the finite angular boundary values Φ^+ and Φ^- inside and outside of Γ respectively. For the vector $\Phi(z)$ to be* *representable by the Cauchy-Lebesgue type integral* (3) *with the jump line* Γ, *it is necessary and sufficient the fulfillment of the equality* (*cp.* [4])

$$\frac{1}{\pi i}\int\limits_{\Gamma} V(t,t_0)\, d_Q\, t\big[\Phi^+(t)-\Phi^-(t)\big] = \Phi^+(t_0)+\Phi^-(t_0). \tag{5}$$

almost everywhere on Γ.

(The equality (5) includes the condition $\Phi^+(t)-\Phi^-(t)\in L(\Gamma)$).

We call the generalized Cauchy-Lebesgue type integral (3) the generalized Cauchy-Lebesgue integral in the domain D^+ (in D^-) if $\Phi^+(t)=\varphi(t)$ ($\Phi^-(t)=-\varphi(t)$) almost everywhere on Γ.

From the Theorem 1 it is easy to conclude the following.

Theorem 2. *Let* $\Phi(z)$ *is a* Q*-holomorphic vector in* D^+ *and* $\Phi^+(t)\in L(\Gamma)$ (*in* D^-, $\Phi(\infty)=0$ *and* $\Phi^-(t)\in L(\Gamma)$) *then for this vector to be representable by the generalized Cauchy-Lebesgue integral in* D^+ (*in* D^-) *it is necessary and sufficient the fulfillment of the equality*

$$\frac{1}{\pi i}\int\limits_{\Gamma} V(t,t_0)\, d_Q\, t\, \Phi^+(t) = \Phi^+(t_0)$$

$$\left(\frac{1}{\pi i}\int\limits_{\Gamma} V(t,t_0)\, d_Q\, t\, \Phi^-(t) = -\Phi^-(t_0)\right)$$

almost everywhere on Γ.

Theorem 3. *Let* $\Phi(z)$ *be a* Q*-holomorphic vector representable by the generalized Cauchy-Lebesgue type integral in* D^+ *and let* $\Phi^+(t)\in L(\Gamma)$. *Then* $\Phi(z)$ *is representable by the generalized Cauchy-Lebesgue integral with respect to its boundary values. The analogous conclusion for the infinite domain* D^- *in case* $\Phi(\infty)=0$ *is also valid.*

Introduce now some classes of Q-holomorphic vectors. We say that Q-holomorphic vector $\Phi(z)$ belongs to the class $E_p(D^+,Q)$ [$(E_p(D^-,Q)$)], $p>1$, if $\Phi(z)$ is representable by the generalized Cauchy-Lebesgue type integral with the density from the class $L_p(\Gamma)$. It is easy to prove from the Theorem 3 it that every Q-holomorphic vector from the class $E_p(D^\pm,Q)$ is representable by the generalized Cauchy-Lebesgue integral with respect to its angular boundary values.

The following theorem is very important for the investigation of the Riemann-Hilbert type problems. One can prove it almost repeating the arguments of Bojarski [1] (see also [3]).

Theorem 4. *Let* D *be a domain of the complex plane bounded by the union of simple closed Liapunov-smooth curves* $\Gamma=\Gamma_0\cup\Gamma_1\cup\cdots\cup\Gamma_m$ *without common points,* $\Gamma_1,\ldots,\Gamma_m$ *are situated outside of each other, but inside of* Γ_0. (Γ_0 *may be missing, then* D *is infinite domain*). *Every* Q*-holomorphic* *vector of the class $E_p(D,Q)$ admits the following representation*

$$\Phi(z)=\frac{1}{\pi i}\int\limits_{\Gamma} V(t,s)\,d_Q\,t\,\mu(t)+iC, \tag{6}$$

where $\mu(t)\in L_p(\Gamma)$ is real vector, C is real constant vector. The vector $\mu(t)$ is defined uniquely within the constant vector or Γ_j, $j\geq 1$, $\mu(t)$ on Γ_0 and C are defined uniquely by the vector $\Phi(z)$.

References

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