On Some Qualitative Issues for the First Order Elliptic Systems in the Plane

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Introduction

The first order system of partial differential equations

$$\frac{\partial u}{\partial x} + A(x,y)\frac{\partial u}{\partial y} + B(x,y)u = 0, \tag{1}$$

where $u = (u_1, u_2, \ldots, u_{2n})$ is $2n$-component desired vector, A, B are given real $2n \times 2n$-matrices depending on two real variables x, y is called elliptic in some domain $G \subset R^2_{(x,y)}$, if

$$\det(A - \lambda I) \neq 0, \tag{2}$$

for every real λ and $(x, y) \in G$; I is an identity matrix. In other words the system (1) is elliptic if the matrix A has no real characteristic numbers in G.

The investigation of such system has a great history. Various particular cases of the system (1) were the object of investigation of Picard, Beltrami, Carleman, Bers, Vekua, Douglis, Bojarski, Hile, Begehr, D. Q. Dai and many other authors.

In the first part of our work we study the problem of validity of the maximum modulus theorem. To this end let us mention some auxiliary explanations. Under the solution of the system (1) we mean the classical solution of the class $C^1(G) \cap C(\overline{G})$.

Denote by

$$\Lambda(A, B) \tag{3}$$

the class of all possible solutions of the system (1); the matrices A and B are called the generating pair of the class (3).

Introduce

$$\rho_u(x,y) = \left[\sum u_k^2(x,y)\right]^{\frac{1}{2}}, \quad (x,y) \in \overline{G} \tag{4}$$

for every u of the class (3). And now raise a question (cf. Bojarski [1]).

Is the inequality

$$\rho_u(x_0,y_0) \leq \max_{(x,y)\in\Gamma} \rho_u(x,y) \tag{5}$$

valid for arbitrary u from (3) and $(x_0,y_0) \in \overline{G}$. Γ is a boundary of the domain G. Of course, in case $n=1$ and

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

the condition (5) if fulfilled.

Consider now $G = \{x^2+y^2 < 1\}$, $n=1$, A is the same matrix, $B = \begin{pmatrix} 2x & 0 \\ 2y & 0 \end{pmatrix}$ and $u = \text{column}(e^{-x^2-y^2}, 0) \in \Lambda(A,B)$.

It is evident, that $\rho_u(0,0) = 1$ and $\rho_u(x,y) = \frac{1}{e}$, i.e. the condition (5) is not fulfilled. In this example the matrix B is not constant matrix. This example shows, that the maximum modulus theorem for minimal dimensional elliptic system is not always true. It is easy to construct the example of higher dimensional system when the condition (5) is disturbed in case A and B are constant. In fact, consider G is the same domain $G = \{x^2+y^2 < 1\}$,

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ -6 & 0 & -2 & 0 \\ 0 & -6 & 0 & -2 \end{pmatrix}$$

and $u = \text{column}(u_1,u_2,u_3,u_4) \in \Lambda(A,B)$, where

$$u_1 = e^x(x\cos y + y\sin y), \qquad u_2 = e^x(y\cos y - x\sin y),$$
$$u_3 = 3(x^2+y^2-1)e^x\cos x, \qquad u_4 = -3(x^2+y^2-1)e^x\sin y.$$

It is clear, that

$$\rho_u(0,0) = 3, \qquad \max_{(x,y)\in\Gamma} \rho_u(x,y) = e$$

and therefore the condition (5) is not fulfilled.

In case the dimension of the system(1)–(3) is minimal, i.e. when $n=1$ and moreover, when

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad b_{kq} \in L_p(G), \quad p > 2.$$

We have with the great effort of very famous mathematicians, in some sense complete theory which is in very close connection with the theory of analytic functions of complex variable. In particular, it is well-known the following fact, that there exists the number $M \geq 1$ (depending only on the matrix B) such, that

$$\rho_u(x_0, y_0) \leq M \max_{(x,y)\in\Gamma} \rho_u(x, y) \tag{6}$$

for every $u \in \Lambda(A, B)$ and $(x_0, y_0) \in G$.

The inequality (6) is weaker than (5), but it is also very interesting problem as was noted by Bojarski in this work "General properties of the solutions of elliptic systems on the plane" in 1960.

Now we describe the sufficiently wide class of the elliptic systems (1)–(3), for which the inequality (6) as well as more strong inequality (5) holds. Consider the case of constant coefficients.

Theorem 1. *Let for the matrices A and B there exists the orthogonal matrix D such, that*

$$D^{-1}AD = \begin{pmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix} \tag{7}$$

$$D^{-1}BD = \begin{pmatrix} d_{11} & -h_{11} & d_{12} & -h_{12} & \cdots & d_{1n} & -h_{1n} \\ h_{11} & d_{11} & h_{12} & d_{12} & \cdots & h_{1n} & d_{1n} \\ d_{21} & -h_{21} & d_{22} & -h_{22} & \cdots & d_{2n} & -h_{2n} \\ h_{21} & d_{21} & h_{22} & d_{22} & \cdots & h_{2n} & -d_{2n} \\ \multicolumn{7}{c}{\dotfill} \\ d_{n1} & -h_{n1} & \cdots & \cdots & \cdots & d_{nn} & -h_{nn} \\ h_{n1} & -d_{n1} & \cdots & \cdots & \cdots & h_{nn} & -d_{nn} \end{pmatrix} \tag{8}$$

where d_{kp}, h_{kp}, $1 \leq k \leq n$, $1 \leq p \leq n$ are arbitrary real numbers and the constructed complex matrix

$$B_0 = \begin{pmatrix} d_{11} + ih_{11} & d_{12} + ih_{12} & \cdots & d_{1n} + ih_{1n} \\ d_{21} + ih_{21} & d_{22} + ih_{22} & \cdots & d_{2n} + ih_{2n} \\ \multicolumn{4}{c}{\dotfill} \\ d_{n1} + ih_{n1} & d_{n2} + ih_{n2} & \cdots & d_{nn} + ih_{nn} \end{pmatrix} \tag{9}$$

is a normal matrix, i.e. $B_0\overline{B}_0^T = \overline{B}_0^T B_0$. Then the inequality (5) holds for any $u \in L(A, B)$, $(x_0, y_0) \in \overline{G}$. Moreover, if the equality holds in some inner point of the domain G then the function ρ_u (but not necessarily vector-function u) is constant.

In above mentioned example, for the case $n = 2$, the conditions (7), (8) are fulfilled, but the constructed complex B_0 is not normal and therefore (5) is violated.

The second part of our work is devoted to the system (1) which has the following complex form

$$w_{\overline{z}} = Q w_z, \tag{10}$$

where Q is given $n \times n$ complex matrix of the class $W_p^1(C)$, $p > 2$ and $Q(z) = 0$ outside of some circle. In this case under the solution of the system (10) we understand the so-called regular solution [4], i.e. $w(z) \in L(\overline{G})$, whose generalized derivatives $w_{\overline{z}}$, w_z belong to $L_r(G')$, $r > 2$, $G' \subset G$ is an arbitrary closed subset. (10) is to be satisfied almost everywhere in D.

Bojarski [2] assumed, that the variable matrix Q in (10) is quasi-diagonal matrix of the special form having the eigenvalues less than 1. Hile noted that what appears to be essential property of the elliptic systems of the form (10) for which one can obtain a useful extension of the analytic function theory is the self-commuting property of the matrix Q, which is

$$Q(z_1)\,Q(z_2) = Q(z_2)\,Q(z_1) \tag{11}$$

for any two points z_1, z_2. Following Hile if Q is self-commuting and if $Q(z)$ has the eigenvalues less than 1 then the system (10) is called generalized Beltrami system. The solutions of such system is called Q-holomorphic vectors.

The matrix valued function $\Phi(z)$ is a generating solution of the system (10) if it satisfies the following properties ([2]):

(i) $\Phi(z)$ is a C^1-solution of (10) in G;

(ii) $\Phi(z)$ is a self-commuting and commutes with Q in G;

(iii) $\Phi(t) - \Phi(z)$ is invertible for all z, t in G, $z \neq t$;

(iv) $\partial_z \Phi(z)$ is invertible for all z in G.

The matrix $V(t,z) = \partial_t \Phi(t)[\Phi(t) - \Phi(z)]^{-1}$ we call the generalized Cauchy kernel for the system(10).

Let now Γ be a union of simple closed non-intersecting Liapunov-smooth curves bounding finite or infinite domain; if Γ is one closed curve then G denotes the finite domain; if Γ consists of several curves then by G^+ denote the connected domain with the boundary Γ. On these curves the positive direction is chosen such, that when passing along Γ, G^+ remains left; the complement of the open set $G^+ \cup \Gamma$ till the whole plane denote by G^-.

Assume the vector $\varphi(t) \in L(\Gamma)$ is given and consider the following integral

$$\Phi(z) = \frac{1}{2\pi i}\int_\Gamma V(t,z)\,d_Q t\,\varphi(t), \tag{12}$$

where $d_Q t = I\,dt + Q\,d\overline{t}$, I is an identity matrix. It is evident, that $\Phi(z)$ is Q-holomorphic vector everywhere outside of Γ, $\Phi(\infty) = 0$.

We call the vector $\Phi(z)$ the generalized Cauchy-Lebesgue type integral for the system (10) with the jump line Γ.

The boundary values of $\Phi(z)$ on Γ are given by the formulas

$$\Phi^{\pm}(t) = \pm\frac{1}{2}\,\varphi(t) + \frac{1}{2\pi i}\int_\Gamma V(\tau,t)\,d_Q\tau\,\mu(\tau), \tag{13}$$

These formulas are to be fulfilled almost everywhere on Γ, provided that $\Phi^{\pm}(t)$ are angular boundary values of the vector $\Phi(z)$ and the integral in (13) is to be understood in the sense of Cauchy principal value.

For the vector $\Phi(z)$ to be representable by the Cauchy-Lebesque type integral (12) with the jump line Γ, it is necessary and sufficient the fulfillment of the equality

$$\frac{1}{\pi i}\int_{\Gamma} V(t,t_0)\, d_Q t\big[\Phi^+(t)-\Phi^-(t)\big] = \Phi^+(t_0)+\Phi^-(t_0) \tag{14}$$

almost everywhere on Γ.

We call the generalized Cauchy-Lebesgue type integral (12) the generalized Cauchy-Lebesgue integral in the domain G^+ (G^-), if $\Phi^+(t)=\varphi(t)$ ($\Phi^-(t)=-\varphi(t)$) almost everywhere on Γ.

Theorem 2. *Let $\Phi(z)$ be a Q-holomorphic vector representable by the generalized Cauchy-Lebesgue type integral in G^+ and let $\Phi^+(t)\in L(\Gamma)$. Then $\Phi(z)$ is representable by the generalized Cauchy-Lebesgue integral with respect to its boundary values. The analogous conclusion for the infinite domain G^- in case $\Phi(\infty)=0$ is also valid.*

Introduce some classes of Q-holomorphic vectors. We say, that Q-holomorphic vector $\Phi(z)$ belongs to the class $E_p(G^+,Q)$ $[E_p^-(G^+,Q)]$, $p>1$, if $\Phi(z)$ is representable by the generalized Cauchy-Lebesgue type integral with the density from the class $L_p(\Gamma)$. It follows easily from the Theorem 2 that every Q-holomorphic vector from $E_p(G^{\pm},Q)$ is representable by the generalized Cauchy-Lebesgue integral with respect to its angular boundary values.

Theorem 3. *Let $\Phi(z)$ be a Q-holomorphic vector representable by the generalized Cauchy-Lebesgue type integral in G^+ (G^-) with the summable density. If the angular boundary values Φ^+ (Φ^-) belong to the class $L_p(\Gamma)$, $p>1$ then $\Phi(z)$ belongs to the class $E_p^+(G^+,Q)$ $[E_p^-(G^-,Q)]$.*

Theorem 4. *Let $\Phi(z)$ be a Q-holomorphic vector representable by the generalized Cauchy-Lebesgue type integral in simple connected domain G (G may be infinite). If $\mathrm{Re}[\Phi(t)]=0$ almost everywhere on the boundary of the domain G then $\Phi(z)=iC$, where C is a real constant vector. (In case G is infinite $C=0$).*

Here we select some useful properties of above mentioned classes which are the natural classes in order to correctly pose and complete analyze the discontinuous boundary value problems for pseudo-holomorphic vectors.

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